Beal's Fuzzy Ideal Structure over the View of Uncertainty Normed Lattice

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Abstract :- In this paper, we construct Beal's fuzzy ideals in terms of (S,T) – normed sublattice. This research delves into the exploration of Beal's fuzzy sublattices and Beal's fuzzy ideals within the context of lattice theory. Through a rigorous analysis of structural theorem concerning these concepts derived from Beal's fuzzy sets. we uncover significant parallels with classical theory. Additionally, we investigate the behavior of Beal's fuzzy ideals under lattice homomorphism. Our findings shed light on the applicability and utility of Beal's fuzzy theory in lattice-based structures offering insights into their properties and relationships.

Keywords: Crisp set, Fuzzy set, Relation, Beal's fuzzy set, Beal's fuzzy sublattice, Prime ideal, Homomorphism, Norm, Order preserving.

1. Introduction

A new chapter in mathematical study opened with the advent of fuzzy set theory [L.A.Zadeh] [13] sparking a plethora of studies with for searching consequences in a variety of fields. Interestingly, subsequent academic work has taken this ground work further in number of areas, [Akrametal] [2], with particular emphasis on group theory and rings. [Ajmal and Thomas] [1] who have introduced the idea of fuzzy sublattices by employing fuzzy set theory in lattice theory, have greatly enhanced this treat. Fuzzy group research has increased as a result of Rosenfeld's [9] new effort. [Atanassov] [3] made a significant break from traditional thought when he created intuitionistic fuzzy sets response to these observations. This extension proposes a framework for dealing with ambiguity, subgroups [Biswas][5] subgroups and subrings [Bakhasach et.al] [4] have been created by recent work in abstract algebra, along with other concepts such as intuitionistic fuzzy numbers [Sureshjani et.al] [10]. In 1998, as part of this work on the traditional of uncertainty.[F.Smarandache] [11] proposed a maximum general theory than that introduced by [Atanassov] [3] called the "Neutrosophic set" this new theory is characterized by a truth membership, an indeterminacy membership and a falsity membership. Responding to the exigencies posed by pervasive imprecision and uncertainty. Yager [12] introduced Pythagorean fuzzy sets. Building upon the foundational work of [Zang and Xu] [14], the conceptional frame work aims to translate nebulous and uncertain circumstances into a rigorous mathematical frame work, there by fascinating the derivation of efficacious solutions [Kumar et.al] [8]. In this present study, a comprehensive examination of Beal's fuzzy ideals and sublattices within the lattice context, elucidating their defining attributes with mathematical riger. Moreover, our investigation extends to an incisive analysis of Beal's fuzzy ideals and lattice homomorphism, there by establishing crucial links between theoretical observation and practical application within the mathematical domain. A generalization of the Beal's conjecture and prize problem defined by [R.Daniel Mauldin] [6] in 1997. In this paper, we construct Beal's fuzzy ideals in terms of (S,T) – normed sublattice. This research delves into the exploration of Beal's fuzzy sublattices and Beal's fuzzy ideals within the context of lattice theory. Through a rigorous analysis of structural theorem concerning these concepts derived from Beal's fuzzy sets. we uncover significant parallels with classical theory. Additionally, we investigate the behavior of Beal's fuzzy ideals under lattice homomorphism. Our findings shed light on the applicability and utility of Beal's fuzzy theory in lattice-based structures offering insights into their properties and relationships.

2.Preliminaries: In this section, we will list a few concepts. As we go through this essay we will denote $L = (A, +, \cdot)$ a lattice where '+' and '·' denote the sup and inf respectively. Here, we will analyse the basic definition of uncertainty set, Beal's fuzzy set and its properties.

Definition 2.1 A partially ordered set (POSET) is a set in which a binary relation $a \leq b$ is defined which satisfies for all a, b, c the following conditions,

