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An extended forward-backwards implicit scheme for a nonlinear mass-spring finite element time-dependent system

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Abstract

This paper reports the development of extended forward-backwards difference numerical solution procedures for a class of nonlinear systems. The theoretical accuracy analysis revealed a high-order accuracy, with an error of order $O(\Delta t^7)$ for displacement, and an error of order $O(\Delta t^5)$ for velocity. This is an improvement over simple second-order accurate implicit integration schemes. The Lyapunov stability analysis showed that the algorithm developed was unconditionally stable, and values of integration parameters, γ_{ijk}, can be determined from stability analyses. A linear system of two-degrees-of-freedom was initially solved to illustrate how to extend the methods to deal with multiple-degrees-of-freedom systems using matrices and vectors. The results of Thomson [16] were confirmed. Two nonlinear finite element problems were successfully solved. The hundred-degrees-of-freedom system confirmed the results of Wang [19]. The graphs of displacement versus time look similar. In addition, phase trajectory plots and a velocity versus displacement graph revealed the property of a closed path for the nonlinear mass-spring system. The accuracy of the results was not compared because such an exercise would require gaining access to the authors' data.

Keywords - Nonlinear dynamical systems; forward-backward difference schemes; Lyapunov stability analysis.

1 Introduction

The study of natural sciences uses experimentation and observation to understand, describe, and predict the natural world. It includes the study of many subjects, such as biology, chemistry, physics, astronomy, and earth science. Integral and differential equations are formulated, which are usually nonlinear. Such system equations include the van der Pol equation, the Lorenz equations, and the Schroedinger equations in quantum physics. The aims and objectives of this study were to accurately solve nonlinear differential equations arising from the mathematical theories that describe and predict the natural world, where closed-form solutions are not possible, so linearised equations and iterative solution procedures are usually employed.

This paper provides details of extended forward-backwards difference numerical solution procedures for a class of nonlinear systems. A linear system of two-degrees-of-freedom is initially solved to illustrate how to extend the methods to deal with multiple-degrees-of-freedom systems using matrices and vectors, which are typically obtained in finite element methods. This paper starts by acknowledging the early contributions of the Newmark trapezoidal scheme [1] that possessed limited accuracy and stability characteristics, but was used for time-integration of nonlinear finite element analysis of solids and structures to further improve solution procedures such as the improved numerical dissipation of Hilber et al. [2], consistent tangent operators of Simo and Taylor [3], time-stepping schemes of Wood [4], simple second order accurate implicit integration schemes of Bathe et al. [5], and finite element methods of Zienkiewicz et al. [6]. More contributions on convergence, stability and accuracy are contained in subsequent sections and in references [6-19].

1.1 Nonlinear vector-valued oscillatory system

The differential equation describing a nonlinear vector-valued oscillatory system may have the following general form:

$$\ddot{x} + f(\dot{x}, x, t) = 0; x(0) = x_0; \dot{x}(0) = \dot{x}_0; t = t_0 \quad (1)$$

The superposed dot on x represents differentiation with respect to time, t, and the double-dot represents the second derivative. Closed-form solutions of most nonlinear systems do not exist. Nonlinear mass-spring finite element time-dependent systems occurring in science and engineering, which generally do not have closed-form solutions, are considered.

1.2 Implicit schemes of Zienkiewicz

Zienkiewicz et al. [6] introduced an implicit generalised Newmark integration scheme from the truncated Taylor series expansion of the displacement function u and its derivatives, repeated here for convenience as follows:

$$u_{n+1} = u_n + \Delta t\dot{u}_n + \cdots + \frac{\Delta t^p}{p!}u_n^{(p)} + \beta_p \frac{\Delta t^p}{p!}\left(u_{n+1}^{(p)} - u_n^{(p)}\right)$$

$$\dot{u}_{n+1} = \dot{u}_n + \Delta t\ddot{u}_n + \cdots + \frac{\Delta t^{p-1}}{(p-1)!}u_n^{(p)} + \beta_{p-1}\frac{\Delta t^{p-1}}{(p-1)!}\left(u_{n+1}^{(p)} - u_n^{(p)}\right)$$

$$u_{n+1}^{(p-1)} = u_n^{(p-1)} + \Delta t u_n^{(p)} + \beta_1 \Delta t \left(u_{n+1}^{(p)} - u_n^{(p)}\right) \quad (2)$$

where $u, \dot{u}, \ddot{u}$ are displacement, velocity and acceleration. Setting $p = 2$ forms the equivalent Newmark scheme [2] which consists of two recurrence equations of displacement and velocity, which, when combined with the governing second-order differential equation (1), gives three simultaneous equations in three unknowns.

