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An Analysis of Simultaneous Approximation on the Roots of Ultraspherical Polynomials

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Abstract: In this paper, we find an interpolatory polynomial $R_m(x)$ satisfying (0;0,1) interpolation with some special kind of boundary conditions at the given knots. The knots being considered here are the zeroes of the Ultraspherical polynomial. In this paper, we find the explicit representation of the fundamental polynomials, their uniqueness, their existence, and the order of convergence.

Keywords: *Lagranges interpolation, Ultraspherical polynomial, Order of convergence.*

1. Introduction

L.G.Pal in 1975 introduced a new kind of interpolation process where a set of distinct points on zeroes of $W_n(x)$ were taken. Here the roots of $W_n'(x)$ were inter-scaled in between the roots of $W_n(x)$. Pal proved that there exists a unique polynomial having degree $2n-1$ for the given arbitrary numbers $(\alpha_{i,n})_{i=1}^{n}$ and $(\beta_{i,n})_{i=1}^{n-1}$. This polynomial satisfies two conditions. This method was applied by Szili[12] for the very first time on infinite interval. He used a mixed of Hermite polynomial and its derivative. His results were sharpened by I.Joo[4] by improvising on the estimates of the fundamental polynomial. In 1998, the problem of (0;0,1) interpolation on the mixed zeroes of the Hermite polynomial $H_n(x)$ and its derivative $H_n'(x)$ was studied by Srivastava and Mathur[6]. Then Lenard[9] studied the modified Pal type interpolation on the zeroes of Laguerre polynomials $L_n^{(k)}(x)$ and $L_n^{(k)\prime}(x)$ along with the Hermite type boundary conditions. (0,1;0) Interpolation problem along with some special type of boundary conditions were studied by Yamini Singh and R.Srivastava[17] in 2019 on the roots of Ultraspherical polynomial.

Ultraspherical polynomials can be described as the solution of the Ultraspherical differential equation with integer n and $\alpha \leq 1/2$. They can also be described as a special case of Jacobi polynomial with $\alpha = \beta$ The Pal type interpolation has been studied by various mathematicians like Xie[14][15], Zhou[16], Gopengaus[2], Joo[5], Hwang[3], Eneduanya[11], Szili[7][8]and many others. Chack, Sharma and Szadaboz[1] studied Pal type interpolation on the zeroes the Laguerre Polynomial. Also,the inverse Pal type interpolation was studied by Szili. Studies done by M.Lenard[10] include the Hermite conditions on the interval [-1,1].

In all of the above mentioned work, polynomials have been used as approximations to given function. One of the major reason for the importance of polynomials is that they are uniformly approximate continuous functions and this can be shown by the classical Weierstrass Theorem which is said to be the basis of Approximation Theory.

According to the Weierstrass theorem if f is defined and is continuous on $[a,b]$ and $\epsilon > 0$ is given,then there exists a polynomial p, defined on [a,b] with the property: $|f(x) - p(x)| < \epsilon$

In this paper we study the (0;0,1) interpolation problem with Hermite type boundary conditions on the interval [-1,1] for $P_{n-1}^{(k)\prime}(x)$ and $P_n^{(k+1)}(x)$ with $k = 1$

2. Problem

Let the set of knots be given as :

$$-1 = x_n < x_{n-1}^* < x_{n-1} < \dots\dots\dots\dots < x_1^* < x_1 < x_0^* < x_0 = 1 \tag{1}$$

where $\{x_i\}_{i=1}^{n-1}$ and $\{x_i^*\}_{i=1}^{n}$ are taken as the roots of $P_{n-1}^{(k)\prime}(x)$ and $P_n^{(k+1)}(x)$ which represent the Ultraspherical polynomials with k=1. On these given knots (1), there exists a unique polynomial $R_m(x)$. This $R_m(x)$ is of degree m which is at-most $3n+3$ and it satisfies the following interpolatory conditions:-

$$R_m(x_i) = y_i; (i = 1,2,\dots\dots n-1) \tag{2}$$

$$R_m(x_i^*) = y_i^*; (i = 1,2,\dots\dots n) \tag{3}$$

$$R_{m\prime}(x_i^*) = y_i^{*\prime}; (i = 1,2,\dots\dots\dots n) \tag{4}$$

with the boundary conditions

$$R_m^l(1) = y_1^l; (l = 0,1) \tag{5}$$

$$R_m^l(-1) = y_{-1}^l; (l = 0,1,2) \tag{6}$$

Here y_i, y_i^*, $y_i^{*\prime}$,y_1^l and y_{-1}^l are arbitrary real numbers. We have provided some results of [13] in the following consecutive sections, followed by some proofs of new results in the next section. The main theorem is also proved.

