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Ricci-Yamabe Solitons on Lorentzian Para-Kenmotsu Manifolds Admitting Generalized Tanaka-Webster Connection

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Abstract:- In this paper, we study the characteristics of Ricci-Yamabe solitons in the context of Lorentzian para-Kenmotsu manifolds with respect to generalized Tanaka-Webster connection. At first, we show that the concircular curvature tensor $C(X,Y)\xi = 0$ does not imply $C^*(X,Y)\xi = 0$. Next, we show that a Lorentzian para-Kenmotsu manifold with respect to ∇^* admitting a Ricci-Yamabe soliton is an η-Einstein manifold and we find the condition for soliton to be shrinking, steady and expanding. we find out if potential vector field V is collinear with ξ then manifold is an η-Einstein manifold. Finally, manifolds satisfying the ξ-concircular flat and φ-concircular semisymmetric conditions have been studied.

***Keywords**:* Lorentzian para-Kenmotsu manifold, Generalized Tanaka-Webster connection, η-Einstein manifold, Ricci-Yamabe soliton.

1. Introduction

The study of geometric flows and their soliton structures has played a pivotal role in differential geometry and theoretical physics, particularly in understanding the evolution of manifolds and their curvature properties. Among the most influential geometric flows are the *Ricci flow* and the *Yamabe flow*, which have been extensively studied due to their connections with Einstein metrics, conformal geometry, and general relativity.

The Ricci flow, introduced by Hamilton [1], is defined by the evolution equation:

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g) \tag{1.1}$$

where g is the Riemannian (or pseudo-Riemannian) metric and $Ric(g)$ denotes its Ricci curvature. A *Ricci soliton* is a self-similar solution to this flow, satisfying:

$$\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_{\mathcal{V}} g + \lambda g = 0, \tag{1.2}$$

where $\mathcal{L}_{\mathcal{V}}$ is the Lie derivative along a vector field $\mathcal{V}$ and λ is a scalar constant. Ricci solitons generalize Einstein metrics and have been classified in various geometric settings [2,3].

On the other hand, the *Yamabe flow*, introduced by Hamilton [2], evolves a metric conformally to achieve constant scalar curvature, governed by:

$$\frac{\partial g}{\partial t} = -Rg, \tag{1.3}$$

where R is the scalar curvature. A *Yamabe soliton* obeys:

$$(R - \lambda)g + \frac{1}{2}\mathcal{L}_{\mathcal{V}} g = 0 \tag{1.4}$$

and has been studied in the context of conformal geometry.

A natural generalization arises by considering a combination of these flows, leading to the *Ricci-Yamabe flow* [4]:

$$\frac{\partial g}{\partial t} = -2\alpha \mathrm{Ric}(g) + \beta R g, \tag{1.5}$$

where α, β are constants. When this flow admits a soliton solution, we obtain the *Ricci- Yamabe soliton* equation:

$$\alpha \mathrm{Ric}(g) - \left(\frac{\beta R}{2} - \lambda\right) g + \frac{1}{2}\mathcal{L}_V g = 0. \tag{1.6}$$

This framework unifies Ricci and Yamabe solitons, providing a richer structure for geometric analysis.

In the context of *Lorentzian para-Kenmotsu manifolds*, which are pseudo-Riemannian analogues of Kenmotsu manifolds, the interplay between curvature and soliton structures becomes particularly intriguing. Recent studies have explored Ricci solitons [6,7] and Yamabe solitons [9] in such settings. However, the investigation of *Ricci-Yamabe solitons* under the *generalized Tanaka-Webster connection*-an extension of the canonical connection in CR-geometry [8] - remains largely unexplored. η-Ricci solitons on para-Sasakian and para-Kenmotsu manifolds was studied by Singh, A., & Kishor, S.,[11,12], Conformal Ricci solitons in Lorentzian para-Sasakian manifolds was studied by Kishor, S. et al [15]. Readers can also see [13, 14, 16].

