
Projectivity in Intuitionistic Fuzzy G-Modules: An Extended Approach

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Abstract*:This article introduces the concepts of projectivity and quasi-projectivity for extended intuitionistic fuzzy G-modules. We establish a condition for finite-dimensional G-modules under which an extended intuitionistic fuzzy G-module is projective with respect to another extended intuitionistic fuzzy G-module. The paper also investigates several important properties of the projectivity of extended intuitionistic fuzzy G-modules with respect to the direct sum of extended intuitionistic fuzzy G-modules*

Keywords: *Extended intuitionistic fuzzy G-modules, projectivity, quasi-projectivity, finite dimensional G-modules, direct sum*

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1. INTRODUCTION

The fundamental concept of projectivity in module theory was initially established by Cartan and Eilenberg (1956), laying a crucial foundation for subsequent theoretical advancements. This foundational work was further extended by Banaschewski (1964) concerning projective modules, which spurred significant developments by various researchers. Notably, Wu and Jans (1967) introduced the concept of quasi-projective modules. Similarly, Isaac (2005) contributed to the theory by developing projective L-modules.

Following the seminal introduction of fuzzy sets by Zadeh (1965), substantial research efforts have been directed towards integrating fuzzy logic into abstract algebra. Rosenfeld (1971) pioneered this interdisciplinary field by introducing fuzzy subgroups within group theory. Since then, the literature on fuzzy algebraic structures has expanded considerably. Key contributions include Negoita and Ralescu's (1975) work on fuzzy submodules of modules and Zahedi and Ameri's (1995) exploration of fuzzy projectivity. Fernandez further enriched this area by investigating their projectivity (2003).

As a significant generalization of fuzzy set theory, Atanassov (1986, 1999) proposed the theory of intuitionistic fuzzy sets, which subsequently opened new avenues for research into algebraic structures under intuitionistic fuzzification. Building upon this framework, Biswas (1989) initiated the study of intuitionistic fuzzy subgroups. Subsequent research by various authors expanded this to include intuitionistic fuzzy subrings, submodules, and related structures (see 2003, 2011, 2012, 2013).

Specifically, Sharma and Kaur (2015) introduced the concept of intuitionistic fuzzy G-modules. They further explored properties such as representation, reducibility, semi-simplicity, injectivity, and fundamental isomorphism theorems in their subsequent works (2016–2017). As a direct continuation of this research trajectory, the present article introduces and investigates the concept of projectivity for extended intuitionistic fuzzy G-modules, presenting it as a dual notion to intuitionistic fuzzy G-module injectivity.

ORGANISATION: The structure of the paper is as follows: Section 2 presents the necessary preliminaries, including fundamental definitions and prior results related to intuitionistic fuzzy sets, G-modules, and some remarks. Section 3 introduces and investigates the concept of projectivity within the framework of extended intuitionistic fuzzy G-modules, highlighting key properties, characterizations, and relevant results. Finally, Section 4 concludes the paper with a summary of findings and outlines potential directions for research in the domain of intuitionistic fuzzy algebraic structures.

2. PRELIMINARIES

Definition 2.1 [5]: Let G be a group and let M be a vector space over a field K. Then M is called a G-module if for every $\mathbf{g \epsilon G}$ **and** $\mathbf{m \epsilon M}$ ∃product (called the action of G on M), gm ∈M satisfies the following axioms:

a. $\mathbf{1_G m = m \ \forall \ m \epsilon M}$ ($\mathbf{1_G}$ **being the identity of G**)

b. (gh)m = g(hm), ∀m ∈M, g, h ∈G,

c. $G(k_1m_1 + k_1m_2) = k_1(gm_1) + k_1(gm_2) \ \forall \ k_1, k_2 \in K;\ m_1, m_2 \in M$ and $g \in G$

Definition 2.2 [5]: Let G be a group and let M be a G-module over thefield K. Let N be a subspace of the vector space M over K. Then N is called a G-submodule of M if an_1+bn_2□N , for all a, b □ K and n_1, n_2□ N.

Definition2.3 [5]: LetMandM□beG-modules.Amappingf:M□□M□iscalledaG-modulehomomorphismif

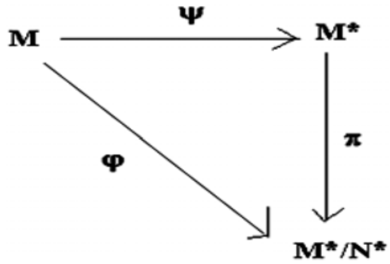
a. f□ k_1m_1□ k_2m_2□ □ k_1f□ m_1□ □ k_2f□ m_2□ ,

b. f□ gm□ □ gf□ m□ ,□k_1,k_2□ K;m,m_1,m_2□ Mandg□G.

Definition2.4 [3]:A G-moduleMis projectiveif foranyG-moduleM□ andanyG-submoduleN□ofM□,everyhomomorphism□ :M□M□/ N□canbeliftedtoahomomorphism□ :M□ M□,i.e.,□∘□ □ □ ,□:M□ □M□/ N□isthecanonicalhomomorphism.Inotherwords,following diagramiscommutative.