1.3 Forward-backwards implicit scheme

Carrying on from these contributions, a forward-backwards difference time-integration scheme was developed by Kaunda [7], using Taylor series, for solutions of nonlinear oscillatory systems. This scheme allows for more accurate implicit generalised one-step multiple-value algorithms [7],[8], repeated here for convenience, and summarised in Table 1.

$$s_{n+1} + \sum_{k=1}^{k=p}\left[\frac{(-1)^k}{k!}\left[\gamma_{1k}\Delta t\frac{d}{dt}\right]^k s_{n+1}\right] = s^* = s_n + \sum_{k=1}^{k=p}\left[\frac{1}{k!}\left[\beta_{1k}\Delta t\frac{d}{dt}\right]^k s_n\right] \quad (3)$$

$$v_{n+1} + \sum_{k=1}^{k=p-1}\left[\frac{(-1)^k}{k!}\left[\gamma_{2k}\Delta t\frac{d}{dt}\right]^k v_{n+1}\right] = v^* = v_n + \sum_{k=1}^{k=p-1}\left[\frac{1}{k!}\left[\beta_{2k}\Delta t\frac{d}{dt}\right]^k v_n\right] \quad (4)$$

Table 1: Forward-backward difference scheme illustration

$forward - difference\ \beta_{ik}\Delta t$	$backward - difference\ \gamma_{ik}\Delta t$	$1\ \Delta t$
$\beta_{ik}\Delta t = (t^* - t_n)\Delta t$	$\gamma_{ik}\Delta t = (t_{n+1} - t^*)\Delta t$	$1\ \Delta t$

where $s = x$ denotes displacement, $v = \dot{x}$ ˙ denotes velocity and $a = \ddot{x}$ ¨represents acceleration. Equations (3) and (4) provide the necessary extra equations to solve the differential equation (1) such that there are three equations in three unknowns. The implicit algorithms presented in [6],[7],[8], permitted to determine and optimise stability and accuracy of the recurrence equations by choosing appropriate tuneable integration parameters, $\beta_p, \gamma_{ik}, \beta_{ik}$. Numerical dissipation or algorithmic damping, mostly desired in finite element methods, may also be incorporated to filter out high frequency responses, as considered in Hilber et al. [2].

1.4 Article organisation

This article is organised as follows: Section 2 develops the solution of nonlinear vector-valued oscillatory systems. These are two-degrees-of-freedom system, and extension to multiple-degree-of-freedom systems using mass, damping and stiffness matrices such as those obtained from finite element methods. Section 3 presents and discusses the results, and Section 4 draws conclusions.

2. Nonlinear vector-valued oscillatory systems

2.1 Extended forward-backward implicit schemes

The displacement is expanded in two forward-backward implicit schemes as shown in equation (5), and summarised in Table 2.

$$s_{n+1} + \sum_{k=1}^{k=p} \left[\frac{(-1)^k}{k!} \left[\gamma_{11k} \Delta t \frac{d}{dt} \right]^k s_{n+1} \right] = s_1^* = s_n + \sum_{k=1}^{k=p} \left[\frac{1}{k!} \left[[\gamma_{12k} + \gamma_{13k}] \Delta t \frac{d}{dt} \right]^k s_n \right]$$

$$s_{n+1} + \sum_{k=1}^{k=p} \left[\frac{(-1)^k}{k!} \left[[\gamma_{11k} + \gamma_{12k}] \Delta t \frac{d}{dt} \right]^k s_{n+1} \right] = s_2^* = s_n + \sum_{k=1}^{k=p} \left[\frac{1}{k!} \left[\gamma_{13k} \Delta t \frac{d}{dt} \right]^k s_n \right] \quad (5)$$

Table 2: Extended forward-backward difference scheme illustration

$forward - difference\ (t_1^* - t_n)$	$backward - difference\ (t_{n+1} - t_1^*)$	$1\ \Delta t$
$(\gamma_{12k} + \gamma_{13k})\Delta t = t_1^* - t_n$	$\gamma_{11k}\Delta t = t_{n+1} - t_1^*$	$1\ \Delta t$
$\gamma_{13k}\Delta t = (t_2^* - t_n)\Delta t$	$(\gamma_{11k} + \gamma_{12k})\Delta t = (t_{n+1} - t_2^*)\Delta t$	$1\ \Delta t$

Two different expansions are used for displacement with the first expansion using the middle point, t_1^* and the second expansion using a different middle point, t_2^*. The last subscript, $k = 1,2,\ldots,p$, is for the summation term. The velocity is similarly expanded as shown in equation (6).

$$v_{n+1} + \sum_{k=1}^{k=p} \left[\frac{(-1)^k}{k!} \left[\gamma_{21k} \Delta t \frac{d}{dt} \right]^k v_{n+1} \right] = v_1^* = v_n + \sum_{k=1}^{k=p} \left[\frac{1}{k!} \left[[\gamma_{22k} + \gamma_{23k}] \Delta t \frac{d}{dt} \right]^k v_n \right]$$

$$v_{n+1} + \sum_{k=1}^{k=p} \left[\frac{(-1)^k}{k!} \left[[\gamma_{21k} + \gamma_{22k}] \Delta t \frac{d}{dt} \right]^k v_{n+1} \right] = v_2^* = v_n + \sum_{k=1}^{k=p} \left[\frac{1}{k!} \left[\gamma_{23k} \Delta t \frac{d}{dt} \right]^k v_n \right] \quad (6)$$

