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On Strongly Multiplicative Labeling of Mycielskian Paths and Cycles

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Abstract:- The strongly multiplicative labeling of a graph G is defined as a labeling from $V(G)$ onto $\{1, 2, . . . , p\}$ such that the label induced on the edges by the product of labels of the end vertices are all distinct. In this paper, we investigate the existence of strongly multiplicative labelings of Mycielskian graph of paths and cycles.

Keywords*: Graph Labeling, Product Magic Labeling, Product Antimagic Labeling, Strongly Multiplicative Labeling.*

1. Introduction

All *graphs* G considered in this paper are finite, undirected, connected graph without any loops or multiple edges. Let $V(G)$ and $E(G)$ be the set of vertices and edges of a graph G , respectively. The *order* and *size* of a graph G is denoted as $p = |V(G)|$ and $q = |E(G)|$ respectively. For general graph theoretic notions we refer Harrary [4].

A *labeling* of a graph G is a one-to-one mapping that carries the set of graph elements onto a set of numbers (usually positive or non-negative integers), called *labels*. There are several types of labeling and a detailed survey of many of them can be found in the dynamic survey of graph labeling by Gallian [3].

In 2000, Figueroa-Centeno, Ichishima and Muntaner-Batle [2] have introduced the concept of product magic labeling and they define a graph G of size q to be *product magic* if there is a labeling f from $E(G)$ onto $\{1, 2, . . . , q\}$ such that at each vertex u , the product of the labels on the edges incident with u is a constant and *product antimagic* if the product of the labels on the edges incident at each vertex u are distinct. They proved the following: a graph of size q is product magic if and only if $q \leq 1$ (that is, if and only if it is K_2 or $\overline{K_n}$ or $K_2 \cup \overline{K_n}$); every path Pn ($n \geq 4$) is product antimagic; every cycle Cn is product antimagic; every 2-regular graph is product antimagic; if G is product antimagic then so are $G + K_1$ and $G \odot \overline{K_n}$. They conjectured that a connected graph of size q is product antimagic if and only if $q \geq 3$.

Kaplan et al. [5] proved that the following graphs are product antimagic: the disjoint union of cycles and paths where each path has at least three edges; connected graphs with n vertices and m edges where $m \geq 4nlnn$; all complete k -partite graphs except K_2 and $K_{1,2}$. In [6], Pikhurko characterizes all large graphs that are product antimagic.

Figueroa-Centeno, Ichishima and Muntaner-Batle [2] also defined a (p, q) -graph G as *product edge-magic* if there is a labeling f from $V(G) \cup E(G)$ onto $\{1, 2, . . . , p + q\}$ such that the value $f(u). f(v). f(uv)$ for every edge $uv \in E(G)$ is a constant and *product edge- antimagic* if the value $f(u). f(v). f(uv)$ for each edge $uv \in E(G)$ are distinct. They proved that $K_2 \cup \overline{K_n}$ is product edge-magic, a graph of size q without isolated vertices is product edge-magic if and only if $q \leq 1$ and every graph other than K_2 and $K_2 \cup \overline{K_n}$ are product edge-antimagic. Because of the total labeling nature, the product edge-antimagic labeling is also called as Product Edge-Antimagic Total (PEAT) labeling.

L.W. Beineke and S.M. Hegde [1] introduced the concept of *strongly multiplicative labeling*, which is defined as a labeling from $V(G)$ onto $\{1, 2, \ldots, p\}$ such that the label induced on the edges by the product of labels of the end vertices are all distinct. They proved that the complete graph Kp is strongly multiplicative if and only if $p \leq 5$, the complete bipartite graph $K_{r,r}$ is strongly multiplicative if and only if $r \leq 4$. They also proved that the cycles, wheels and trees are strongly multiplicative.

In this paper, we investigate the existence of strongly multiplicative labelings of Mycielskian graph of paths and cycles.

2. Property of Strongly Multiplicative Labeling

The following theorem gives a relation between strongly multiplicative labeling and product edge antimagic total (PEAT) labeling of a graph G.

Theorem 2.1: *If a graph G has a strongly multiplicative labeling, then it is PEAT.*

Proof: Let G be strongly multiplicative.