Figure1

Definition2.5 [3]: LetM,M□beG-modules.ThenMisM□-projectiveif foranyG-submoduleN□ofM□,anyhomomorphism□ :M□M□/ N□can belifted toa homomorphism□ :M□ M□.



Remark2.1 [3]:

a. AG-moduleMisprojectiveifandonlyifMisM□projectivefor everyG-moduleM□

b. LetMandM□beG-modulessuchthatM/is M□-projective.Let N□beanyG-submoduleof M□.ThenMisN□-projectiveandMisM□/ N□-projective

c. A direct sum $M=\bigoplus^{n} M_i$ is M* -projective if andonly if M_i is M* -projective for every i, where M, M_i , M* are G-modules.

Remark 2.2 [3]:

a. Let M, Mi($1 \leq i \leq n$) be G-modules. Then $M=\bigoplus^{n} M_i$-projective if and only if M is Mi-projective for all i

b. A G-module M is quasi-projective if M is M-projective

c. Two G-modules M and M∗are said to be relativelyprojective if M is M∗-projective and M∗is M-projective

Definition 2.6 [1]: Let X be a non-empty set. An intuitionistic fuzzy set (IFS) A of X is an object of the form A = { x, $\mu_A(x)$, $\nu_A(x)$: x ∈ X}, where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ define the degree of membership and degree of non-membership of the element x ∈ X respectively and for any x ∈ X , we have $\mu_A(x) + \nu_A(x) \leq 1$.

Remark 2.3 [1]:

a. When $\mu_A(x) + \nu_A(x) = 1$, i.e., $\nu_A(x) = 1 - \mu_A(x)$, ∀x ∈ X. Then Ais called a fuzzy set.

b. For convenience, we write the IFS A = {x, $\mu_A(x)$, $\nu_A(x)$: x ∈ X}, by A = (μ_A, ν_A).

Definition 2.7 [17]: Let G be a group and M be a G-module over K, which is a subfield of C. Then an intuitionistic fuzzy G-module on M is an intuitionistic fuzzy set A = (μ_A, ν_A) of M such that following conditions are satisfied:

a. $\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y)$, ∀a, b ∈ K and x, y ∈ M, and

b. $\mu_A(gm) \geq \mu_A(m)$ and $\nu_A(gm) \leq \nu_A(m)$, ∀g ∈ G; m ∈ M.

3.PROJECTIVITY OF EXTENDED INTUITIONISTIC FUZZY G-MODULE

Definition 3.1: Let X be a non-empty set. An extended intuitionistic fuzzy set (EIFS) A of X is an object of the form A = { x, $\mu_A(x)$, $\nu_A(x)$: x ∈X}, where $\mu_A: X \rightarrow [0, 1]$ and $\nu_A: X \rightarrow [0, 1]$ define the degree of membership and degree of non-membership of the element x ∈X respectively and for any x ∈X , we have ${\mu_A}^2 + {\nu_A}^2 \leq 1$. For convenience, we write the EIFS A = {x, μ_A (x), ν_A (x) :x ∈X}, by A = (μ_A, ν_A).

Definition 3.2: If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$be Extended Intuitionistic fuzzy (EIF) G-modules of G-modules E and E*, respectively. A function f : E → E* is said to be a function from A to B if ${\alpha_B}^2 of = \alpha_A$and ${\beta_B}^2 of = \beta_A$

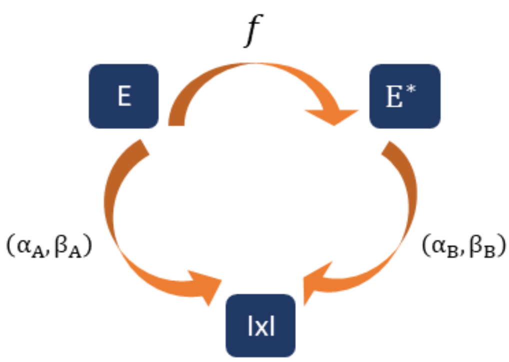


Figure2

Suppose E and E*be G-modules and let Ebe E*-projective module.Then for every homomorphism ϕ:E →E^*/F*can be lifted to ϕ: E→E*, such that πoψ = ϕ, where F*is G-submodule of E*and π: E*→E*/F*is a natural homomorphism. If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$be EIF G-modules of G-modules E and E∗respectively and$(\alpha_{B|_{F^*}}, \beta_{B|_{F^*}})$be the EIF G-module on F*Then A is saidto be B-projective if the following diagram is commutative,

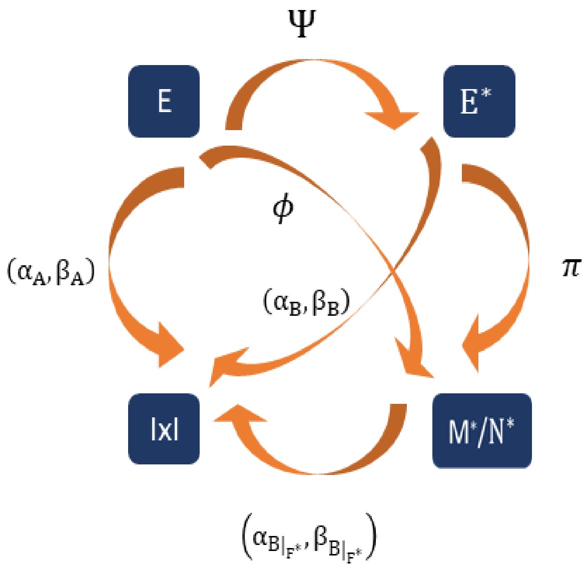


Figure3

i.e., we have

$\alpha_A{}^2 = \alpha_B \, o \psi$ and $\beta_A{}^2 = \beta_B \text{ o } \psi$;