Similarly, the acceleration is expanded as shown in equation (7).

$$a_{n+1} + \sum_{k=1}^{k=p} \left[\frac{(-1)^k}{k!} \left[\gamma_{31k} \Delta t \frac{d}{dt} \right]^k a_{n+1} \right] = a_1^* = a_n + \sum_{k=1}^{k=p} \left[\frac{1}{k!} \left[[\gamma_{32k} + \gamma_{33k}] \Delta t \frac{d}{dt} \right]^k a_n \right]$$

$$a_{n+1} + \sum_{k=1}^{k=p}\left[\frac{(-1)^k}{k!}\left[[\gamma_{31k} + \gamma_{32k}]\Delta t\,\frac{d}{dt}\right]^k a_{n+1}\right] = a_2^* = a_n + \sum_{k=1}^{k=p}\left[\frac{1}{k!}\left[\gamma_{33k}\Delta t\,\frac{d}{dt}\right]^k a_n\right] \quad (7)$$

The recurrence equations for displacement and velocity are then combined with the governing differential equation to form a set of simultaneous equations written in matrix form.

It may be shown, for example, that for the upper summation limit set to p = 2, the recurrence equations can be used in solving a second order differential equation as shown in equation (8).

$$\begin{bmatrix} K & D & M \\ 1 & c_{22} & c_{23} \\ 0 & 1 & c_{33} \end{bmatrix}\begin{Bmatrix} s_{n+1} \\ v_{n+1} \\ a_{n+1} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & d_{22} & d_{23} \\ 0 & 1 & d_{33} \end{bmatrix}\begin{Bmatrix} s_n \\ v_n \\ a_n \end{Bmatrix} + \begin{Bmatrix} F_{n+1} \\ 0 \\ 0 \end{Bmatrix} \quad (8)$$

Similarly, for the upper summation limit set to p = 3, which includes terms of the first derivative of acceleration, $\dot{a}$, the recurrence equations can be used in solving a second order differential equation as shown in equation (9).

$$\begin{bmatrix} K & D & M & 0 \\ 0 & K & D & M \\ 1 & e_{32} & e_{33} & e_{34} \\ 0 & 1 & e_{43} & e_{44} \end{bmatrix}\begin{Bmatrix} s_{n+1} \\ v_{n+1} \\ a_{n+1} \\ \dot{a}_{n+1} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & f_{32} & f_{33} & f_{34} \\ 0 & 1 & f_{43} & f_{44} \end{bmatrix}\begin{Bmatrix} s_n \\ v_n \\ a_n \\ \dot{a}_n \end{Bmatrix} + \begin{Bmatrix} F_{n+1} \\ \dot{F}_{n+1} \\ 0 \\ 0 \end{Bmatrix} \quad (9)$$

The second equation results from the derivative of the first equation with respect to time, t, and the last two recurrence equations consist of the displacement and velocity, respectively.

Similarly, for the upper summation limit set to p = 4, which includes terms of the second derivative of acceleration, $\ddot{a}$, the recurrence equations can be used in solving a second order differential equation as shown in equation (10).

$$\begin{bmatrix} K & D & M & 0 & 0 \\ 0 & K & D & M & 0 \\ 0 & 0 & K & D & M \\ 1 & g_{42} & g_{43} & g_{44} & g_{45} \\ 0 & 1 & g_{53} & g_{54} & g_{55} \end{bmatrix}\begin{Bmatrix} s_{n+1} \\ v_{n+1} \\ a_{n+1} \\ \dot{a}_{n+1} \\ \ddot{a}_{n+1} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & h_{42} & h_{43} & h_{44} & h_{45} \\ 0 & 1 & h_{53} & h_{54} & h_{55} \end{bmatrix}\begin{Bmatrix} s_n \\ v_n \\ a_n \\ \dot{a}_n \\ \ddot{a}_n \end{Bmatrix} + \begin{Bmatrix} F_{n+1} \\ \dot{F}_{n+1} \\ \ddot{F}_{n+1} \\ 0 \\ 0 \end{Bmatrix} \quad (10)$$

The third equation results from the derivative of the second equation with respect to time, t, and the last two recurrence equations consist of the displacement and velocity, respectively.