The paper is structured as follows: First section is Introduction. Second section is Preliminaries, where we give a brief introduction of Lorentzian para-Kenmotsu manifolds.

We divide the section 3 into four subsections : in subsection 3.1, we obtain the various relationship between concircular curvature tensor with respect to Levi-Civita connection and with respect to generalized Tanaka-Webster connection ∇^*. In section 3.2, Ricci-Yamabe solitons in Lorentzian para-Kenmotsu manifold $\mathcal{M}$ with respect to ∇^* have been discussed. Section 3.4 deals with Lorentzian para-Kenmotsu manifold with ξ-concircularly flat and φ-concircularly semisymmetric conditions. Section 3.4 is devoted to the study of $\mathcal{C}^*(\xi, X) \cdot Ric^* = 0$ condition.

2. Preliminaries

Let $\mathcal{M}$ be an n-dimensional smooth manifold equipped with a Lorentzian metric g. A Lorentzian *para-Kenmotsu manifold* $(\mathcal{M}, \varphi, \xi, \eta, g)$ is a Lorentzian almost paracontact manifold satisfying the following conditions:

$$\varphi^2 = I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$\varphi\xi = 0, \eta(\varphi X) = 0, \tag{2.4}$$

$$g(X, \xi) = \eta(X), \tag{2.5}$$

$$\Phi(X, Y) = \Phi(Y, X) = g(\varphi X, Y), \tag{2.6}$$

for all vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, where φ is a (1,1)-tensor field, ξ is the *Reeb vector field*, and η is the associated 1-form.

The Lorentzian para-Kenmotsu condition is characterized by the properties:

$$(\nabla_X \varphi)Y = -g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{2.7}$$

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.8}$$

where ∇ is the Levi-Civita connection of the metric g. Furthermore, on a Lorentzian para-Kenmotsu manifold $\mathcal{M}$, the following relation hold:

$$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y), \tag{2.9}$$

Furthermore, on a Lorentzian para-Kenmotsu manifold $\mathcal{M}$, the following results hold:

$$g(R(X, Y)Z, \xi) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.10}$$

3. $$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.11}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.12}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.13}$$

$$Q\xi = (n-1)\xi, \tag{2.14}$$

$$Ric(\varphi X, \varphi Y) = Ric(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.15}$$

The *generalized Tanaka-Webster connection* ∇^* on a Lorentzian para-Kenmotsu manifold is defined by [5]:

$$\nabla^*_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi - \eta(X)\varphi Y, \tag{2.16}$$

for all vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$.

Using (2.8) and (2.9) in (2.16), we get

$$\nabla^*_X Y = \nabla_X Y - g(X, Y)\xi + \eta(Y)X - \eta(X)\varphi Y. \tag{2.17}$$

If the Riemann curvature tensor, Ricci tensor, scalar curvature and Ricci operator with respect to the generalized Tanaka-Webster connection is denoted by $\mathcal{R}^*, Ric^*, R^*$ and Q^* respectively. Then from [10], we have

$$\nabla^*_X \xi = -2X - 2\eta(X)\xi, \tag{2.18}$$

$$\begin{aligned} R^*(X, Y)Z = {} & \mathrm{R}(X, Y)Z + 3g(Y, Z)X - 3g(X, Z)Y + 2\eta(X)g(Y, Z)\xi \\ & -2\eta(Y)g(X, Z)\xi - 2\eta(X)\eta(Z)Y + 2\eta(Y)\eta(Z)X \\ & -2\eta(X)\eta(Z)\varphi Y + 2\eta(Y)\eta(Z)\varphi X, \end{aligned} \tag{2.19}$$

$$\mathcal{R}^*(X, Y)\xi = 2\eta(Y)X - 2\eta(X)Y + 2\eta(X)\varphi Y - 2\eta(Y)\varphi X, \tag{2.20}$$

$$Ric^*(X, Y) = Ric(X, Y) + (3n-5)g(X, Y) + (2n + 2\psi - 4)\eta(X)\eta(Y), \tag{2.21}$$