$P_1: For\ all\ X, a \leq b.$

$P_2: If\ a \leq b\ and\ b \leq a, then\ a = b.$

$P_3: If\ a \leq b\ and\ b \leq c, then\ a \leq c.$

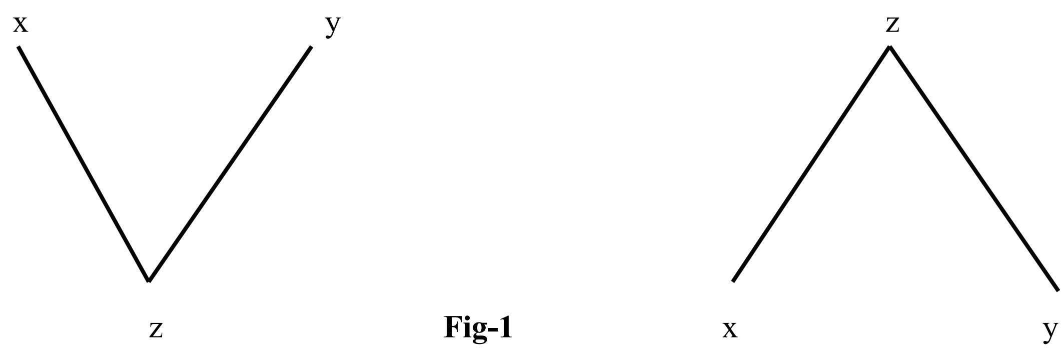

Fig-1 Least upper bound (LUB) and Greatest Lower bound (GLB)

Definition 2.2 (i) A mapping $f: P \to Q$ is called order preserving, if $a \leq b$ implies $f(a) \leq f(b)$.

(ii) A mapping $g: P \to Q$ is called order reversing (antitone) if and only if $a \leq b$ and $g(b) \leq g(a)$.

Definition 2.3 A lattice is a poset in which $a \wedge b = \inf(a, b)$ *and* $a \vee b = \sup(a, b)$ exists for any pair of elements a and b. A sublattice of a lattice 'L' is a subset of X and L such that $a, b \in X$ implies $a \wedge b \in X$ and $a \vee b \in X$. A lattice 'L' is complete is said to be distributive, if

$D_1: a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$D_2: a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, for all $a, b, c \in L$.

Definition 2.4 A Beal's fuzzy set (BFS) on a universal set 'A' is a set of 3-tuples the form $\triangle = \{(x, J_\triangle(x), K_\triangle(x)\}$ where $J_\triangle(x)$ and $K_\triangle(x)$ represents the membership and non-membership degrees of $x \in A$ and $J_\triangle(x)$ and $K_\triangle(x)$ satisfy $0 \leq J_\triangle(x) + K_\triangle(x) \leq 1$ for all $x \in A$ $m, n \in N = \{4,5,6, ..\}$ the indeterminacy $\prod_\triangle(x) = \sqrt[m+n]{1 - (J_\triangle(x))^m - (K_\triangle(x))^n}$.

Example 2.5 Let $J_\triangle(x) = 0.9$ and $K_\triangle(x) = 0.7$ $m = 4, n = 5$ $\prod_\triangle(x) = \sqrt[9]{1 - (0.9)^4 - (0.7)^5}$

$$= \sqrt[9]{1 - 0.6561 - 0.16807}$$

$$= 0.0195$$

and at the same $0 \leq (0.9)^4 + (0.7)^5 \leq 1$

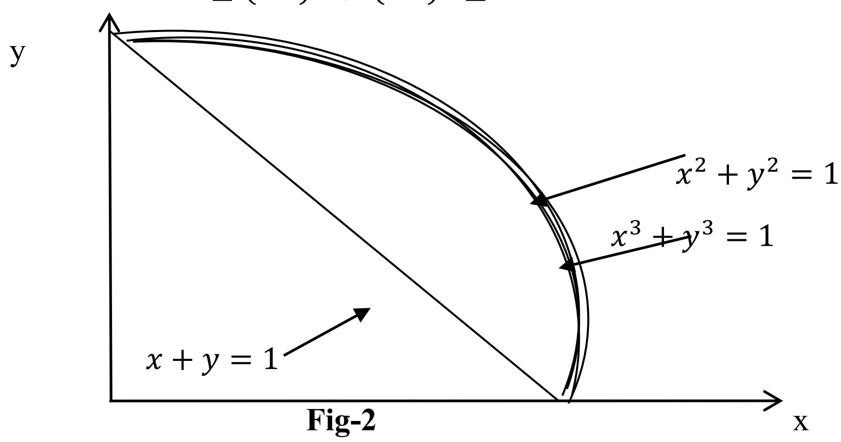

Fig-2 Grade spaces of Beal's fuzzy set (m ≥ 4 , n ≥ 4)

Remark 2.6 Beal's fuzzy set is a generalization of fermat's fuzzy set .

Remarks 2.7. Fermat's fuzzy set is a generalization of intuitionistic fuzzy set. Below are some of the Beal's fuzzy sets operations.