Then by definition, there exits a labeling $f : V(G) \to \{1,2,\ldots,q\}$ such that for each edge

$uv \in E(G)$, $\pi(uv) = f(u) \times f(v)$ are distinct.

Let the edges of G be $e_1, e_2, \ldots, e_q$ such that $\pi_f(e_1) < \pi_f(e_2) < \cdots < \pi_f(e_q)$.

Let us define a bijection $g : V(G) \cup E(G) \to \{1,2,\ldots,q+p\}$ such that

(i) for each vertex $u \in V(G)$, $g(u) = f(u)$

(ii) for each edge $e_i \in E(G)$, $g(e_i) = p+i$ where $1 \leq i \leq q$.

It is clear that, the labeling g maps the vertices of G to $\{1,2,\ldots,p\}$.

Also,

$$\pi_f(e_1) < \pi_f(e_2) < \cdots < \pi_f(e_q)$$

$$\Rightarrow \pi_f(e_1) \times (p+1) < \pi_f(e_2) \times (p+2) < \cdots < \pi_f(e_q) \times (p+q)$$

$$\Rightarrow \pi_g(e_1) < \pi_g(e_2) < \cdots < \pi_g(e_q)$$

Thus, under the total labeling g, $\pi_g(e)$ are distinct for every $e \in E(G)$.

Hence, g is a PEAT labeling of G. □

3. Mycielskian Graph of Paths and Cycles

The *Mycielskian graph* of a graph G denoted by $\mu(G)$ is defined as a graph which contains G itself as a subgraph, together with $n+1$ additional vertices namely a vertex v_i corresponding to each vertex u_i of G, and an extra vertex w. Each vertex v_i is connected by an edge to w, so that these vertices form a subgraph in the form of a star $K_{1,n}$. In addition, for each edge $u_i u_j$ of G, the Mycielskian of the graph includes two edges, $u_i v_j$ and $v_i u_j$. Thus, if G has n vertices and m edges, then $\mu(G)$ has $2n+1$ vertices and $3m+n$ edges.

Theorem 3.1: *For $n \geq 3$, the Mycielskian graph of paths P_n has a strongly multiplicative labeling.*

Proof: Let $G = \mu(P_n)$ with

$V(G) = \{w, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and

$E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$,

Where $E_1 = \{u_i u_{i+1} : 1 \le i \le n-1\}$

$E_2 = \{v_i u_{i+1} : 1 \le i \le n-1\}$

$E_3 = \{v_i u_{i-1} : 2 \le i \le n\}$

$E_4 = \{w v_i : 1 \le i \le n\}$.

Define $f : V(G) \to \{1, 2, \ldots, 2n+1\}$ as

$f(u_i) = 2n + 2 - i; 1 \le i \le n$ $f(v_i) = i + 1; 1 \le i \le n$ and $f(w) = 1$.

Let us define an induced function $\pi : E(G) \to N$, such that $\pi(uv) = f(u) \times f(v)$

Claim: For any $uv \in E(G)$, $\pi(uv)$ are distinct.

For $1 \le i \le n$,

$\pi(wv_i) = f(w) \times f(v_i) = i + 1 = \{2, 3, \ldots, n+1\}$.

For $1 \le i \le n-1$,

$$\begin{aligned}
\pi(u_i u_{i+1}) &= f(u_i) \times f(u_{i+1}) = (2n+2-i)(2n+2-(i+1)) \\
&= (2n+2-i)(2n+1-i) \\
&= \{(2n+1)2n, 2n(2n-1), (2n-1)(2n-2), \ldots, (n+3)(n+2)\} \\
\pi(v_i u_{i+1}) &= f(v_i) \times f(u_{i+1}) = (i+1)(2n+2-(i+1)) \\
&= (i+1)(2n+1-i) \\
&= \{2(2n), 3(2n-1), \ldots, n(n+2)\}.
\end{aligned}$$

For $2 \le i \le n$,

$$\begin{aligned}
\pi(v_i u_{i-1} &= f(v_i) \times f(u_{i-1}) = (i+1)(2n+2-(i-1)) \\
&= (i+1)(2n+3-i) \\
&= \{3(2n+1), 4(2n-1), \ldots, (n+1)(n+3)\}.
\end{aligned}$$

Thus, for any $uv \in E(G)$, it is clear that $\pi(uv)$ are distinct.