$\alpha_B{}^2 = \alpha_{B|_{F^*}}{}^2 \text{ o } \pi$ and $\beta_B{}^2 = \beta_{B|_{F^*}}{}^2 \text{ o } \pi$

$$\alpha_A{}^2 = \alpha_{B|_{F^*}}{}^2 \text{ o } \phi \text{ and } \beta_A{}^2 = \beta_{B|_{F^*}}{}^2 \text{o } \phi$$

Let m∈ E. Then ψ(m) ∈E^*.

Case (a): When ψ(m) ∈F*, then π(ψ(m)) = F^*so that

$$\alpha_{B|_{F^*}}{}^2\left(\pi(\psi(m))\right) = \alpha_{B|_{F^*}}{}^2(F^*) = 1$$

$\alpha_B{}^2(\psi(m)) = (\alpha_{B|_{F^*}}{}^2 \text{ o } \pi)\ \psi(m) = \alpha_{B|_{F^*}}{}^2\ (\pi\ \psi(m)) = 1 \geq \alpha_{B|_{F^*}}{}^2\ \psi(m)$

$(\alpha_{B|_{F^*}}{}^2 \text{ o } \psi)(m)) = \alpha_A{}^2(m)$. Similarly, $\beta_B{}^2((\psi(m)) \leq \beta_A{}^2(m)$

Case (b): When ψ(m) ∈E*/F*, then

$$\alpha_B{}^2(\psi(m)) = (\alpha_{B|_{F^*}}{}^2 \text{ o } \pi)(\psi(m)) = \alpha_{B|_{F^*}}{}^2(\pi\text{o } \psi)(m) = \alpha_{B|_{F^*}}{}^2\phi(m)$$

$\alpha_{B|_{F^*}}{}^2 \text{ o } \phi(m) = \alpha_A{}^2(m)$

Similarly, $\beta_B{}^2((\psi(m)) = \beta_A{}^2(m)$.Hence,

$\alpha_A{}^2(m) \leq \alpha_B{}^2(\psi(m))\,;\ \beta_A{}^2(m) \geq \beta_B{}^2((\psi(m))\ \ \forall \psi \in \text{Hom}(E, E^*)$ and $m \in E$

Definition 3.3. Let E and E*be G-modules. If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$be EIF G-modules of G-modules E and E*. Then A is B-projective if

a) Eis E*-projective and

b) $\alpha_A{}^2(m) \leq \alpha_B{}^2(\psi(m));\ \beta_A{}^2(m) \geq \beta_B{}^2((\psi(m))\ \ \forall \psi \in \text{Hom}(E, E^*)$ and $m \in E$

Example 3.1: If G = {1, −1, i, −i}, E = C and E*= C^n, then E andE* are G-modules; and E is E*-projective. Define Extended intuitionistic fuzzy sets A and B on E and E*by

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$$\alpha_A(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{4} & \text{if } x \neq 0 \\ \frac{1}{8} & \text{if otherwise} \end{cases} \text{and} \beta_A(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{8} & \text{if } x \neq 0 \\ \frac{1}{4} & \text{if otherwise} \end{cases}$$

And

$$\alpha_B(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{3}{8} & \text{if } x \neq 0 \\ \frac{1}{4} & \text{if otherwise} \end{cases} \text{and} \beta_B(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{4} & \text{if } x \neq 0 \\ \frac{3}{8} & \text{if otherwise} \end{cases}$$

Then A and B are EIF G-modules on E and E*, respectivelyfor A, maximal chain of submodule of C is {0} ⊂R⊂C and for B, maximal chain of submodules of $C^n$is {0} ⊂R^n⊂C^n.

Also,$\alpha_A{}^2(m) \leq \alpha_B{}^2(\psi(m))$; $\beta_A{}^2(m) \geq \beta_B{}^2((\psi(m))$ $\forall\, \psi \in \mathrm{Hom}(E, E^*)$ and $m \in E$.Therefore, A is B-projective.