Defiinition 2.8 Consider a non - empty crisp set E and let $\triangle_1 = \{(x, J_1(x), K_1(x))/x \in E\}$ and

$\triangle_2 = \{(x, J_2(x), K_2(x))/ x \in E\}$ be two Beal's fuzzy subset of E. Then

(i) $\triangle_1 \subset \triangle_2$ if and only of $J_1^m \leq J_2^m$ and $K_1^n \geq K_2^n$.

(ii) $\triangle_1 = \triangle_2$ if and only if $\triangle_1 \subset \triangle_2$ and $A_1 \supset A_2$.

(iii) $\triangle_1^c = (K_1, J_1)$

(iv) $\triangle_1 \cap \triangle_2 = (J_1^m \wedge J_2^m, K_1^n \vee K_2^n)$

(v) $\triangle_1 \cup \triangle_2 = (J_1^m \vee J_2^m, K_1^n \wedge K_2^n)$.

Theorem 2.9 The set of Beal's fuzzy membership grades (BMG) $(m, n \geq 4)$ is larger than the set of intuitionistic membership grades (IMG) , Pythagorean membership grades and fermatean membership grades.

Proof: Since for $p, q \in [0,1]$, we have

$p^m \leq p^{m-1} \leq \cdots \leq p^3 \leq p^2 \leq p$ and

$q^m \leq q^{m-1} \leq \cdots \leq q^3 \leq q^2 \leq q$

therefore, we can get

$$p + q \leq 1$$

Implies $p^2 + q^2 \leq 1$

Implies $p^3 + q^3 \leq 1$

Implies $p^4 + q^4 \leq 1$

. .

. .

. .

Implies $p^{m-1} + q^{n-1} \leq 1$

Implies $p^m + q^n \leq 1$

then, the space of Beal's fuzzy membership grades is larger than the space of intuitionistic membership grades, Pythagorean membership grades and fermatean membership grades. This development can be evidently recognized from the above **(Fig-2).**

Definition 2.10 The function S: $[0,1] \times [0,1] \longrightarrow [0,1]$ is called a S-norm , if it satisfies the following conditions

$(i) S(x, 1) = x$ (Border Condition)

(ii) $S(x, y) = S(y, x)$ (Commutativity)

(iii) $S(S(x, y), z) = S\big(x, S(y, z)\big)$ (Associativity)

$(iv) S(x, y) \leq S(x, z)$ if $y \leq z$ (Monotonicity) for all $x, y, z \in [0,1]$.

If norm S has the property, $S(x, y) = \min\{x, y\}$, then

$(i) S(x, 0) = 0$

$(ii) S\big(S(u, v), S(x, y)\big) = S(S(u, x), S(v, y))$ for all $x, y, u, v \in [0,1]$

$(iii) S(x, x) = x$.

Definition 2.11 The function T: $[0,1] \times [0,1] \longrightarrow [0,1]$ is called a T-norm, if it satisfies the following condition

$(i) T(x, 1) = x$

$(ii) T(x, y) = T(y, x)$

$(iii) T(T(x, y), z) = T\big(x, T(y, z)\big)$

$(iv) T(x, y) \leq T(x, z)$ if $y \leq z$ for all $x, y, z \in [0,1]$.

If norm T has the property, $T(x, y) = \max\{x, y\}$, then

$(i) T(x, 0) = 0$

$(ii) T\big(T(u, v), T(x, y)\big) = T(T(u, x), S(v, y))$ for all $x, y, u, v \in [0,1]$.

$(iii) T(x, x) = x$.

Definition 2.12 A Beal's fuzzy set $\delta = \{(x, J_\delta(x), K_\delta(x))/x \in E\}$ of E is called a Beal's fuzzy set relation of E if each of the following holds for every $x_1, x_2 \in E$,

(BFR_1): $J_\delta^m(x_1 - x_2) \geq T\{J_\delta^m(x_1), J_\delta^m(x_2)\}$ and

$K_\delta^n(x_1 - x_2) \leq \delta\{K_\delta^n(x_1), K_\delta^n(x_2)\}$.

(BFR_2) : $J_\delta^m(x_1, x_2) \geq S\{J_\delta^m(x_1), J_\delta^m(x_2)\}$ and

$K_\delta^m(x_1, x_2) \geq T\{K_\delta^n(x_1), K_\delta^n(x_2)\}$ the Beal's fuzzy subset $'\delta'$ is recogonized as the Beal'sfuzzy index of E.