Where $\psi = trace(\varphi)$.

$$Q^*X = QX + (3n-5)X + (2n + 2\psi - 4)\eta(X)\xi, \tag{2.22}$$

$$R^{*} = R + (3n-4)(n-1) - 2\psi. \tag{2.23}$$

1.1 Definition

A manifold $\mathcal{M}$ is called an η-Einstein manifold if its Ricci tensor Ric can be expressed as

$$Ric(X,Y) = pg(X,Y) + q\eta(X)\eta(Y), \tag{2.24}$$

where p and q are scalar functions on $\mathcal{M}$. In particular, if $q = 0$, then (2.24) *represents an Einstein manifold.*

Now, putting $X = Y = \xi$ in (2.24) and using (2.13), we get

$$p - q = (n-1) \tag{2.25}$$

Contracting (2.24) along X and Y, we get

$$pn - q = R \tag{2.26}$$

From (2.25) and (2.26), we get $p = \frac{R}{n-1} - 1$ and $q = \frac{R}{n-1} - n$

Thus (2.24) becomes

$$\mathrm{Ric}(X,Y) = \left(\frac{R}{n-1} - 1\right) g(X,Y) + \left(\frac{R}{n-1} - n\right) \eta(X)\eta(Y) \tag{2.27}$$

From (2.26), we have

$$QX = \left(\frac{r}{n-1} - 1\right) X + \left(\frac{r}{n-1} - n\right) \eta(X) \tag{2.28}$$

Differentiating (2.28) covariantly with respect to Y, we have

$$(\nabla_Y Q)X = \frac{Y(r)}{n-1}(X + \eta(X)\,\xi) - \left(\frac{r}{n-1} - n\right)\{g(X,Y)\,\xi + 2\,\eta(X)\eta(Y)\,\xi + \eta(X)\,Y\} \tag{2.29}$$

By contracting (2.29) over Y, we get

$$\frac{n-3}{2(n-1)} X(r) = \frac{1}{n-1}\eta(X)\,\xi(r) + (n(n-1) - r)\,\eta(X). \tag{2.30}$$

Putting $X = \xi$ in (2.30), we get

$$\xi(r) = 2\big(r - n(n-1)\big). \tag{2.31}$$

Using (2.31) into (2.30), we get

$$X(r) = -2\big(r - n(n-1)\big)\eta(X). \tag{2.32}$$

4. Main Results

3.1 Concircular Curvature Tensor in $\mathcal{M}$ with respect to ∇^{*}

Concircular curvature tensor $\mathcal{C}$ on a Lorentzian para-Kenmotsu manifold $\mathcal{M}$ is defined by

$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{R}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{3.1}$$

For all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$.

Now , the concircular curvature tensor with respect to the generalized Tanaka-Webster connection is given by

$$\mathcal{C}^*(X,Y)Z = \mathcal{R}^*(X,Y)Z - \frac{R^*}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{3.2}$$

For all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$.

Taking the inner product of (3.2) with W, we have

$$\mathcal{C}^*(X,Y,Z,W) = \mathcal{R}^*(X,Y,Z,W) - \frac{R^*}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \tag{3.3}$$

where $\mathcal{C}^*(X,Y,Z,W) = g(\mathcal{C}^*(X,Y)Z,W)$ and $\mathcal{R}^*(X,Y,Z,W) = g(\mathcal{R}^*(X,Y)Z,W)$.

Now, interchanging X and Y in (3.3), we get

$$\mathcal{C}^*(Y,X,Z,W) = \mathcal{R}^*(Y,X,Z,W) - \frac{R^*}{n(n-1)}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]. \tag{3.4}$$

Adding (3.3) and (3.4), we get

$$\mathcal{C}^*(X,Y,Z,W) + \mathcal{C}^*(Y,X,Z,W) = \mathcal{R}^*(X,Y,Z,W) + \mathcal{R}^*(Y,X,Z,W). \tag{3.5}$$

From (2.19), it is clear that $\mathcal{R}^*(X,Y,Z,W) + \mathcal{R}^*(Y,X,Z,W) = 0$.