Theorem 3.1: Let E and E* be G-modules and A, B be EIF G-modules on E and E* respectively such that A is B-projective. If E* is a G-submodules of E*and C is an EIF G-module on F*, then A is C-projective if B/F*⊆ C

Proof. Since A is B-projective. Therefore

a) E is E* -projective and

b) $\alpha_A{}^2(m) \leq \alpha_B{}^2(\psi(m))$; $\beta_A{}^2(m) \geq \beta_B{}^2((\psi(m))$ $\forall\, \psi \in \mathrm{Hom}(E, E^*)$ and $m \in E$

Since E is E* -projective and F* is a G-submodule of E*, by Remark 2.1 we have E is F* -projective. Let ϕ ∈Hom(E, F*) and η :F* → E* be the inclusion homomorphism. Then,

ηo ϕ = ψ ∈Hom(E, E*)and by (b), we get

$$\alpha_A{}^2(m) \leq \alpha_B{}^2\big((\eta \circ \phi)(m)\big) = \alpha_B{}^2\Big(\eta\,\big(\phi(m)\big)\Big) = \alpha_B{}^2(\phi(m))$$

And

$$\beta_A{}^2(m) \geq \beta_B{}^2\big((\eta \circ \phi)(m)\big) = \beta_B{}^2\Big(\eta\,\big(\phi(m)\big)\Big) = \beta_B{}^2(\phi(m))$$

∀m ∈E, ∀ ϕ ∈Hom(E, F*) ....(i)

Since ϕ(m) ∈F* and B| F*⊆C

$\alpha_B{}^2\big(\phi(m)\big) \leq \alpha_c{}^2(\phi(m))and\beta_B{}^2\big(\phi(m)\big) \geq \beta_C{}^2(\phi(m))$ .... (ii)

From (i) and (ii), we have

$$\alpha_A{}^2(m) \leq \alpha_c{}^2(\phi(m))\text{and}\beta_A{}^2(m) \geq \beta_C{}^2(\phi(m))\forall m \in E,\ \forall\, \phi \in \mathrm{Hom}(E, F^*)$$

Hence, A is C-projective.

Theorem 3.2: Let A and B be EIF G-modules on the G-modules E and E*, respectively. Let Br (r ∈ [0, 1]) be the EIF G-modules on E* defined by $\alpha_{Br}{}^2(m) = \alpha_B{}^2(m) \wedge r$and $\beta_{Br}{}^2(m) = \beta_B{}^2(m) \vee r$ $\forall m \in E*$.If A is Br -projective for some r ∈ [0, 1]. Then A is B-projective.

Proof. Assume that A is Br -projective for some r ∈ [0, 1]. Then

a) E is E∗ -projective and

b) $(\alpha_A{}^2(m) \leq \alpha_{B_r}{}^2(\psi(m))$; $\beta_A{}^2(m) \geq \beta_{B_r}{}^2((\psi(m))$ $\forall\, \psi \in \mathrm{Hom}(E, E^*)$ and $m \in E$

Since Br ⊆B for all r ∈ [0, 1], i.e$\alpha_{B_r}{}^2(\psi(m)) \leq \alpha_B{}^2(\phi(m))$and $\beta_{B_r}{}^2((\psi(m)) \geq \beta_B{}^2$; and hence by (b), we have $\alpha_A{}^2(m) \leq \alpha_B{}^2(\phi(m))$ and $\beta_A{}^2(m) \geq \beta_B{}^2(\phi(m)) \forall \psi \in \mathrm{Hom}(E, E*)$ and $m \in E$.Hence A is B-projective.

Theorem 3.3: Let E be a G-module and $E = \bigoplus_{i=1}^{n} E_i$where E_i's are G-submodules of E. Let A be an EIF G-module on E andB_i's be EIF G-modules on $E_i$and let $B = \bigoplus_{i=1}^{n} B_i$.Then A is B- projective if and only if A is B_i- projective for all i $(1 \leq i \leq n)$

Proof. Assume that A is B-projective. Then

a) $E = \bigoplus_{i=1}^{n} E_i$ is projective and

b) $\alpha_A{}^2(m) \leq \alpha_B{}^2(\psi(m))$; $\beta_A{}^2(m) \geq \beta_B{}^2((\psi(m))$ $\forall \psi \in \mathrm{Hom}(E, E^*)$ and $m \in E$

To prove that A is Bi -projective for i= 1, 2, ...,n.From (a) and Remark 2.2, we have that E is Ei-projective for all i= 1, 2, ...,n. Let ϕ∈Hom(E, Ei) and η : Ei⟶ E be the inclusion homomorphism. Then ψ = η o ϕ :E ⟶ E is a homomorphism, and hence by (b),

$$\alpha_A{}^2(m) \leq \alpha_B{}^2(\psi(m)) = \alpha_B{}^2(\eta\, o\phi)(m)) = \alpha_B{}^2(\eta\,(\phi(m)) = \alpha_B{}^2(\eta\,(\phi(m)) = \alpha_B{}^2((\phi(m))$$

And

$$\beta_A{}^2(m) \geq \beta_B{}^2(\psi(m)) = \beta_B{}^2(\eta\, o\phi)(m)) = \beta_B{}^2(\eta\,(\phi(m)) = \beta_B{}^2(\eta\,(\phi(m)) = \beta_B{}^2((\phi(m)) \quad \forall \phi \in \mathrm{Hom}(E, Ei) \qquad \ldots\ldots\ldots \quad \text{(i)}$$