Definition 2.13 Let Δ_1 and Δ_2 be two Beal's fuzzy subset of E. The product Δ_1 and Δ_2 is defined by $\Delta_1 \circ \Delta_2 = \{(x, J_{\Delta_1 \circ \Delta_2}(x), K_{\Delta_1 \circ \Delta_2}(x))/ x \in E\}$,

where

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$$\left(J_{\Delta_1 \circ \Delta_2}(x)\right)^m = S\{T\{\left(J_{\Delta_1}(x_1)\right)^m, \left(J_{\Delta_2}(x_2)\right)^m\}/ x_1, x_2 \in E, x_1x_2 = x\}$$

and

$$\left(K_{\Delta_1 \circ \Delta_2}(x)\right)^n = T\{S\{\left(J_K(x_1)\right)^n, \left(K_{\Delta_2}(x_2)\right)^n\}/ x_1, x_2 \in E, x_1x_2 = x\}.$$

3. Beal's fuzzy sublattice and Ideals

In this section, we are discussed Beal's fuzzy lattice, Beal's fuzzy ideal and their characterizations in details.

Definition 3.1 Let 'M' be a lattice and $L = \{(x, J(x), K(x)/ x \in M\}$ is a Beal's fuzzy subset of M. So 'L' is considered as a Beal's fuzzy sublattice of 'M' if the following conditions are valid.

(BFL_1): $J^m(m_1 + m_2) \geq T\{J^m(m_1), J^m(m_2)\}$ and

: $K^n(m_1+m_2) \leq S\{K^n(m_1), K^n(m_2)\}$ for any $m_1m_2 \in M$.

(BFL_2): $J^m(m_1m_2) \geq T\{J^m(m_1), J^m(m_2)\}$ and

: $K^n(m_1m_2) \leq S\{K^n(m_1), K^n(m_2)\}$ for any $m_1m_2 \in M$.

Example 3.2 Consider the lattice 'L' of "divisors of 6" that is L = {1 , 2 , 3 , 6}.

Let $A = \{(x, J_A(x), K_A(x)/ x \in L\}$ be given (1,0.6,0.3) , (2,0.5,0.7) , (3,0.5 ,0.4) ,(6 ,0.2,0.4) then 'A' is Beal's fuzzy sublattice of L.

Definition 3.3 A Beal's fuzzy set 'D' of A is called a Beal's fuzzy ideal of 'A' if $u \leq v$ in A implies $J^m(u) \geq J^m(v)$ and $K^n(u) \leq K^n(v)$.

Definition 3.4 Consider a Beal's fuzzy ideal 'D' of A. So 'D' is called a Beal's fuzzy prime ideal of 'A' if $J^m(uv) \leq S\{J^m(u), J^m(v)\}$ and $K^n(uv) \geq T\{K^n(u), K^n(v)\}$.

Definition 3.5 Consider a Beal's fuzzy set 'D' in A and $\gamma_1, \gamma_2 \epsilon [0,1]$, then $(\gamma_1, \gamma_2) - cut$ or (γ_1, γ_2) –level set of D , denoted by $D_{(\gamma_1,\gamma_2)}$ is the crisp set,

$$D_{(\gamma_1,\gamma_2)} = \{u \in A / J^m(u) \geq \gamma, and\ K^n(u) \leq \gamma_2\}$$

Based on the above definition, the following results have to be obtained .

Proposition 3.6 Let 'D' be a Beal's fuzzy ideal of A .Then the following proposition are equivalent;

(i) 'D' is a Beal's fuzzy prime ideal of A.

$(ii) D(t_1, t_2) = (S\{J^m(t_1), J^m(t_2)\}, T\{K^n(t_2, K^n(t_2)\})$ for any $t_1, t_2 \in A$.

$(iii) D(t_1, t_2) = D(t_1)\ or\ D(t_2) for\ any\ t_1, t_2 \in A.$

we describe Beal's fuzzy sublattice, Beal's fuzzy ideals, Beal's fuzzy prime ideals and in terms of level subset in the results that follows.

Theorem 3.7 Let 'D' be a Beal's fuzzy set of A. Then 'D' is a Beal's fuzzy sublattice of A if and only if for each $(\gamma_1,\gamma_2) \in Im(D), D(\gamma_1,\gamma_2)$ is a sublattice of A. (Here D (γ_1,γ_2) is called a level subset of A).