Hence, $\mathcal{C}^*(X,Y,Z,W) + \mathcal{C}^*(Y,X,Z,W) = 0$.
Now, interchanging Z and W in (3.3), we have

$$\mathcal{C}^*(X,Y,W,Z) = \mathcal{R}^*(X,Y,W,Z) - \frac{R^*}{n(n-1)}[g(Y,W)g(X,Z) - g(X,W)g(Y,Z)]. \tag{3.6}$$

Adding (3.3) and (3.6), we get

$$\mathcal{C}^*(X,Y,Z,W) + \mathcal{C}^*(X,Y,W,Z) = \mathcal{R}^*(X,Y,Z,W) + \mathcal{R}^*(X,Y,W,Z). \tag{3.7}$$

From (2.19), we have

$$\begin{aligned}\mathcal{R}^*(X,Y,Z,W) + \mathcal{R}^*(X,Y,W,Z) &= -2\eta(X)\eta(Z)g(\varphi Y,W) + 2\eta(Y)\eta(Z)g(\varphi X,W)\\ &\quad -2\eta(X)\eta(W)g(\varphi Y,Z) + 2\eta(Y)\eta(W)g(\varphi X,Z).\end{aligned} \tag{3.8}$$

So, it is clear from (3.7) and (3.8) $\mathcal{C}^*(X,Y,Z,W) + \mathcal{C}^*(X,Y,W,Z) \neq 0$.

Since,

$$\mathcal{C}^*(Z,W,X,Y) = \mathcal{R}^*(Z,W,X,Y) - \frac{R^*}{n(n-1)}[g(W,X)g(Z,Y) - g(Z,X)g(W,Y)]. \tag{3.9}$$

Subtracting (3.9) from (3.3), we have

$$\mathcal{C}^*(X,Y,Z,W) - \mathcal{C}^*(Z,W,X,Y) = \mathcal{R}^*(X,Y,Z,W) - \mathcal{R}^*(Z,W,X,Y) \tag{3.10}$$

From (2.19), we have

$$\mathcal{R}^*(X,Y,Z,W) - \mathcal{R}^*(Z,W,X,Y) = 2\eta(Y)\eta(Z)g(\varphi X,W) - 2\eta(X)\eta(W)g(\varphi Y,Z). \tag{3.11}$$

So, it is clear from (3.10) that $\mathcal{C}^*(X,Y,Z,W) \neq C^*(Z,W,X,Y)$.

Then, following can be stated :

Theorem 1 : In an n-dimensional Lorentzian para-Kenmotsu manifold $\mathcal{M}$ admitting generalized Tanaka-Webster connection, the following relations hold :

$$\mathcal{C}^*(X,Y,Z,W) = -\mathcal{C}^*(Y,X,Z,W)$$

$$\mathcal{C}^*(X,Y,Z,W) \neq \mathcal{C}^*(X,Y,W,Z)$$

$$\mathcal{C}^*(X,Y,Z,W) \neq \mathcal{C}^*(Z,W,X,Y)$$

for any $X, Y, Z, W \in \mathfrak{X}(\mathcal{M})$.

Next, we study the equivalence of ξ-concircularly flatness with respect to ∇ and ∇^*.

Using (2.19), (2.23) into (3.2), we have

$$\mathcal{C}^*(X,Y)Z = \mathcal{C}(X,Y)Z + 3g(Y,Z)X - 3g(X,Z)Y + 2\eta(X)g(Y,Z)\xi - 2\eta(Y)g(X,Z)\xi - 2\eta(X)\eta(Z)Y + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)\varphi Y + 2\eta(Y)\eta(Z)\varphi X - \frac{[(3n-4)(n-1)-2\Psi]}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \quad (3.12)$$

Putting $Z = \xi$ into (3.12), we get

$$\mathcal{C}^*(X,Y)\xi = \mathcal{C}(X,Y)\xi - \left(2 - \frac{4}{n} - \frac{2\Psi}{n(n-1)}\right)(\eta(Y)X - \eta(X)Y) + 2(\eta(X)\varphi Y - \eta(Y)\varphi X) \quad (3.13)$$

Then, we have the following result:

Theorem 2 : In a Lorentzian para-Kenmotsu manifold $\mathcal{M}$, ξ- concircular flatness with respect to the Levi-Civita connection does not imply the concircular flatness with respect to generalized Tanaka-Webster connection.