Since $\phi \in \mathrm{Hom}(E, Ei), \phi(m) \in Ei \subseteq E$

$$\phi(m) = 0 + 0 + \ldots + \phi(m) + \ldots + 0.$$

Therefore,

$$\alpha_B{}^2(\phi(m)) = \alpha_B{}^2\,(0 + 0 + \cdots + \phi(m) + \cdots + 0)$$
$$= \alpha_{B_1}{}^2(0) \wedge \alpha_{B_2}{}^2(0) \wedge \ldots \alpha_{B_i}{}^2(\phi(m)) \ldots \alpha_{B_n}{}^2(0)$$
$$= \alpha_{B_i}{}^2(\phi(m))$$
$$\beta_B{}^2(\phi(m)) = \beta_B{}^2\,(0 + 0 + \cdots + \phi(m) + \cdots + 0)$$
$$= \beta_{B_1}{}^2(0) \wedge \beta_{B_2}{}^2(0) \wedge \ldots \beta_{B_i}{}^2(\phi(m)) \ldots \beta_{B_n}{}^2(0)$$
$$= \beta_{B_i}{}^2(\phi(m)$$

Thus,

$$\alpha_B{}^2(\phi(m)) = \alpha_{B_i}{}^2(\phi(m) \text{ and } \beta_B{}^2(\phi(m)) = \beta_{B_i}{}^2(\phi(m) \;\forall \phi \in \mathrm{Hom}(E, Ei) \text{ and } m \in E$$

Therefore, from (1) we get $\alpha_A{}^2(m) \leq \alpha_{B_i}{}^2(\phi(m))$ and$\beta_A{}^2(m) \geq \beta_{B_i}{}^2(\phi(m) \forall \phi \in \mathrm{Hom}(E, Ei)$and $m \in E$. Therefore, A is Bi -projective for all i.

Conversely: Assume that A is Bi -projective for all i. Then for each i=1, 2,...,n. We have

c) E is Ei-projective and

d) $\alpha_A{}^2(m) \leq \alpha_{B_i}{}^2(\phi(m))$ and$\beta_A{}^2(m) \geq \beta_{B_i}{}^2(\phi(m)$

$\forall \phi \in \mathrm{Hom}(E, Ei)$and $m \in E$. To prove that A is B-projective. By (c) and Remark 2.2, $E = \bigoplus_{i=1}^{n} E_i$-Projective. Let ψ ∈Hom(E, E). Then ψ(m) ∈E, for every m ∈E, and so

$$\psi(m) = m_1 + m_2 + \ldots + m_n, \text{ where } m_i \in E_i, \forall i = 1, 2, \ldots, n. \qquad \ldots \text{(ii)}$$

Let π_i: E $\to$ $E_i$be the projection mapping ($1 \le i \le n$). Then $\pi_i(\psi(m))= m_i$, for all i= 1, 2, ..., n. and hence by (ii) we get

$$\psi(m) = \pi_1(\psi(m)) + \pi_2(\psi(m)) + \ldots + \pi_n(\psi(m))$$

$= (\pi_1 \circ \psi)(m) + (\pi_2 \circ \psi)(m) + \ldots + (\pi_n \circ \psi)(m)$.

Let $\phi_i = \pi_i \circ \psi$. Then $\phi i \in$Hom(E, Ei) and therefore

$$\psi(m) = \phi_1(m) + \phi_2(m) + \ldots + \phi_n(m). \qquad \ldots(iii)$$

Now,

$$\alpha_B{}^2(\Psi(m)) = \alpha_B{}^2(\phi 1(m) + \phi 2(m) \ldots + \phi n(m))$$

$$\alpha_B{}^2(\Psi(m)) \ge \wedge\left\{\alpha_{B_i}{}^2(\phi_i(m)) : i = 1,2,\ldots n\right\}$$

$$\ge \alpha_A{}^2(m)$$

Similarly,

$$\beta_B{}^2(\Psi(m)) = \beta_B{}^2(\phi 1(m) + \phi 2(m) \ldots + \phi n(m))$$

$$\alpha_B{}^2(\Psi(m)) \le \vee\left\{\alpha_{B_i}{}^2(\phi_i(m)) : i = 1,2,\ldots n\right\}$$

$$\le \beta_A{}^2(m)$$

Thus, $\alpha_B{}^2(\Psi(m)) \ge \alpha_A{}^2(m)$ and$\alpha_B{}^2(\Psi(m)) \le \beta_A{}^2(m) \forall m \in M$ and $\psi \in$Hom(E, E). Therefore, A is B-projective.

Remark 3.1: Let $E = \bigoplus_{i=1}^{n} E_i$ whereE_i's are G-submodules of E. LetBi′s are EIF G-modules on Ei such that $B = \bigoplus_{i=1}^{n} B_i$ Then Bis quasi-projective if and only if B is Bi -projective, for all i.

Remark 3.2: Let Ei′s be G-modules. Then the direct sum$\bigoplus_{i=1}^{n} E_i$quasi-projective if and only if Ei is E j -projective, for every i, j ∈{1, 2,...,n}.

Remark 3.3: Let E be a G-module. For a positive integer n, E^n=E ⊕E ⊕....⊕E is quasi-projective if and only if E is quasi-projective.

Theorem 3.4: Let E = $E_1 \oplus E_2$, where $E_1$and E_2be G-submodules of E. Let Bi′s be EIF G-modules on Ei($1 \le i \le n$) such that B = $B_1 \oplus B_2$. Then B is quasi-projective if and only if Bi is Bj–projective for every i, j ∈ {1, 2,...,n}.