3. Preliminaries

Some well known results of the Ultraspherical Polynomials which are represented by $P_n^{(k)}(x)$ where $P_n^{(k)}(x) = P_n^{(k,k)}(x)$ $(k > -1, n \geq 0)$ are used. These polynomials satisfy the following properties:-

$$(1-x^2)P_n^{(k)\prime\prime}(x) - 2x(k+1)P_n^{(k)\prime}(x) + n(n+2k+1)P_n^{(k)}(x) = 0, \tag{7}$$

$$P_n^{(k)'}(x) = \frac{n+2k+1}{2} P_{n-1}^{(k+1)}(x), \tag{8}$$

$$|P_n^{(k)}(x)| = O(n^k), x \in [-1,1], \tag{9}$$

$$(1-x^2)^{\frac{k}{2}+\frac{1}{4}}\left|P_n^{(k)}(x)\right| = O\left(\frac{1}{\sqrt{n}}\right). \tag{10}$$

Here we also have some properties of fundamental polynomials of the Lagranges interpolation which are given as:-

$$l_j(x) = \frac{P_{n-1}^{(k)\prime}(x)}{P_{n-1}^{(k)\prime\prime}(x_j)(x-x_j)}, \tag{11}$$

$$l_j^*(x) = \frac{P_n^{(k+1)}(x)}{P_n^{(k+1)\prime}(x_j^*)\left(x-x_j^*\right)}. \tag{12}$$

Also we have:

$$l_j(x) = \frac{\tilde{h}_n^{(k)}}{\left(1-x_j^2\right)\left[P_n^{(k)\prime\prime}(x_j)\right]^2} \sum_{\nu=0}^{n-1} \frac{1}{h_\nu^{(1)}} P_\nu^{(k)\prime}\left(x_j\right) P_\nu^{(k)\prime}(x), \tag{13}$$

where

$$h_\nu^{(k)} = \frac{2^{2k+1}}{(2\nu+2k+1)} \frac{\Gamma(2(\nu+k+1))}{\Gamma(\nu+1)\Gamma(\nu+2k+1)} \begin{cases} \sim \frac{1}{\nu} & (\nu > 0) \\ = c_2 & (\nu = 0) \end{cases}, \tag{14}$$

here c_2 depends only on k.

Also $x_1 > x_2 >$...............x_n which are the roots of the Ultraspherical polynomial $P_n^{(k)}(x)$, satisfies the following conditions:

$$\left(1 - x_j^2\right) \sim \begin{cases} \frac{j^2}{n^2} & (x_j \geq 0) \\ \frac{(n-j)^2}{n^2} & (x_j < 0) \end{cases} \tag{15}$$

$$\left|P_n^{(k)\prime}(x_j)\right| \sim \begin{cases} \frac{n^{k+2}}{j^{k+3/2}} & (x_j \geq 0) \\ \frac{n^{k+2}}{(n-j)^{k+3/2}} & (x_j < 0) \end{cases} \tag{16}$$

4. Explicit Representation of Interpolatory Polynomial

We write $R_m(x)$ satisfying (2) to (6) as:

$$R_m(x) = \sum_{j=1}^{n-1} A_j(x) y_j + \sum_{j=1}^{n} B_j(x) y_j^* + \sum_{j=1}^{n} C_j(x) y_j^{*\prime} + \sum_{j=0}^{1} D_j(x) y_1^{(l)} + \sum_{j=0}^{2} E_j(x) y_{-1}^{(l)}, \tag{17}$$