Corollary 1: In a Lorentzian para-Kenmotsu manifold ξ-concircular flatness with respect to ∇

And $\nabla^*$are equivalent, if X, Y are orthogonal to ξ.

3.2 Ricci-Yamabe solitons in $\mathcal{M}$ with ∇^*

Let the metric of $\mathcal{M}$ with respect to ∇^* be Ricci-Yamabe soliton, then from (1.6), we have

$$(\mathcal{L}^*_V g)(X,Y) + 2\alpha Ric^*(X,Y) + (2\lambda - \beta R^*)g(X,Y) = 0, \quad (3.14)$$

for any $X, Y \in \mathfrak{X}(\mathcal{M})$.

Now, from the definition of Lie derivative, we have

$$(\mathcal{L}^*_V g)(X,Y) = g(\nabla^*_X V, Y) + g(X, \nabla^*_Y V). \quad (3.15)$$

Using (2.17) in (3.15), we have

$$(\mathcal{L}^*_V g)(X,Y) = (\mathcal{L}_V g)(X,Y) - g(X,V)\eta(Y) - g(Y,V)\eta(X) - \eta(X)g(\varphi V,Y) - \eta(Y)g(X,\varphi V) + 2\eta(V)g(X,Y). \quad (3.16)$$

Now, using (2.21}, (2.23) and (3.16) into (3.14), we have

$$(\mathcal{L}_V g)(X,Y) - g(X,V)\eta(Y) - g(Y,V)\eta(X) - \eta(X)g(\varphi V,Y) - \eta(Y)g(X,\varphi V) + 2\eta(V)g(X,Y) + 2\alpha\left(\left(\frac{R}{n-1} - 1\right)g(X,Y) + \left(\frac{R}{n-1} - n\right)\eta(X)\eta(Y)\right) + 2\alpha(3n-5)g(X,Y) + 2\alpha(2n + 2\Psi - 4)\eta(X)\eta(Y) + [2\lambda - \beta(R + (3n-4)(n-1) - 2\Psi)]g(X,Y) = 0 \quad (3.17)$$

Putting $X = Y = \xi$ into (3.17), we have

$$(\mathcal{L}_V g)(\xi,\xi) - 4\alpha n + 4\alpha + 4\alpha\Psi - 2\lambda + \beta R + \beta(3n-4)(n-1) - 2\beta\Psi = 0. \quad (3.18)$$

The aftermath of the Lie derivative of $g(\xi,\xi) = -1$ is

$$(\mathcal{L}_V g)(\xi,\xi) = -2\eta(\mathcal{L}_V \xi), \quad (3.19)$$

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relations (3.18) and (3.19) infer

$$2\eta(\mathcal{L}_V\xi) = 4\alpha(1+\Psi-n) - 2\lambda + \beta[R + (3n-4)(n-1) - 2\Psi]. \qquad (3.20)$$

Thus, we have the following result:

Theorem 3 : Let a Lorentzian para-Kenmotsu manifold with ∇^* admits Ricci-Yamabe soliton then $\mathcal{L}_V\xi$ is orthogonal to ξ if

$$\lambda = 2\alpha(1+\Psi-n) + \frac{\beta}{2}[R + (3n-4)(n-1) - 2\Psi]. \qquad (3.21)$$

Now, let us assume $V = \xi$ in (3.14), we have

$$(\mathcal{L}^*_\xi g)(X,Y) + 2\alpha Ric^*(X,Y) + (2\lambda - \beta R^*)g(X,Y) = 0. \qquad (3.22)$$