Proof: Assume that B is quasi-projective. Then

a) E is E-projective and

b) $\alpha_B{}^2(m) \le \alpha_B{}^2(\Psi(m))$and $\beta_B{}^2(m) \ge \beta_B{}^2(\Psi(m))$; $\forall \psi \in$Hom(E, E) and m ∈ E.

Also From (a) and Remark 3.2, Ei is E j -projective for every i, j ∈ {1, 2}.

To show that Bi is Bj -projective for all i, j. Since B is quasi-projective, from Remark 3.1, B is Bi -projective ∀i =1, 2. .... (i)

$$\alpha_B{}^2(m) \le \alpha_{B_i}{}^2(\phi(m)) \; ; \forall \phi \in \mathrm{Hom}(E, Ei) \text{ and } m \in E \qquad \ldots\ldots\ldots\ldots(ii)$$

Let $\psi \in$Hom(E_1, E_2) and let π_1 : E $\to$ E_1 be the projection map. Then $\psi \circ \pi_1$:E $\to$ $E_2$is a homomorphism. Then from (ii), we get

$$\alpha_B{}^2(m) \le \alpha_{B_2}{}^2(\psi \circ \pi 1)(m) \text{ and } \beta_B{}^2(m) \ge \beta_{B_2}{}^2((\psi \circ \pi 1)(m)) \; \forall m \in E$$

If m = $m_1 \in E_1$, then

$$\alpha_B{}^2(m_1) \leq \alpha_{B_2}{}^2(\psi \circ \pi 1)(m_1) = \alpha_{B_2}{}^2\left(\psi\left(\pi 1(m_1)\right)\right) = \alpha_{B_2}{}^2(\psi(m_1))$$

And

$$\beta_B{}^2(m_1) \geq \beta_{B_2}{}^2\left((\psi \circ \pi 1)(m_1)\right) = \beta_{B_2}{}^2\left(\psi\left(\pi 1(m_1)\right)\right) = \beta_{B_2}{}^2(\psi(m_1)) \forall m \in E$$

$$\alpha_B{}^2(m_1) = \alpha_B{}^2(m_1 + 0) = \alpha_{B_1}{}^2(m_1) \wedge \alpha_{B_2}{}^2(0) = \alpha_{B_1}{}^2(m_1)$$

Similarly,

$$\beta_B{}^2(m_1) = \beta_B{}^2(m_1 + 0) = \beta_{B_1}{}^2(m_1) \wedge \beta_{B_2}{}^2(0) = \beta_{B_1}{}^2(m_1)$$

$$\alpha_{B_1}{}^2(m_1) \leq \alpha_{B_2}{}^2(\psi(m_1)) \text{ and } \beta_{B_1}{}^2(m_1) \geq \beta_{B_2}{}^2(\psi(m_1)) \quad \forall \psi \in \mathrm{Hom}(E_1, E_2) \text{ and } m_1 \in E_1$$

Therefore, B_1 is B_2 -projective. Similarly, we can show that B_2 is B_1-projective. Also we have to show that B_1 is B_1-projective.Let$\theta_1 \epsilon \mathrm{Hom}(E_1, E_1)$, then $(\theta_1 \circ \pi 1)(m_1) \epsilon \mathrm{Hom}(E, E_1)$. Since B is B_1-projective, therefore

$$\alpha_B{}^2(m) \leq \alpha_{B_1}{}^2(\theta_1 \circ \pi 1)(m) \text{ and } \beta_B{}^2(m) \geq \beta_{B_1}{}^2\left((\theta_1 \circ \pi 1)(m)\right) \forall m \in E$$

If $m = m_1 \in E_1$, then

$$\alpha_B{}^2(m_1) \leq \alpha_{B_2}{}^2(\theta_1 \circ \pi 1)(m_1) = \alpha_{B_2}{}^2\left(\theta_1\left(\pi 1(m_1)\right)\right) = \alpha_{B_2}{}^2(\theta_1(m_1))$$

And

$$\beta_B{}^2(m_1) \geq \beta_{B_2}{}^2\left((\theta_1 \circ \pi 1)(m_1)\right) = \beta_{B_2}{}^2\left(\theta_1\left(\pi 1(m_1)\right)\right) = \beta_{B_2}{}^2(\theta_1(m_1)) \forall \theta_1 \epsilon \mathrm{Hom}(E_1, E_2), \forall m \in E$$

But $\alpha_B{}^2(m_1) = \alpha_{B_1}{}^2(m_1)$ and $\beta_B{}^2(m_1) = \beta_{B_1}{}^2(m_1) \theta_1 \epsilon \mathrm{Hom}(E_1, E_1)$ and $\forall m \in E$. Therefore, B_1 is B_1 -projective. Similarly, we can show that B_2 is B_2 -projective.Hence we get Bi is Bj-projective for every i, j ∈ {1, 2}.