Proof: Suppose that D is a Beal's fuzzy sublattice. Consider $D(\gamma_1,\gamma_2)$ any non-empty level subset of D and let $t_1, t_2 \in (\gamma_1,\gamma_2)$.

Then $J^m(t_1) \geq \gamma_1,\ K^n(t_1) \leq \gamma_2$ *and* $J^m(t_2) \geq \gamma_1,\ K^n(t_2) \leq \gamma_2$.

Thus $J^m(t_1+t_2) \geq T\{J^m(t_1), J^m(t_2)\} \geq \gamma_1$ and $K^n(t_1+t_2) \leq S\{K^n(t_1), K^n(t_2)\} \geq \gamma_2$

and $J^m(t_1t_2) \geq T\{J^m(t_1), J^m(t_2)\} \geq \gamma_1$ and $K^n(t_1t_2) \leq S\{K^n(t_1), K^n(t_2)\} \geq \gamma_2$

Then $t_1+t_2 \in D(\gamma_1,\gamma_2)$ and $t_1t_2 \in D(\gamma_1,\gamma_2)$.

Hence $D(\gamma_1,\gamma_2)$ is a sublattice of A.

Conversely,

Assume that $D(\gamma_1,\gamma_2)$ is a sublattice of A. Let $t_1, t_2 \in D(\gamma_1,\gamma_2)$. we can assume that

$t_1 + t_2 = t_1$ *and* $t_1t_2 = t_2$ without loss of generality, then

$J^m(t_1+t_2) \geq \gamma_1 = T\{J^m(t_1), J^m(t_2)\}$ and

$K^n(t_1+t_2) \leq \gamma_2 = S\{K^n(t_1), K^n(t_2)\}$ and

$J^m(t_1t_2) \geq \gamma_1 = T\{J^m(t_1), J^m(t_2)\}$ and

$K^n(t_1t_2) \leq \gamma_2 = S\{K^n(t_1), K^n(t_2)\}$.

'D' is a Beal's fuzzy sublattice as a result this concludes the evidence.

Theorem 3.8 Let 'D' be a Beal's fuzzy sublattice of A. Then 'D' is a Beal's fuzzy ideal of A if and only if for each $(\gamma_1,\gamma_2) \in Im(D),\ D(\gamma_1,\gamma_2)$ is an ideal of A.

Equivalently, for each 'D' is a Beal's fuzzy set of A, D is a Beal's fuzzy ideal if and only each non-empty level subset $D(\gamma_1,\gamma_2)$ is an ideal. In this case $D(\gamma_1,\gamma_2)$ is called a level subset of ideal of A.

Theorem 3.9 Let 'D' be a Beal's fuzzy ideal of A. Then 'D' is a Beal's fuzzy prime ideal of A if and only if for each $(\gamma_1,\gamma_2) \in Im(D), D(\gamma_1,\gamma_2)$ is a prime ideal of A.

Proof: Suppose 'D' is a Beal's fuzzy prime ideal of A . Let $(\gamma_1,\gamma_2) \in Im(D)$ and

$t_1, t_2 \in D(\gamma_1,\gamma_2)$ then $J^m(t_1.t_2) \geq \gamma_1$ *and* $K^n(t_1.t_2) \leq \gamma_2$. But by proposition (3.6) we have $D(t_1.t_2) = D(t_1)$ *or* $D(t_1.t_2) = D(t_2)$

Thus $J^m(t_1) \geq \gamma_1,\ K^n(t_2) \leq \gamma_2$ *or* $J^m(t_1) \geq \gamma_1,\ K^n(t_2) \leq \gamma_2$

So $t_1 \in D(\gamma_1,\gamma_2)$ is a prime ideal. Suppose each level ideal $D(\gamma_1,\gamma_2)$ is prime. Assume that D is not the Beal's fuzzy prime ideal. Then by proposition (3.6) , their exists $t_1t_2 \in A$ such that $D(t_1.t_2) \neq D(t_2)$ and $D(t_1.t_2) \neq D(t_2)$.

Since D is a Beal's fuzzy ideal of A.

$J^m(t_1.t_2) \geq J^m(t_1)$, $J^m(t_1.t_2) \geq J^m(t_2)$ and

$K^n(t_1.t_2) \geq K^n(t_1)$, $K^n(t_1.t_2) \geq K^n(t_2)$.