The elements of [3x3] matrices C and D, corresponding to setting p = 2, are defined in the set of equations (11).

$$c_{22} = -\frac{1}{2}(2\gamma_{111} + \gamma_{121})\Delta t$$

$$c_{23} = -\frac{1}{4}[\gamma_{112}^2 + (\gamma_{112} + \gamma_{122})^2]\Delta t^2$$

$$c_{33} = -\frac{1}{2}(2\gamma_{211} + \gamma_{221})\Delta t$$

$$d_{22} = \frac{1}{2}(\gamma_{121} + 2\gamma_{131})\Delta t$$

$$d_{23} = \frac{1}{4}[\gamma_{132}^2 + (\gamma_{122} + \gamma_{132})^2]\Delta t^2$$

$$d_{33} = \frac{1}{2}(2\gamma_{231} + \gamma_{221})\Delta t \quad (11)$$

The elements of [4x4] matrices E and F, corresponding to setting p = 3, are defined in the set of equations (12).

$$e_{32} = -\frac{1}{2}(2\gamma_{111} + \gamma_{121})\Delta t$$

$$e_{33} = \frac{1}{4}[\gamma_{112}^2 + (\gamma_{112} + \gamma_{122})^2]\Delta t^2$$

$$e_{34} = -\frac{1}{12}[\gamma_{113}^3 + (\gamma_{113} + \gamma_{123})^3]\Delta t^3$$

$$e_{43} = -\frac{1}{2}(2\gamma_{211} + \gamma_{221})\Delta t$$

$$e_{44} = \frac{1}{4}[\gamma_{212}^2 + (\gamma_{212} + \gamma_{222})^2]\Delta t^2$$

$$f_{32} = \frac{1}{2}(\gamma_{121} + 2\gamma_{131})\Delta t$$

$$f_{33} = \frac{1}{4}[(\gamma_{122} + \gamma_{132})^2 + \gamma_{132}^2]\Delta t^2$$

$$f_{34} = \frac{1}{12}[(\gamma_{123} + \gamma_{133})^3 + \gamma_{133}^3]\Delta t^3$$

$$f_{43} = \frac{1}{2}(\gamma_{221} + 2\gamma_{231})\Delta t$$

$$f_{44} = \frac{1}{4}[(\gamma_{222} + \gamma_{232})^2 + \gamma_{232}^2]\Delta t^2 \quad (12)$$

The elements of [5x5] matrices G and H, corresponding to setting p = 4, are defined in the set of equations (13).

$$g_{42} = -\frac{1}{2}(2\gamma_{111} + \gamma_{121})\Delta t$$

$$g_{43} = \frac{1}{4}[\gamma_{112}^2 + (\gamma_{112} + \gamma_{122})^2]\Delta t^2$$

$$g_{44} = -\frac{1}{12}[\gamma_{113}^3 + (\gamma_{113} + \gamma_{123})^3]\Delta t^3$$

$$g_{45} = \frac{1}{48}[\gamma_{114}^4 + (\gamma_{114} + \gamma_{124})^4]\Delta t^4$$

$$g_{53} = -\frac{1}{2}(2\gamma_{211} + \gamma_{221})\Delta t$$

$$g_{54} = \frac{1}{4}[\gamma_{212}^2 + (\gamma_{212} + \gamma_{222})^2]\Delta t^2$$

$$g_{55} = -\frac{1}{12}[\gamma_{213}^3 + (\gamma_{213} + \gamma_{223})^3]\Delta t^3$$

$$h_{42} = \frac{1}{2}(\gamma_{121} + 2\gamma_{131})\Delta t$$

$$h_{43} = \frac{1}{4}[(\gamma_{122} + \gamma_{132})^2 + \gamma_{132}^2]\Delta t^2$$

$$h_{44} = \frac{1}{12}[(\gamma_{123} + \gamma_{133})^3 + \gamma_{133}^3]\Delta t^3$$

$$h_{45} = \frac{1}{48}[(\gamma_{124} + \gamma_{134})^4 + \gamma_{134}^4]\Delta t^4$$

$$h_{53} = \frac{1}{2}(\gamma_{221} + 2\gamma_{231})\Delta t$$

$$h_{54} = \frac{1}{4}[(\gamma_{222} + \gamma_{232})^2 + \gamma_{232}^2]\Delta t^2$$

$$h_{55} = \frac{1}{12}[(\gamma_{223} + \gamma_{233})^3 + \gamma_{233}^3]\Delta t^3 \quad (13)$$

Values of integration parameters, γ_{ijk}, are determined from stability and accuracy analyses.

2.2 Determination of integration parameters for implicit schemes

2.2.1 Lyapunov stability analysis

Applying Lyapunov stability of system matrix equations for asymptotic stability of the one-step multiple-value algorithms [7],[8], or to ensure that the matrix E is non-singular, Sylvester's theorem is applied (a Hermitian matrix, E, is positive-semidefinite if and only if all principal minors of E are non-negative). This enforces the positive-semidefinite properties of the dynamics matrices using the theorem given in matrix equations as in the set of equations (14).

$$e_{11} > 0$$

$$\begin{vmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{vmatrix} \geq 0$$

$$\mathrm{B} \begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{vmatrix} \geq 0 \quad (14)$$

Sylvester's theorem is also applied to check the positive-semidefinite properties of the dynamics matrices, M, D, K, especially those matrices that need an inverse. For the above matrix equations, matrices C, E, and G, corresponding to setting p = 2, 3, 4, need to have inverses to compute kinematic variables, $s_{n+1}, v_{n+1}, a_{n+1}, etc.$ Matrices E and G, by inspection, present better properties of stability by the Lyapunov method. Hence, the values of integration parameters, γ_{ijk}, may be determined from stability analyses.