Conversely: Assume that Bi is Bj-projective for every i, j ∈ {1, 2}. Then

c) $E_i$is E_j-projective for every i, j ∈ {1, 2} and

d) $\alpha_{B_i}{}^2(m) \leq \alpha_{B_j}{}^2(\Psi(m))$ and $\beta_{B_i}{}^2(m) \geq \beta_{B_j}{}^2(\Psi(m)) \forall \psi \in \mathrm{Hom}(E_i, E_j)$, and $m \in Ei$

By (c) and Remark 3.2, $E = E_1 \oplus E_2$ –projective. Therefore by Remark 2.2, E is Ei-projective for i= 1, 2. Now to prove that B is Bi -projective.Let Let$\theta_1 \epsilon \mathrm{Hom}(E, E_1)$ and let $\phi_1 : E_2 \rightarrow E$ be the inclusion homomorphism.Then $\theta_1 \mathrm{O} \phi_1$: $E_2 \rightarrow E_1$is a homomorphism. Since $B_2$is B_1 - projective,

$$\alpha_{B_2}{}^2(m_2) \leq \alpha_{B_1}{}^2\left((\theta_1 \mathrm{O} \phi 1)(m_2)\right) \text{and } \beta_{B_2}{}^2(m_2) \geq \beta_{B_1}{}^2\left((\theta_1 \mathrm{O} \phi 1)(m_2)\right) \forall \theta 1 \in \mathrm{Hom}(E, E1), \quad m_2 \in E_2$$

$$\alpha_{B_2}{}^2(m_2) \leq \alpha_{B_1}{}^2\left(\theta_1\left(\phi 1(m_2)\right)\right) \text{and } \beta_{B_2}{}^2(m_2) \geq \beta_{B_1}{}^2\left(\theta_1\left(\phi 1(m_2)\right)\right) \forall \theta 1 \in \mathrm{Hom}(E, E1), \quad m2 \in E2$$

i.e

$$\alpha_{B_2}{}^2(m_2) \leq \alpha_{B_1}{}^2(\theta_1(m_2)) \text{ and } \beta_{B_2}{}^2(m_2) \geq \beta_{B_1}{}^2(\theta_1(m_2)) \forall \theta 1 \in \mathrm{Hom}(E, E_1), \ m_2 \in E_2$$

……………….. (iii)

Since $\theta_1 \epsilon \mathrm{Hom}(E, E_1)$and $\eta_1 = \eta_{1|E_1} \in \mathrm{Hom}(E_1, E_1)$. Also, B_1 is B_1-projective, we have

$$\alpha_{B_1}{}^2(m_1) \leq \alpha_{B_1}{}^2(\eta_1(m_1))\,;\ \beta_{B_1}{}^2(m_1) \geq \beta_{B_1}{}^2(\eta_1(m_1)) \quad \forall \eta_1 \in \mathrm{Hom}(E1, E1)$$

And $m1 \in E1$

$$\alpha_{B_1}{}^2(m_1) \leq \alpha_{B_1}{}^2(\theta_1(m_1))\,;\ \beta_{B_1}{}^2(m_1) \geq \beta_{B_1}{}^2(\theta_1(m_1)) \quad \forall \theta_1 \in \mathrm{Hom}(E, E_1)$$

and$m1 \in E1$ ………………….(iv)

From (iii) and (iv), we get,

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$$\forall\, m_i \in Ei \text{ and } \theta_1 \in Hom(E, E_1)$$

$$\alpha_{B_1}{}^2(m_1) \wedge \alpha_{B_2}{}^2(m_2) \leq \alpha_{B_1}{}^2\big(\theta_1(m_1)\big) \wedge \alpha_{B_1}{}^2\big(\theta_1(m_2)\big) \text{and}$$

$$\beta_{B_1}{}^2(m_1) \vee \beta_{B_2}{}^2(m_2) \geq \beta_{B_1}{}^2\big(\theta_1(m_1)\big) \vee \beta_{B_1}{}^2\big(\theta_1(m_2)\big)$$

If $m = m_1 + m_2 \in E = E_1 \oplus E_2$, then we have

$$\alpha_B{}^2(m) = \alpha_{B_1}{}^2(m_1) \wedge \alpha_{B_2}{}^2(m_2) \text{ and } \beta_B{}^2(m) = \beta_{B_1}{}^2(m_1) \vee \beta_{B_2}{}^2(m_2)$$

Hence, $\alpha_B{}^2(m) \leq \alpha_{B_1}{}^2\big(\theta_1(m_1)\big) \wedge \alpha_{B_1}{}^2\big(\theta_1(m_2)\big)$and

$$\beta_B{}^2(m) \geq \beta_{B_1}{}^2\big(\theta_1(m_1)\big) \vee \beta_{B_1}{}^2\big(\theta_1(m_2)\big);\quad \theta_1 \in Hom(E, E_1) \qquad \ldots\ldots\ldots\ldots\ldots \text{(v)}$$

Since B_1 is an intuitionistic fuzzy G-module on E_1, we have $\forall x, y \in E_1$, $a, b \in K$,

$$\alpha_{B_1}{}^2(ax + by) \geq \alpha_{B_1}{}^2(x) \wedge \alpha_{B_1}{}^2(y) \text{and}$$

$$\beta_{B_1}{}^2(ax + by) \geq \beta_{B_1}{}^2(x) \vee \beta_{B_1}{}^2(y) \ldots\ldots\ldots\ldots\text{(vi)}$$