Let $D(t_1, t_2) = (\gamma_1, \gamma_2)$. Then $t_2 \notin D(\gamma_1, \gamma_2)$ but $t_1 \notin D(\gamma_1, \gamma_2)$ and $t_2 \notin D(\gamma_1, \gamma_2)$.

This runs counter to the idea that $D(\gamma_1, \gamma_2)$ is prime.

Hence 'D' is a Beal's fuzzy prime ideal of A.

Theorem 3.10 If D_1 and D_2 are two Beal's fuzzy sublattice of a lattice A_1, then $D_1 \cap D_2$ is a Beal's fuzzy sublattice of A.

Proof: Consider $D_1 = \{ (x, J_1(x), K_1(x))/x \in A\}$ and $D_2 = \{ (x, J_2(x), K_2(x))/x \in A\}$ are two Beal's fuzzy subset of A. Then $D_1 \cap D_2 = \{ (t, J_{D_1 \cap D_2}(t), K_{D_1 \cap D_2}(t)/ t \in A\}$ where

$J^m_{D_1 \cap D_2}(t) = T\{ J^m_1(t), J^m_2(t) \}$ and $K^n_{D_1 \cap D_2}(t) = S\{ K^n_1(t), K^n_2(t) \}$.

$$J^m_{D_1 \cap D_2}(\theta_1 + \theta_2) = T\{ J^m_1(\theta_1 + \theta_2), J^m_1(\theta_1 + \theta_2) \}$$

$$\geq T\{ T\{ J^m_1(\theta_1), J^m_1(\theta_2)\}, T\{ J^m_2(\theta_1), J^m_2(\theta_2)\}$$

$$= T\{ T\{ J^m_1(\theta_1), J^m_2(\theta_2)\}, T\{ J^m_2(\theta_1), J^m_2(\theta_2)\}$$

$$= T\left\{ J^m_{D_1 \cap D_2}(\theta_1), J^m_{D_1 \cap D_2}(\theta_2)\right\}$$

As D_1 and D_2 are Beal's fuzzy sublattices of A .

(i.e) $J^m_{D_1 \cap D_2}(\theta_1 + \theta_2) \geq T\left\{ J^m_{D_1 \cap D_2}(\theta_1), J^m_{D_1 \cap D_2}(\theta_2)\right\}$ for all $\theta_1, \theta_2 \in A$.

Similarly, we get

$J^m_{D_1 \cap D_2}(\theta_1 \theta_2) \geq T\left\{ J^m_{D_1 \cap D_2}(\theta_1), J^m_{D_1 \cap D_2}(\theta_2)\right\}$ for all $\theta_1, \theta_2 \in A$.

Also

$$K^n_{D_1 \cap D_2}(\theta_1 + \theta_2) = S\{ K^n_1(\theta_1 + \theta_2), K^n_2(\theta_1 + \theta_2) \}$$

$$\leq S\left\{ S\{K^n_1(\theta_1), K^n_1(\theta_2)\}, S\{ K^n_2(\theta_1), K^n_2(\theta_2)\}\right\}$$

$$= S\left\{ S\{K^n_1(\theta_1), K^n_2(\theta_2)\}, S\{ K^n_1(\theta_1), K^n_2(\theta_2)\}\right\}$$

$$= S\{\left\{ K^n_{D_1 \cap D_2}(\theta_1), K^n_{D_1 \cap D_2}(\theta_2)\right\}$$

As D_1 and D_2 are Beal's fuzzy sublattices of A . Similar evidence support the Beal's fuzzy ideal.

4. Homomorphism and Beal's fuzzy sublattice

In this section, the idea of homomorphism and Beal's fuzzy sublattice is analysed.

Definition: 4.1 Consider $D_1 = \{(x, J_1(x), K_1(x))/x \in A\}$ and $D_2 = \{(x, J_2(x), K_2(x))/x \in B\}$ two Beal's fuzzy subsets on A and B respectively. Let $\emptyset$ be a mapping from A and B. Then $\emptyset(D)$ is a Beal's fuzzy subset on B and defined by

$\emptyset(D)(x) = (\emptyset(J_1^m)), \emptyset(K_1^m), (x))$ for all $x \in B$.

where $\emptyset(J_1^m)(x) = \begin{cases} S\{J_1^m(u)/u \in \emptyset^{-1}(x)\}, & if\ \emptyset^{-1}(x) \neq \chi \\ 0, & if\ \emptyset^{-1}(x) = \chi \end{cases}$

$$\emptyset(K_1^n)(x) = \begin{cases} T\{K_1^n(v)/ v \in \emptyset^{-1}(x)\}, & if\ \emptyset^{-1}(x) \neq \chi \\ 0, & if\ \emptyset^{-1}(x) = \chi \end{cases}$$