Hence, *f* is a strongly multiplicative labeling of $\mu(P_n)$, $n \ge 3$. □

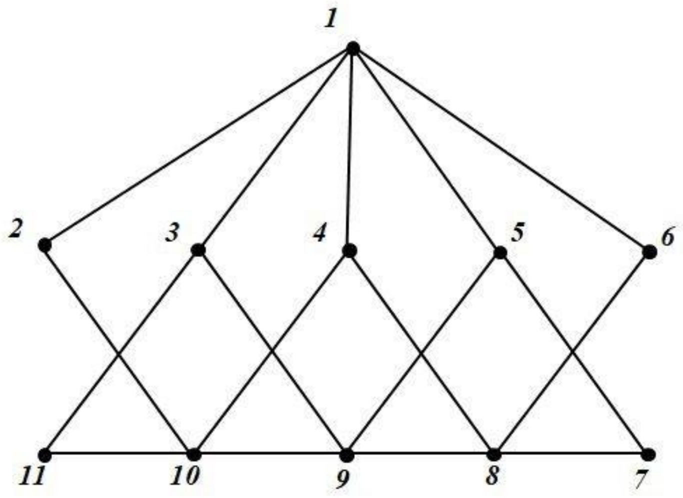

Figure 1. Strongly multiplicative labeling of $\mu(P_5)$

Theorem 3.2: *For $n \geq 3$, the Mycielskian graph of cycles C_n has a strongly multiplicative labeling.*

Proof: Let $G = \mu(C_n)$ with

$V(G) = \{w, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and

$E(G) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$,

Where $E_1 = \{u_i u_{i+1} : 1 \leq i \leq n-1\}$

$E_2 = \{v_i u_{i+1} : 1 \leq i \leq n-1\}$

$E_3 = \{v_i u_{i-1} : 2 \leq i \leq n\}$

$E_4 = \{w v_i : 1 \leq i \leq n\}$

$E_5 = \{u_n u_1, v_1 u_n, v_n u_1\}$.

Define $f : V(G) \rightarrow \{1, 2, \ldots, 2n+1\}$ as in Theorem 3.1 .

Let us define an induced function $\pi : E(G) \rightarrow N$, such that $\pi(uv) = f(u) \times f(v)$.

Already in the proof of Theorem 3.1 , it is proved that for any $uv \in E_1 \cup E_2 \cup E_3 \cup E_4$, $\pi(uv)$ are distinct.

Claim: For any $uv \in E_5$, $\pi(uv)$ are also distinct.

$\pi(u_n u_1) = f(u_n) \times f(u_1) = (n+2)(2n+1)$

$\pi(v_1 u_n) = f(v_1) \times f(u_n) = 2(n+2)$

$\pi(v_n u_1) = f(v_n) \times f(u_1) = (n+1)(2n+1)$.

Thus, for any $uv \in E_5$, $\pi(uv)$ are distinct and they are also distinct from $\pi(uv)$, $uv \in E_1 \cup E_2 \cup E_3 \cup E_4$.

Thus, for any $uv \in E(G)$, it is clear that $\pi(uv)$ are distinct.

Hence, f is a strongly multiplicative labeling of $\mu(C_n)$, $n \geq 3$. □

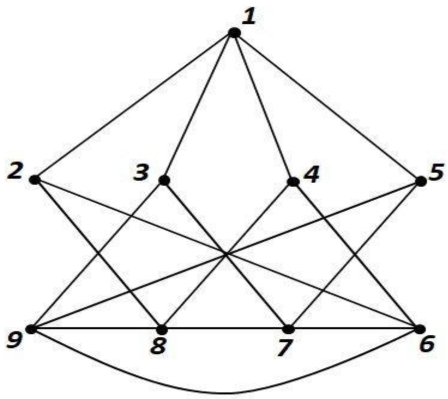

Figure 2. Strongly multiplicative labeling of $\mu(C_4)$

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