Since, $\theta_1 \in Hom(E, E_1)$, $x = \theta_1(m_1) \in E_1$ and $y = \theta_1(m_2) \in E_1$. From (vi),

$$\alpha_{B_1}{}^2(1.x + 1.y) = \alpha_{B_1}{}^2(1.\theta_1(m_1) + 1.\theta_1(m_2)$$

$$\geq \alpha_{B_1}{}^2\big(\theta_1(m_1)\big) \wedge \alpha_{B_1}{}^2\big(\theta_1(m_2)\big)$$

i.e. $\quad \alpha_{B_1}{}^2(\theta_1(m_1) + \theta_1(m_2) \geq \alpha_{B_1}{}^2\big(\theta_1(m_1)\big) \wedge \alpha_{B_1}{}^2\big(\theta_1(m_2)\big)$

i.e. $\quad \alpha_{B_1}{}^2(\theta_1(m_1 + m_2) \geq \alpha_{B_1}{}^2\big(\theta_1(m_1)\big) \wedge \alpha_{B_1}{}^2\big(\theta_1(m_2)\big)$

i.e. $\quad \alpha_{B_1}{}^2(\theta_1(m)) \geq \alpha_{B_1}{}^2\big(\theta_1(m_1)\big) \wedge \alpha_{B_1}{}^2\big(\theta_1(m_2)\big)$

Similarly,

$$\beta_{B_1}{}^2(\theta_1(m)) \leq \beta_{B_1}{}^2\big(\theta_1(m_1)\big) \vee \beta_{B_1}{}^2\big(\theta_1(m_2)\big) \qquad \ldots\ldots\ldots\text{(vii)}$$

From (v) and (vii), we get

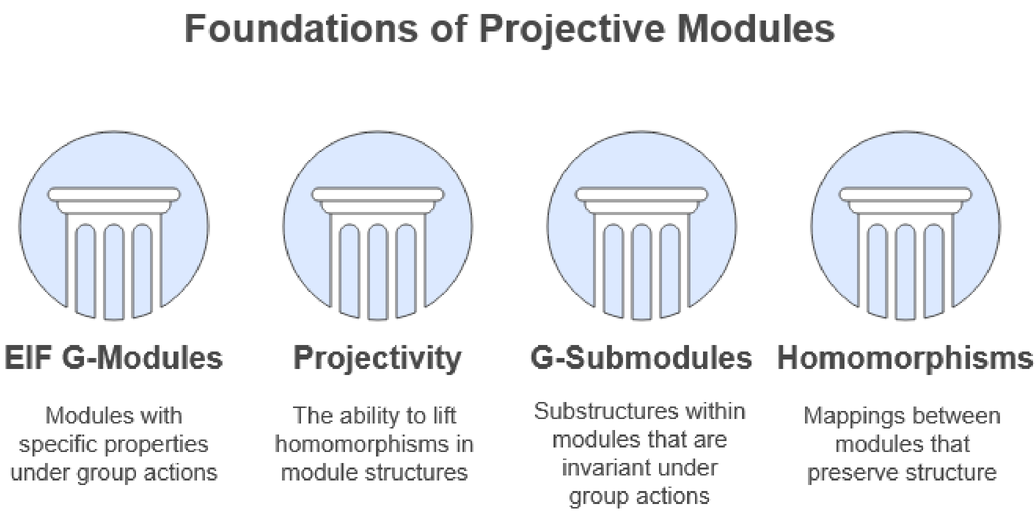
$$\alpha_B{}^2(m) \leq \alpha_{B_1}{}^2(\theta_1(m)) \text{and} \beta_B{}^2(m) \geq \beta_{B_1}{}^2\big(\theta_1(m)\big); \theta_1 \in Hom(E, E_1)$$

Therefore, B is B_1-projective. Similarly, we can show that B is B_2 -projective. So, B is Bi -projective for each i= 1, 2. Hence by Remark 3.3, B is quasi projective.

Remark 3.4: Let $E = \oplus_{i=1}^{n} E_i$ whereE_i's are G-submodules of E. LetBi's are EIF G-modules on Ei such that $B = \oplus_{i=1}^{n} B_i$ Then Bis quasi-projective if and only if B is Bi is Bj-projective, for every i, j ∈ {1, 2,...,n}.

4. CONCLUSION

In this paper, we introduced the concepts of projectivity and quasi-projectivity in the context of intuitionistic fuzzy G-modules. We provided several illustrative examples that reveal the underlying structure of these modules. Furthermore, we examined the notion of relative projectivity (and quasi-projectivity) of one intuitionistic fuzzy G-module with respect to another, offering insights into their interrelations and functional properties.The foundational structure of projective modules rests on four key elements: EIF G-Modules, Projectivity, G-Submodules, and homomorphisms. EIF G-Modules introduce group-action-based properties that enrich module analysis. Projectivity ensures the lifting of homomorphisms, supporting structural exactness. G-Submodules capture invariant substructures under group actions, while homomorphisms preserve structure across module mappings. Together, these elements offer a cohesive framework for studying projective modules within algebraic theory



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