Here $A_j(x)$ and $B_j(x)$ are the fundamental polynomials of first kind, $C_j(x)$ is the fundamental polynomial of second kind and $D_j(x)$ and $E_j(x)$ are the fundamental polynomials corresponding to the Hermite boundary conditions having degree at-most $3n + 3$. These are uniquely determined by the following conditions:

for $j = 1,2.\dots n - 1$

$$\begin{cases} A_j(x_i) = \delta_{ji}, & (i = 1,2,\dots,n-1) \\ A_j(x_i^*) = 0, & (i = 1,2,\dots,n) \\ A_j'(x_i^*) = 0, & (i = 1,2\dots,n) \\ A_j^l(1) = 0, & (l = 0,1) \\ A_j^l(-1) = 0, & (l = 0,1,2) \end{cases} \tag{18}$$

for $j = 1,2\dots n$

$$\begin{cases} B_j(x_i) = 0, & (i = 1,2,\dots,n-1) \\ B_j(x_i^*) = \delta_{ji}, & (i = 1,2,\dots,n) \\ B_j'(x_i^*) = 0, & (i = 1,2\dots,n) \\ B_j^l(1) = 0, & (l = 0,1) \\ B_j^l(-1) = 0, & (l = 0,1,2) \end{cases} \tag{19}$$

for $j = 1,2\dots n$

$$\begin{cases} C_j(x_i) = 0, & (i = 1,2,\dots,n-1) \\ C_j(x_i^*) = 0, & (i = 1,2,\dots,n) \\ C_j'(x_i^*) = \delta_{ji}, & (i = 1,2\dots,n) \\ C_j^l(1) = 0, & (l = 0,1) \\ C_j^l(-1) = 0, & (l = 0,1,2) \end{cases} \tag{20}$$

for $j = 0,1$

$$\begin{cases} D_j(x_i) = 0, & (i = 1,2,\dots,n-1) \\ D_j(x_i^*) = 0, & (i = 1,2,\dots,n) \\ D_j'(x_i^*) = 0, & (i = 1,2\dots,n) \\ D_j^l(1) = \delta_{ji}, & (l = 0,1) \\ D_j^l(-1) = 0, & (l = 0,1,2) \end{cases} \tag{21}$$

for $j = 0,1,2$

$$\begin{cases} E_j(x_i) = 0, & (i = 1,2,\dots,n-1) \\ E_j(x_i^*) = 0, & (i = 1,2,\dots,n) \\ E_j'(x_i^*) = 0, & (i = 1,2\dots,n) \\ E_j^l(1) = 0, & (l = 0,1) \\ E_j^l(-1) = \delta_{ji}, & (l = 0,1,2) \end{cases} \tag{22}$$

The explicit form has been given in the following lemmas:

Lemma 1 : $A_j(x)$, which is the fundamental polynomial for $j = 1,2 \ldots . n-1$, and satisfies the interpolatory conditions (18) is given by:

$$A_j(x) = \frac{[P_n^{(k+1)}(x)]^2 l_j(x)(1-x^2)^2 x}{[P_n^{(k+1)}(x_i)]^2(1-x_i^2)^2 x_i} \tag{23}$$

Lemma2: $B_j(x)$, which is the fundamental polynomial for $j = 1,2 \ldots . n$, satisfying the interpolatory conditions (19) is given by is given by:

$$B_j(x) = \frac{[P_{n-1}^{(k)'}(x)]^2 l_j^*(x)(1-x^2)^3}{(1-x_j^*)^2[P_{n-1}^{(k)'}(x_j^*)]^2} \tag{24}$$

Lemma3: $C_j(x)$, which is the fundamental polynomial for $j = 1,2 \ldots . n$, and satisfies the interpolatory conditions (20) is given by:

$$C_j(x) = \frac{P_n^{(k+1)'}(x_i^*)(x_i^{*2}-1)B_j(x)+P_{n-1}^{(k)'}(x)l_j^*(x)(1-x^2)}{P_{n-1}^{(k)''}(x_j^*)(1-x_j^{*2})-2x_j^* P_{n-1}^{(k)'}(x_j^*)+P_{n-1}^{(k)'}(x_j^*)l_j^{*'}(x_j^*)(1-x_j^{*2})} \tag{25}$$

Lemma4: $D_j(x)$, the fundamental polynomial for $j = 0,1$ corresponding to the boundary conditions satisfying the interpolatory condition (21) is given by:

$$D_j(u) = 3Q_x[P_{n-1}^{(k)'}]^2 P_n^{(k+1)}(x)(1-x^2)(1-x)^j \tag{26}$$

where degree of $Q(x)$ is 3-j

Lemma5: $E_j(x)$, the fundamental polynomial for $j = 0,1,2$ corresponding to the boundary conditions satisfying the interpolatory condition (22) is given by:

$$E_j(x) = P_{n-1}^{(k)'}(x)\left[P_n^{(k+1)}(x)\right]T(x)(1-x^2)^j[P_n^{(k+1)}(x)-2] \tag{27}$$

where degree of T(x) is $4-2j$.