From (2.18), we have

$$(\mathcal{L}^*_\xi g)(X,Y) = g(\nabla^*_X\xi, Y) + g(X, \nabla^*_Y X) = -4g(X,Y) - 4\eta(X)\eta(Y). \qquad (3.23)$$

Using (3.23) into (3.22), we have

$$Ric^*(X,Y) = -\frac{(2\lambda - \beta R^* - 4)}{2\alpha}g(X,Y) + \frac{2}{\alpha}\eta(X)\eta(Y), \quad \alpha \neq 0, \qquad (3.24)$$

which shows that manifold $\mathcal{M}$ is an η-Einstein manifold.

Putting $X = Y = \xi$ in (3.24), we get

$$\lambda = 2\alpha(1+\Psi-n) + \frac{\beta R^*}{2}. \qquad (3.25)$$

Thus, using (3.24) and (3.25), we can state the following:

Theorem 4 : Let the Lorentzian para-Kenmotsu manifold $\mathcal{M}$ with respect to ∇^* admits Ricci-Yamabe soliton then $\mathcal{M}$ is an η-Einstein manifold and soliton constant λ is given by

$$\lambda = 2\alpha(1+\Psi-n) + \frac{\beta R^*}{2}.$$

Now, we can have the following corollary:

Corollary 2 : Let $\mathcal{M}$ admits a Ricci-Yamabe soliton then soliton is expanding,steady an shrinking according as $R^* \succ \frac{4\alpha(1+\Psi-n)}{\beta}$, $R^* = -\frac{4\alpha(1+\Psi-n)}{\beta}$ and $R^* < -\frac{4\alpha(1+\Psi-n)}{\beta}$ respectively.

Now, Let us assume that $\mathcal{M}$ admits a Riccci-Yamabe soliton where the potential vector field V is pointwise collinear i.e. $V = b\xi$, with b being a function on $\mathcal{M}$. Then from (3.14), we have

$$(Xb)\eta(Y) + (Yb)\eta(X) + bg(\nabla^*_X\xi, Y) + bg(X, \nabla^*_Y\xi) + 2\alpha Ric^*(X,Y)$$
$$+(2\lambda - \beta R^*)g(X,Y) = 0. \qquad (3.26)$$

Using (2.18) in (3.26), we get

$$(Xb)\eta(Y) + (Yb)\eta(X) - 4bg(X,Y) - 4b\eta(X)\eta(Y) + 2\alpha Ric^*(X,Y)$$
$$+(2\lambda - \beta R^*)g(X,Y) = 0. \qquad (3.27)$$

Putting $Y = \xi$ in (3.27) and using (2.21), we get

$$-(Xb) + (\xi b)\eta(X) + 2\alpha(2n - 2\Psi - 2)\eta(X) + (2\lambda - \beta R^*)\eta(X) = 0. \qquad (3.28)$$

Putting $X = \xi$, into (3.28), we get

$$(\xi b) = -\alpha(2n - 2\Psi - 2) + \left(\frac{\beta R^*}{2} - \lambda\right). \quad (3.29)$$

Using the relation (3.29) into (3.28), we have

$$(Xb) = 2\alpha(n - \Psi - 1)\eta(X) + \left(\lambda - \frac{\beta R^*}{2}\right)\eta(X) \quad (3.30)$$

Relation (3.30), infers

$$db = \left[2\alpha(n - \Psi - 1) + \lambda - \frac{\beta R^*}{2}\right]\eta \quad (3.31)$$

Applying d on both sides of (3.31), we get

$$\left[2\alpha(n - \Psi - 1) + \lambda - \frac{\beta R^*}{2}\right]d\eta = 0 \quad (3.32)$$

Since, $d\eta \neq 0$, so from (3.32), we get

$$\lambda = \frac{\beta R^*}{2} - 2\alpha(n - \Psi - 1). \quad (3.33)$$

Using the value of λ from (3.33) into (3.31), we get $db = 0$, which shows that b is constant. Therefore, from (3.27), we get

$$Ric^*(X,Y) = \frac{1}{2\alpha}(4b - 2\lambda + \beta R^*)g(X,Y) + \frac{2b}{\alpha}\eta(X)\eta(Y), \quad \alpha \neq 0 \quad (3.34)$$

Thus, we have the following result:

Theorem 5 : Let a Lorentzian para-Kenmotsu manifold $\mathcal{M}$ admits a Ricci-Yamabe soliton such that potential vector field V is collinear with ξ then manifold is an η-Einstein manifold.