2.2.2 Accuracy analysis

The next stage is to compare XXXX with the one-step four-value method of Kaunda [7],[8] that was rigorously derived using theoretical accuracy analyses given in Hildebrand [11], with error accuracies of order, $\mathrm{E_s} = -\frac{\Delta t^7}{100800}\mathrm{s}^{(7)}(\zeta_\mathrm{s})$ for displacement and $\mathrm{E_v} = \frac{\Delta t^5}{720}\mathrm{v}^{(5)}(\zeta_\mathrm{v})$ for velocity. This may be shown in the set of equations (15), such that,

$$\gamma_{111} = \gamma_{121} = \gamma_{131} = \frac{1}{3}$$

$$\gamma_{211} = \gamma_{221} = \gamma_{231} = \frac{1}{3}$$

$$\gamma_{311} = \gamma_{321} = \gamma_{331} = \frac{1}{3}\gamma_{411} = \gamma_{421} = \gamma_{431} = \frac{1}{3} \quad (15)$$

may be suitable in conjunction with the set of equations (16):

$$\gamma_{112} = \gamma_{122} = \gamma_{132} = \sqrt{\frac{6}{35}}$$

$$\gamma_{212} = \gamma_{222} = \gamma_{232} = \sqrt{\frac{4}{25}}$$

$$\gamma_{312} = \gamma_{322} = \gamma_{332} = \sqrt{\frac{2}{15}}$$

$$\gamma_{113} = \gamma_{123} = \gamma_{133} = \sqrt[3]{\frac{2}{21}}$$

$$\gamma_{213} = \gamma_{223} = \gamma_{233} = \sqrt[3]{\frac{1}{15}}$$

$$\gamma_{114} = \gamma_{124} = \gamma_{134} = \sqrt[4]{\frac{24}{595}} \quad (16)$$

Hence, the values of integration parameters, γ_{ijk}, may also be determined from accuracy analyses. A compromise between the values from stability and accuracy analyses may be necessary for optimal selection of integration parameters.

In addition to checking the determinants and ranks of matrices to ensure that a matrix inverse exists, an eigenvalue problem usually needs to be solved for matrices, for example, G and H above, to calculate the eigenvalues, λ, with the corresponding natural frequencies, ω, from which the fundamental frequency, f, and period, τ, can be obtained, which may then be used to set an appropriate time step. From a calculated period, τ, by a rule of thumb, it is recommended to set the time step as $\Delta t \leq \frac{\tau}{20}$. In addition, an eigenvalue problem needs to be solved for the mass and stiffness matrices, M and K.

3. Results and discussion of results

3.1 Two-degrees-of-freedom systems (2-dof)

A linear second-order two-degrees-of-freedom system subjected to a history of loading $f_i(t)$, and the initial conditions of displacement $x_i(0)$, and velocity $\dot{x}_i(0)$, forms a vector-valued function, and is shown in Figure 1. The differential equation is given in equation (17).

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{s}_1 \\ \ddot{s}_2 \end{Bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{Bmatrix} \dot{s}_1 \\ \dot{s}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} s_1 \\ s_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \quad (17)$$

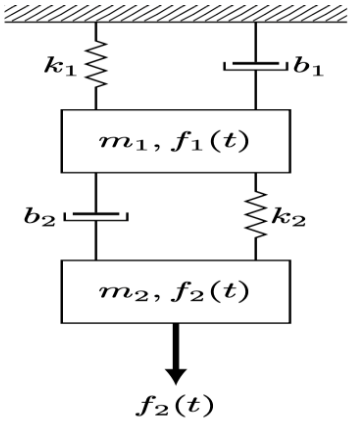

Figure 1: 2-dof Mass-spring-damper system

A linear two-degrees-of-freedom system was taken from Thomson [16] with:

$m_{11} = m_1 = 100$; $m_{12} = m_{21} = 0$; $m_{22} = m_2 = 25$; $b_{11} = b_{12} = b_{21} = b_{22} = 0$; $k_{11} = 54000$; $k_{12} = k_{21} = -18000$; $f_1 = 0$; and $f_2 = 400$. The eigenvalues were found as, $\lambda_1 = 258.9$ with the corresponding fundamental natural frequency, $\omega_1 = 16.09$ rad/s, and $\lambda_2 = 1001$ with the corresponding natural frequency, $\omega_2 = 31.6$ rad/s. Figure 2 shows a graph of displacements versus time. The results given in Thomson [16] are confirmed.