For $j = 2$,

$$E_j(x) = -2x(k+1)!\,P_{n-1}^{(k)'}(x)P_n^{(k+1)}(x)[P_n^{(k+1)}(x)-2] \tag{28}$$

Existence: By Lemma 1 to Lemma 5, it can be clearly seen that the polynomial $R_m(x)$ satisfies the conditions (2) to (6). So there exists an interpolatory polynomial $R_m(x)$ of degree $3n+3$.

5. Order Of Convergence

Theorem1: For $C_j(x)$, its first derivative on$[-1,1]$ for $n \geq 2$, holds as:

$$\sum_{j=1}^{n} C_j'(x) = O(n^{9/2}) \tag{29}$$

Proof: Differentiation (25), we get

$$\sum_{j=1}^{n} C_j'(x) = \eta_1 + \eta_2, \tag{30}$$

now we have

$$\eta_1 \leq \sum_{j=1}^{n} \frac{|P_{n-1}^{(k)\prime}(x_i^*)|(x_i^{*2}-1)|B_j(x)|}{|P_{n-1}^{(k)\prime\prime}(x_j^*)|(1-x_j^{*2})-2x_j^*|P_{n-1}^{(k)\prime}(x_j^*)|+|P_{n-1}^{(k)\prime}(x_j^*)||l_j^{*\prime}(x_j^*)|(1-x_j^{*2})} \tag{31}$$

Similarly we get

$$\eta_2 \leq \sum_{j=1}^{n} \frac{|P_{n-1}^{(k)\prime}|(x)|l_j^*(x)|(1-x^2)}{|P_{n-1}^{(k)\prime\prime}(x_j^*)|(1-x_j^{*2})-2x_j^*|P_{n-1}^{(k)\prime}(x_j^*)|+|P_{n-1}^{(k)\prime}(x_j^*)||l_j^{*\prime}(x_j^*)|(1-x_j^{*2})} \tag{32}$$

by using (9), (10), (13), (15) and (16) , we get:

$$\eta_1 = O(n^{9/2})$$

Similarly, using the same process we can find the order of η_2 as:

$$\eta_2 = O(n^3)$$

Hence the given theorem is proved.

Theorem2: The first derivative of the second kind of fundamental polynomials on [-1,1] holds as:

$$\sum_{j=1}^{n-1} \left|(1-x^2)^2 B_j'(x)\right| = O\left(n^{k+\frac{7}{2}}\right). \tag{33}$$

Proof: Differentiating (24),we get

$$\sum_{j=1}^{n-1} \left|(1-x^2)^2 B_j'(x)\right| = \zeta_1 + \zeta_2 + \zeta_3, \tag{34}$$

$$B_j'(x) = \sum_{j=1}^{n} [-6x(1-x^2)[P_{n-1}^{(k)}{}'(x)]^2 l_j^*(x) + 2(1-x^2)^3 P_{n-1}^{(k)}{}'(x) P_{n-1}^{(k)}{}''(x) l_j^*(x) + (1-x^2)^3 [P_{n-1}^{(k)\prime}]^2(x)\, l_j^*(x)] / [P_{n-1}^{(k)}(x_i^*)]^2 (1-x_i^{*2})^3 \tag{35}$$

we get:

$$\zeta_1 \leq \sum_{j=1}^{n} \frac{-6x(1-x^2)|P_{n-1}^{(k)\prime}(x)|^2|l_j^*(x)|}{|P_{n-1}^{(k)\prime}(x_i^*)|^2|(1-x_i^{*2})|} \tag{36}$$

Now by using (9), (10), (13), (15) and (16) , we get:

$$\zeta_1 = O(n^{5/2})$$

Similarly we can find the orders of ζ_2 and ζ_3 as:

$$\zeta_2 = O(n^{5/2})$$

$$\zeta_3 = O(n^{7/2})$$

Hence the given theorem is proved.

Theorem3: The first kind of fundamental polynomials on [-1,1] holds as:

$$\sum_{j=1}^{n-1} \left|(1-x_i)^2 A_j(x)\right| = O\left(n^{9/2}\right). \quad (37)$$

Proof: From (23), we get

$$\sum_{j=1}^{n-1} \left|(1-x_i)^2 A_j(x)\right| = \epsilon_1 + \epsilon_2 + \epsilon_3, \quad (38)$$

where the decomposition (13) is used in for $l_j(x)$,we get:

$$|\varepsilon_1| \leq \frac{(1-x^2)l_j(x)[P_n^{(k)}(x)]^2[1-4x^2]}{[P_n^{(k)}(x_i)]^2(1-x_i^2)x_i} \quad (39)$$

and by using (9), (10), (13), (15) and (16) we get:

$$\epsilon_1 = O(n^{9/2})$$

Similarly we can find the orders of ϵ_2 and ϵ_3.