3.3 $\mathcal{M}$ with ξ-Concircularly Flat and φ-Concircularly Semisymmetric conditions

Let us assume that the Lorentzian para-Kenmotsu manifold $\mathcal{M}$ with respect to ∇^*

is ξ-Concircularly flat. Then from (3.2), we have

$$\mathcal{R}^*(X,Y)\xi = \frac{R^*}{n(n-1)}[\eta(Y)X - \eta(X)Y]. \quad (3.35)$$

Using (2.20) into (3.35), we have

$$R^* = 2n(n-1)\left[1 - \frac{(\eta(Y)\varphi X - \eta(X)\varphi Y)}{(\eta(Y)X - \eta(X)Y)}\right]. \quad (3.36)$$

Thus, we have the following result:

Theorem 6 : Let $\mathcal{M}$ be an n-dimensional Lorentzian para-Kenmotsu manifold equipped with the generalized Tanaka-Webster connection ∇^*. If there exists vector fields $X, Y \in \mathfrak{X}(M)$ such that $\eta(Y)X - \eta(X)Y \neq 0$, then the scalar curvature R^* of ∇^* is given by

$$R^* = 2n(n-1)\left[1 - \frac{(\eta(Y)\varphi X - \eta(X)\varphi Y)}{(\eta(Y)X - \eta(X)Y)}\right].$$

Now, let us consider a Lorentzian para-Kenmotsu manifold $\mathcal{M}$ with respect to the generalized Tanaka-Webster connection ∇^* is φ-concircularly semi-symmetric, i.e.

$$\mathcal{C}^* \cdot \varphi = 0. \tag{3.37}$$

From (3.37), we have

$$\mathcal{C}^*(X,Y)\varphi Z - \varphi\mathcal{C}^*(X,Y)Z = 0. \tag{3.38}$$

From (3.2), we have

$$\mathcal{C}^*(X,Y)\varphi Z = \mathcal{R}^*(X,Y)\varphi Z - \frac{R^*}{n(n-1)}[g(Y,\varphi Z)X - g(X,\varphi Z)Y] \tag{3.39}$$

And,

$$\varphi\mathcal{C}^*(X,Y)Z = \varphi\mathcal{R}^*(X,Y)Z - \frac{R^*}{n(n-1)}[g(Y,Z)\varphi X - g(X,Z)\varphi Y]. \tag{3.40}$$

Using (3.39), (3.40) into (3.38), we have

$$\mathcal{R}^*(X,Y)\varphi Z - \varphi\mathcal{R}^*(X,Y)Z - \frac{R^*}{n(n-1)}[g(Y,\varphi Z)X - g(X,\varphi Z)Y - g(Y,Z)\varphi X + g(X,Z)\varphi Y] = 0 \tag{3.41}$$

Putting $Z = \xi$ into (3.41) and using (2.1), (2.2), (2.4), we have

$$R^* = 2n(n-1)\left[1 - \frac{(\eta(X)Y - \eta(Y)X)}{(\eta(X)\varphi Y - \eta(Y)\varphi X)}\right], \tag{3.42}$$

provided $\eta(X)\varphi Y - \eta(Y)\varphi X \neq 0$.

Hence, we have the following result:

Theorem 7 : Let $\mathcal{M}$ be an n-dimensional Lorentzian para-Kenmotsu manifold equipped with the generalized Tanaka-Webster connection ∇^*. If there exists vector fields $X, Y \in \mathfrak{X}(M)$ such that $\eta(X)\varphi Y - \eta(Y)\varphi X \neq 0$ then the scalar curvature R^* of ∇^* is given by relation (3.42).