Also $\phi^{-1}(D)$ is a Beal's fuzzy set of A such that

$\phi^{-1}(D_2)(u) = (\phi^{-1}(J_2^m)(u), \phi^{-1}(K_2^n)(u))$ for all $x \in A$ where

$\phi^{-1}(J_2^m)(u) = J_2^m(\phi(u))$ and $\phi^{-1}(K_2^n)(u) = K_2^n(\emptyset(u))$.

In particular, if $\phi : A \longrightarrow A'$ is a lattice homomorphism, D_1 is a Beal's fuzzy sublattice of A and D_2 is a Beal's fuzzy sublattice of A', then $\phi^{-1}(D_1)$ is called homomorphism image of D_1 under ϕ and $\phi^{-1}(D_2)$ is called the homomorphism preimage of D_2, where A and A' denote lattices, respectively. The following theorems to be proved relative to the homomorphism.

Theorem 4.2. If $f: A \longrightarrow A'$ is a lattice epimorphism and D_1 is a Beal's fuzzy ideal of A, then $f(D_1)$ is a Beal's fuzzy ideal of A'.

Proof: Consider $D_1 = \{(x, J_1(x), K_1(x))/x \in A\}$ be a Beal's fuzzy ideal of A.
So $f(D) = \{(\theta_1, f(J_1)(\theta_2), f(K_1)(\theta_2)/\theta_2 \in A'\}$.

Let $\theta_1, P \in A'$

$$\begin{aligned} f(J_1^m)(\theta_1 + P) &= S\{J_1^m(\theta_2)/\theta_2 \in f^{-1}(\theta_1 + P)\} \\ &= S\{J_1^m(K + l)/ K \in f^{-1}(\theta_1)\ and\ l \in f^{-1}(P)\} \\ &\geq S\{T\{J_1^m(K), J_1^m(l)\}/K \in f^{-1}(\theta_1)\ and\ l \in f^{-1}(P)\} \\ &= T\{S\{J_1^m(K)/K \in f^{-1}(\theta_1)\}, S\{J_1^m(l)/ l \in f^{-1}(P)\}\} \\ &= T\{f(J_1^m)(\theta_1), f(J_1^m)(P)\} \text{ (Since } D_1 \text{ is a Beal's fuzzy ideal of A)} \end{aligned}$$

Also

$$\begin{aligned} f(K_1^n)(\theta_1 + P) &= T\{K_1^n(\theta_2)/\theta_2 \in f^{-1}(\theta + P)\} \\ &\leq T\{K_1^n(K + l)/ K \in f^{-1}(\theta_1) and\ l \in f^{-1}(P) \\ &\leq \{S\{K_1(K), K_1^n(l)\}/ K \in f^{-1}(\theta_1)\ and\ l \in f^{-1}(P)\} \\ &= S\{f(K_1^n)(\theta_1), f(K_1^n)(P)\} \text{ (Since } D_1 \text{ is a Beal's fuzzy ideal of A).} \end{aligned}$$

Also

$$f(K_1^n)(\theta_1.P) = T\{K_1^n(\theta_2)/\ \theta_2 \in f^{-1}(\theta_1, P)\}$$
$$\leq T\{K_1^n(K, l)/\ K \in f^{-1}(\theta_1) and\ l \in f^{-1}(P)\}$$
$$\leq T\{T\{K_1^n(K), K_1^n(l)\}/\ K \in f^{-1}(\theta_1) and\ l \in f^{-1}(P)\}$$
$$= T\{(TK_1^n)(K)/\ K \in f^{-1}(\theta_1), (TK_1^n)(l)/\ l \in f^{-1}(P)\}$$

(Since D_1 is a Beal's fuzzy ideal of A)

Hence $f(D_1)$ is a Beal's fuzzy ideal of A'.

Theorem 4.3 Let $f : A \longrightarrow A'$ be a lattice homomorphism and D_2 is a Beal's fuzzy ideal of A'.Then $f^{-1}(D_2)$ is a Beal's fuzzy ideal of A.