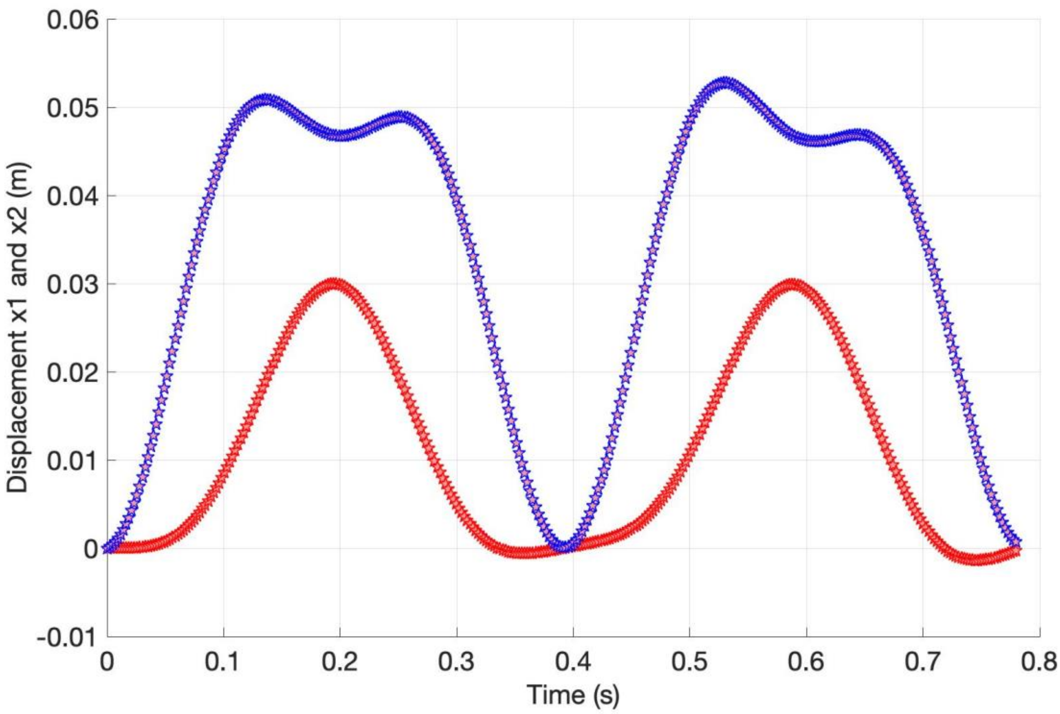

Figure 2: Graph of displacements (m) vs time (t)

3.2 Multiple-degrees-of-freedom systems (m-dof)

A finite element example was taken from Wang et al. [19], which was a multiple-degree-of-freedom mass-spring system with up to 1500 nonlinear springs, whose details are shown in Table 3.

Table 3: Multiple-degree-of-freedom nonlinear mass-spring system

Mass (kg)	Spring (N/m)	Force (N)
$m_1 = 1$	$k_1 = k$	$f_1 = sin(t)$
m_2	k_2	$f_2 = sin(t)$
m_3	k_3	$f_3 = sin(t)$
$\cdots$	$\cdots$	$\cdots$
m_n	k_n	$f_n = sin(t)$
	$k = 10^5\ N/m$	$\alpha = -2$

$m_i = 1\ kg;\ \alpha = -2;\ k_i = k[1 + \alpha(\mu_i - \mu_{i-1})^2];\ 2 \leq i \leq n;\ n \equiv number\ of\ dof$

Figure 3 shows a graph of displacement vs time for a 10-dof system solved using a fixed time step of $\Delta t_0 = 1e - 03$ (s) for the duration of simulation of 10 π(s) or 5 periodic cycles. The corresponding phase trajectory is shown in Figure 4 as a closed path (see Jordan et al. [17]), with a duration of 2 π(s).