Hence the given theorem is proved.

6. Main Theorem

Let $m = 3n + 3$ and let $\{x_i\}_{i=1}^{n-1}$ and $\{x_i^*\}_{i=1}^{n}$ be the roots of the Ultraspherical polynomials $P_{n-1}^{(k)'}(x)$ and $P_n^{(k)}(x)$ respectively, with $k = 1$, if $f \in C^r[-1,1]$ $(r \geq k+1, n \geq 2r-k+2)$, then the interpolatory polynomial is given by:

$$R_m(x;f) = \sum_{i=1}^{n-1} f(x_i)A_i(x) + \sum_{i=1}^{n} f(x_i^*)B_i(x) + \sum_{i=1}^{n} f'(x_i^*)C_i(x) + \sum_{j=0}^{1} f^j(1)D_j(x) + \sum_{j=0}^{2} f^j(-1)E_j(x) \quad (40)$$

satisfies

$$|f'(x) - R'_m(x;f)| = w(f^{(r)};\tfrac{1}{n})O(n^{9/2-r}) \quad (41)$$

for $x \in [-1,1]$, where the fundamental polynomials $A_j(x)$, $B_j(x)$, $C_j(x)$, $D_j(x)$ and $E_j(x)$ are given in $(23)-(28)$

Proof: For $k = 0$ we refer to proof by Xie and Zhou [15] and we prove the case for $k \geq 1$. Let $f \in C^r[-1,1]$, then by the theorem of Gopengauz [2] for every $m > 4r + 5$ there exists a polynomial $p_m(x)$ of degree at most m such that for $j = 0 \ldots r$

$$|f^{(j)}(x) - p_m^{(j)}(x)| \leq M_{r,j}(\frac{\sqrt{1-x^2}}{m})^{r-j} w(f^{(r)}; \frac{\sqrt{1-x^2}}{m}) \tag{42}$$

where $w(f^{(r)}; .)$ denotes the modulus of continuity of the function $f^{(r)}(x)$ and the constants $M_{r,j}$ depend only on r and j. Further,

$$f^{(j)}(\pm 1) = p_m^{(j)}(\pm 1) \quad (j = 0 \ldots r).$$

By the uniqueness of the interpolational polynomials $R_m(x; f)$,

it is clear that $R_m(x; p_m) = p_m(x)$.

Hence for $x \epsilon [-1,1]$

$$| f'(x) - R'_m(x ; f) | \leq | f'(x) - p_m'(x) | + | R'_m(x ; p_m) - R'_m(x ; f) |$$

$$\leq | f'(x) - p'_m(x) | + \sum_{i=1}^{n-1} | f(x_j) - p_m(x_j) | \, | A'_j(x) | + \sum_{i=1}^{n} | f(x_j^*) - p_m(x_j^*) | \, |B'_j(x)|$$

$$+ \sum_{i=1}^{n} |f'(x_j^*) - p'_m(x_j^*)| \, |C'_j(x)|$$

using (40) and (42), applying the estimates (29), (33) and (37), we obtain

$$|f'(x) - R'_m(x; f)| = w(f^{(r)}; \frac{1}{n}) O(n^{9/2-r})$$

which is the proof of main theorem.

By using this main theorem that has been mentioned above, we can give the conclusion of the convergence theorem.

7. Conclusion

Let $\{x_i\}$ and $\{x_i^*\}$ be the roots of the Ultraspherical polynomial $P_{n-1}^{(k)'}(x)$ and $P_n^{(k+1)}(x)$ respectively with $k = 1$. If $f \in C^{k+2}$ [-1,1], $f^{k+2} \in Lipa$, $\alpha > \frac{1}{2}$, then $R_m(x; f)$ and $R'_m(x; f)$ uniformly converge to $f(x)$ and $f'(x)$, respectively on $[-1,1]$ as $n \to \infty$ where $m = 3n + 3$. Then there exists a polynomial $R_m(x)$ which satisfies conditions (2) to (6) and is our required polynomial.

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