3.4 $\mathcal{M}$ satisfying $\mathcal{C}^*(\xi, X) \cdot Ric^* = 0$

Let the n-dimensional Lorentzian para-Kenmotsu manifold $\mathcal{M}$ associated with generalized Tanaka-Webster connection ∇^* satisfies the relation

$$\mathcal{C}^*(\xi, X) \cdot Ric^* = 0. \tag{3.43}$$

From (3.43), we have

$$Ric^*(\mathcal{C}^*(\xi,X)Y,Z) + Ric^*(Y,\mathcal{C}^*(\xi,X)Z) = 0. \tag{3.44}$$

From (3.2), we have

$$\mathcal{C}^*(\xi,Y)Z = \mathcal{R}^*(\xi,Y)Z - \frac{R^*}{n(n-1)}\{g(Y,Z)\xi - \eta(Z)Y\}. \tag{3.45}$$

From (2.19), we have

$$\mathcal{R}^*(\xi,Y)Z = 2g(Y,Z)\xi - 2\eta(Z)Y + 2\eta(Z)\varphi Y. \tag{3.46}$$

Using (3.46) into (3.45), we have

$$C^*(\xi, X)Y = 2g(X,Y)\xi - 2\eta(Y)X + 2\eta(Y)\varphi X - \frac{R^*}{n(n-1)}\{g(X,Y)\xi - \eta(Y)X\}. \quad (3.47)$$

Using (3.47) into (3.44), we have

$$2g(X,Y)Ric^*(\xi,Z) - 2\eta(Y)Ric^*(X,Z) + 2\eta(Y)Ric^*(\varphi X,Z) - \frac{R^*}{n(n-1)}g(X,Y)Ric^*(\xi,Z)$$
$$+\frac{R^*}{n(n-1)}\eta(Y)Ric^*(X,Z) + 2g(X,Z)Ric^*(Y,\xi) - 2\eta(Z)Ric^*(X,Y) + 2\eta(Z)Ric^*(\varphi X,Y)$$
$$-\frac{R^*}{n(n-1)}g(X,Z)Ric^*(Y,\xi) + \frac{R^*}{n(n-1)}\eta(Z)Ric^*(X,Y) = 0. \quad (3.48)$$

Using (2.21) into (3.48), we get

$$2(2n-2\Psi-2)g(X,Y)\eta(Z) - 2\eta(Y)Ric^*(X,Z) + 2\eta(Y)\{Ric(\varphi X,Z) + (3n-5)g(\varphi X,Z)\}$$
$$-\frac{R^*}{n(n-1)}(2n-2\Psi-2)g(X,Y)\eta(Z) + \frac{R^*}{n(n-1)}\eta(Y)Ric^*(X,Z)$$
$$+2(2n-2\Psi-2)g(X,Z)\eta(Y) - 2\eta(Z)Ric^*(X,Y) + 2\eta(Z)\{Ric(\varphi X,Y)$$
$$+(3n-5)g(\varphi X,Y)\}$$
$$-\frac{R^*}{n(n-1)}(2n-2\Psi-2)\eta(Y) + \frac{R^*}{n(n-1)}\eta(Z)Ric^*(X,Y) = 0. \quad (3.49)$$

Taking $Z = \xi$ into (3.49) and using (2.4), we have

$$Ric^*(X,Y) = (2n-2\Psi-2)g(X,Y) - \frac{2n(n-1)}{R^* - 2n(n-1)}\{S(\varphi X,Y) + (3n-5)g(\varphi X,Y)\}. \quad (3.50)$$

Thus, we can state the following:

Theorem 8 : If a n-dimesional Lorentzian para-Kenmotsu manifold admitting generalized Tanaka-Webster connection satisfies the condition $C^*(\xi,X)\cdot Ric^* = 0$. Then Ricci tensor is given by relation (3.50).

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