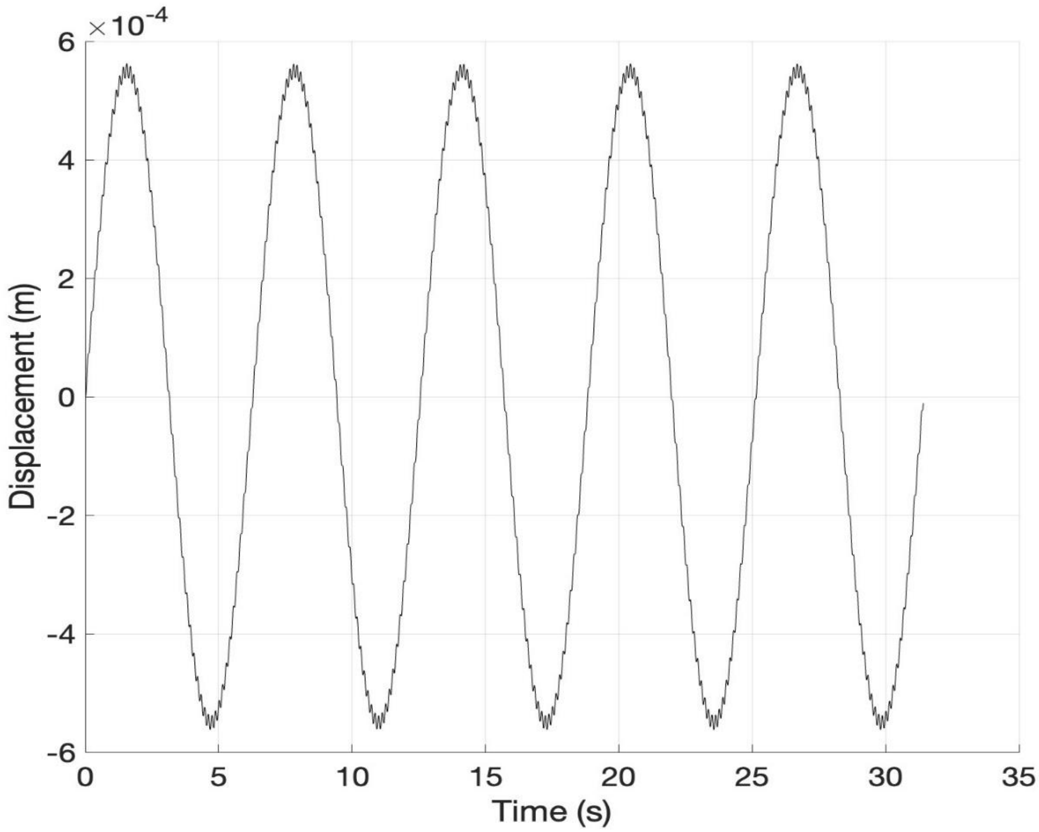

Figure 3: Displacement vs time

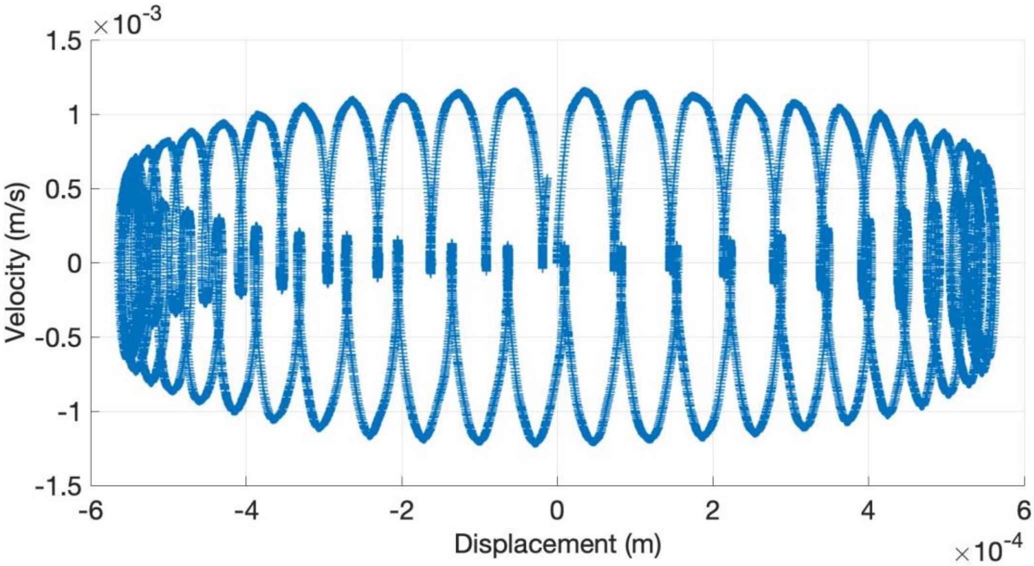

Figure 4: Velocity vs displacement

Figure 5 shows a graph of displacement vs time for a 100-dof system for the same duration of simulation of 10 π(s). The corresponding phase trajectory is shown in Figure 6 as a closed path (see Jordan et al.[17]), with a duration of 2 π(s).