Theorem 4.4 Let $f : A \longrightarrow A'$ be an onto mapping and D_1 and D_2 are Beal's fuzzy sets of the lattices A and A' respectively. So

$(i)\ f(f^{-1}(D_2)) = D_2$

$(ii)\ D_1 \subseteq f^{-1}(f(D_1))$

Proof: (i) we have $f(f^{-1}(J_2^m)(t)) = S\{f^{-1}(J_2^m)(\theta_2)\ /\ \theta_2 \in f^{-1}(t)\}$

$$= S\{J_2^m(f(\theta_2)/\ \theta_2 \in A, f(\theta_2) = t\}$$
$$= J_2^m(t)$$

Similarly, $f(f^{-1}(K_2)(t)) = K_2^n(t)$.

Therefore $f(f^{-1}(D_2)) = D_2$

(ii) Also, we have $f^{-1}(f(J_1^m)(\theta_2) = f(J_1^m)(f(\theta_2))$

$$= S\{J_1^m(\theta_2)/\ \theta_2 \in f^{-1}(f(\theta_2))\}$$
$$\geq J_1^m(\theta_2)$$

and

$$f^{-1}(f(K_1^n)(\theta_2) = f(K_1^n)(f(\theta_2)) = f(K_1^n)(f(\theta_2))$$
$$= T\{K_1^n(\theta_2)/\ \theta_2 \in f^{-1}(f(\theta_2))\}$$
$$\leq K_1^n(\theta_2)$$

Hence $D_1 \subseteq f^{-1}(f(D_1))$.

Definition 4.5 Let $f : A \longrightarrow A'$ be a function from D_1 is f-invariant. Then $f^{-1}(f(D_1)) = D_1$.

Theorem 4.6 Let $f : A \longrightarrow A'$ be a function from a lattice 'A' to another A' and D_1, D_2 are two Beal's fuzzy subsets of A and D_1' , D_2' are Beal's fuzzy subset of A'. Then

$(i) D_1 \subseteq D_2 \Rightarrow f(D_1) \subseteq f(D_2)$

(ii) $D_1' \subseteq D_2' \Rightarrow f^{-1}(D_1') \subseteq f^{-1}(D_2')$.

Proof: Let $D_1 = \{ (\theta_2, J_1(\theta_2), K_1(\theta_2)) / \theta_2 \in A \}$ and $D_2 = \{ (\theta_2, J_2(\theta_2), K_2(\theta_2)) / \theta_2 \in A \}$ be two Beal's fuzzy subset of A. Then

$D_1 \subseteq D_2 \Rightarrow J_1^m(\theta_2) \leq J_2^m(\theta_2)$ and $K_1^n(\theta_2) \geq K_2^n(\theta_2)$.

So $f(D_1) = \{(P, f(J_1^m)(P), f(K_1^n)(P) / P \in A'\}$

and $f(D_2) = \{(P, f(J_2^m)(P), f(K_2^n)(P) / P \in A'\}$

Now

$$\begin{aligned} f(J_1^m)(t) &= S\{J_1^m(\theta_2)/\theta_2 \in f^{-1}(t)\} \\ &\leq S\{J_2^m(\theta_2)/\theta_2 \in f^{-1}(t)\} \\ &= f(J_2^m)(t) \ as \ J_1^m(\theta_2) \leq J_2^m(\theta_2) \end{aligned}$$

Also,

$$\begin{aligned} f(K_1^n)(t) &= T\{K_1^n(\theta_2)/\theta_2 \in f^{-1}(t)\} \\ &\geq T\{K_1^n(\theta_2)/\theta_2 \in f^{-1}(t)\} \\ &= f(K_1^n)(t) \ as \ K_1^n(\theta_2) \geq K_1^n(\theta_2) \end{aligned}$$

Hence , $f(D_1) \subseteq f(D_2)$. Likewise, we can demonstrate other condition (ii).

Conclusion:

To summarize up our study investigated the complex characteristics of Beal's fuzzy ideals and sublattice. we define operations for these fuzzy ideals and proved their preservation in distributive Lattices. we also derived through examination of Beal's fuzzy ideals homomorphism images and pre-images which resulted in the creation of invariant Beal's fuzzy sets. This work eliminated in establishing a correspondence theorem that connects the f-invariant Beal's fuzzy ideals of a lattice to its homomorphism image and characterization of Beal's fuzzy lattice over ideals.

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Future work: In this work, our efforts extending various uncertainty sets like Neutrosophic set, Circular intuitionistic fuzzy set to further develop Lattice theory, building on the foundations laid in this study.

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