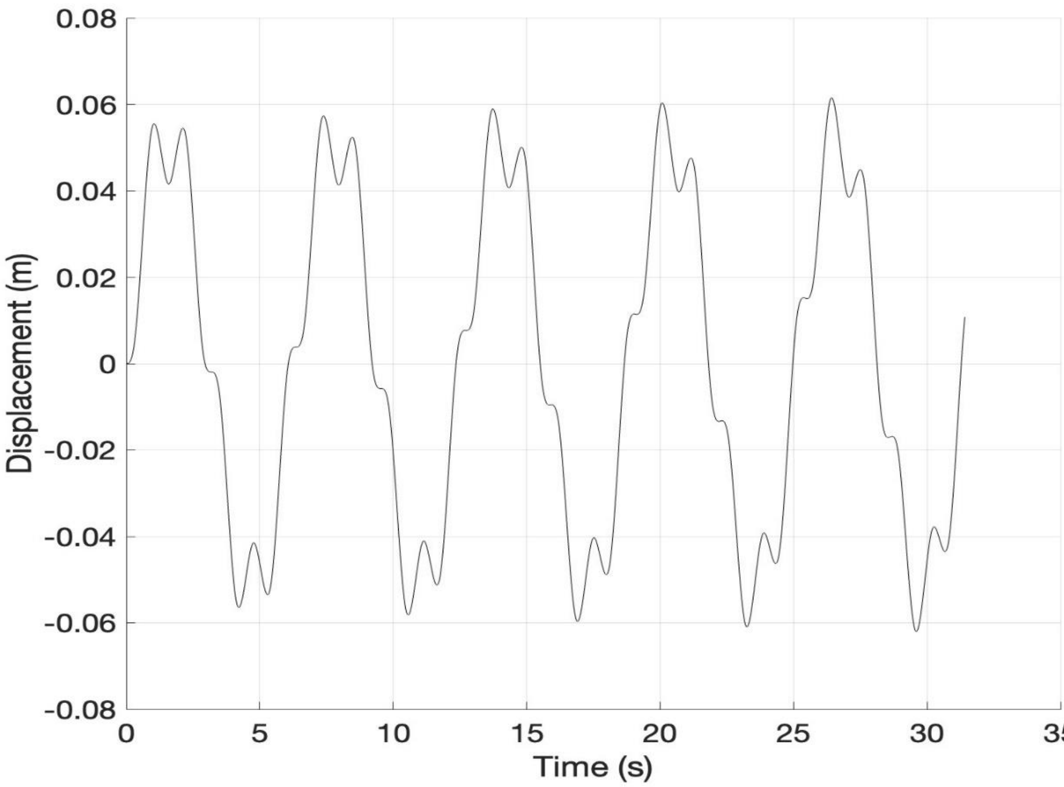

Figure 5: Displacement vs time

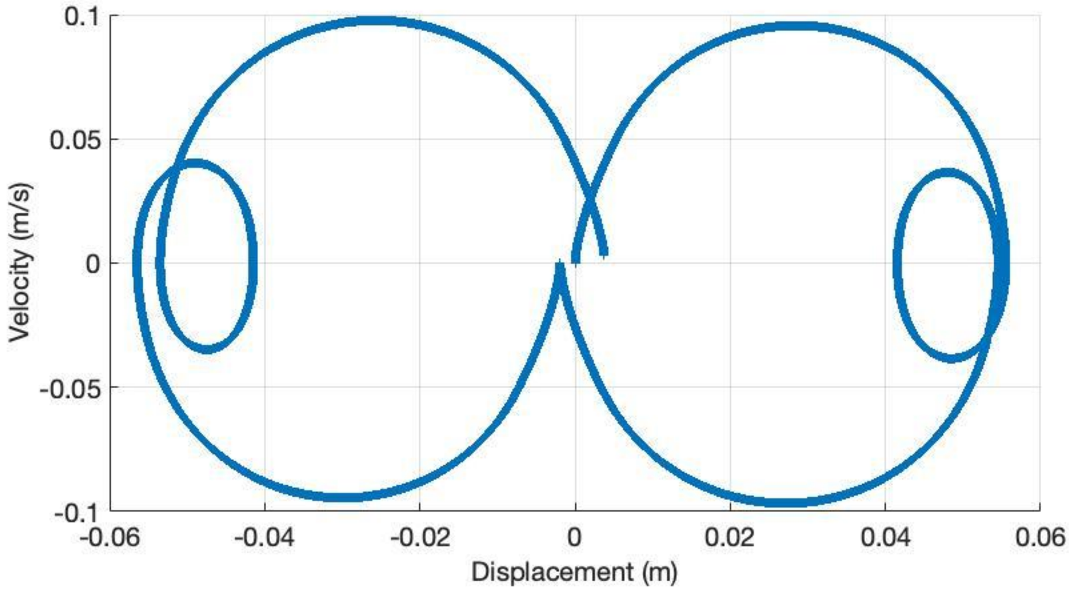

Figure 6: Velocity vs displacement

The results shown in Figure 5 agree with those of Wang et al. [19].

3. **Conclusions**

This paper has developed extended forward-backward difference numerical solution procedures for a class of nonlinear systems. The theoretical accuracy analysis revealed a high order accuracy, with an error of order $O(\Delta t^7)$ for displacement, and an error of order $O(\Delta t^5)$ for velocity. This is an improvement from simple second order accurate implicit integration schemes. The Lyapunov stability analysis showed that the algorithm developed was unconditionally stable, and values of integration parameters, γ_{ijk}, may be determined from stability analyses.

A linear system of two-degrees-of-freedom was initially solved to illustrate how to extend the methods to deal with multiple-degrees-of-freedom systems using matrices and vectors. The results of Thomson [16] were confirmed.

Two nonlinear finite element problems were successfully solved. The hundred-degrees-of-freedom system confirmed the results of Wang [19]. The graphs of displacement versus time look similar. In addition, phase trajectory plots and a velocity versus displacement graph revealed the property of a closed path for the nonlinear mass-spring system. The accuracy of the results was not compared because such an exercise would have required getting access to the authors' data.

Declaration of competing interest

The author declares that there have been no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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