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Toeplitz Determinants of Bi-Starlike and Bi-Convex Functions of Order β and their geometric behavior

Chinthamani. S[1*], P. Lokesh[2]

[1*] Department of Mathematics, Stella Maris College (Autonomous), Chennai, TN – 600086

[2] Department of Mathematics, AdhiParasakthi College of Engineering, Kalavai, TN – 632506

*Corresponding Author : chinthamani@stellamariscollege.edu.in

Abstract A regular function $f(z)$ defined in $\mathfrak{D}: |z| < 1$ is said to be univalent in $\mathfrak{D}$, if it assumes no value more than once in it. But $f(z)$ becomes Bi-univalent function in $\mathfrak{D}$ if $f(z)$ and its inverse are univalent in $\mathfrak{D}$. These Bi-univalent functions are of great interest in recent research of Geometric Function Theory. The main focus of this work is to bring out the upper bound of the symmetric Toeplitz determinants $T_2(2)$ and $T_3(1)$ for the classes $\mathcal{S}_\sigma{}^*(\beta)$ and $\mathcal{K}_\sigma(\beta)$. The novelty of this work lies in analyzing their geometric behavior. MSC 2020: Primary 30C45, Secondary 11C20

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1 Introduction

Let $\mathcal{S}$ indicate the class of $f(z)$ which takes the power series form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

$z = x + iy,$ which convergent in $\mathfrak{D}: |z| < 1$. See [2, 8, 9, 11] for brief history of univalent and analytic functions. The Koebe $\frac{1}{4}$ theorem [4] guarantees that the image of the unit circular disk $\mathfrak{D}$ under every $f(z) \in S$ contains a disk of radius 1/4.

So, every $f \in S$ has inverse $f^{-1}(z)$ such that $f^{-1}f\,(z) = z,\ |z| < 1$ and

$f(f^{-1}(w)) = w\ (|w| < r_0(f),\ r_0(f) \geq \frac{1}{4})$ where the inverse function is defined to be

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots .$$

The function $f(z)$ is said to belong to σ, the class of bi-univalent functions, if and only if (i) $f(z)$ belongs to the class S and (ii) there exists a function $g(z)$ such that $f(g(z)) = g(f(z)) = z$ in some neighborhood of the origin. The functions $e^z - 1,\ \frac{z}{(1-z)}$ are some examples for bi-univalent functions in $\mathfrak{D}$ but the amicable Koebe function $\frac{z}{(1-z)^2}$ is not a bi-univalent function.

Lewin. M [13] pioneered the estimation of coefficient bounds for bi-univalent functions in 1967. H.M Srivastava et al. in 2010 (see [18]) renewed the investigation in this direction. Seminal work on the class σ can be found in the articles [12, 13, 17]. Among univalent functions, the most popular subclasses are $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ $(0 \leq \beta < 1)$, the classes of starlike and convex functions of order β. By definition we have,

$$S^*(\beta) = \left\{ f \in S : R\left(\frac{zf'(z)}{f(z)}\right) > \beta; |z| < 1; 0 \leq \beta < 1 \right\}$$

$$\mathcal{K}(\beta) = \left\{ f \in S : R\left(1 + \frac{zf''(z)}{f(z)}\right) > \beta; |z| < 1; 0 \leq \beta < 1 \right\}$$

And we get $\mathcal{S}^* = \mathcal{S}^*(0)$. When $\beta = 0$, $S^*(\beta)$ and $\mathcal{K}(\beta)$ become S^* and $\mathcal{K}$ respectively, which are the familiar classes of starlike and convex functions of order 0.

The classes $\mathcal{S}_\sigma^*(\beta)$ and $\mathcal{K}_\sigma(\beta)$ are Bi-starlike and Bi-convex of order β $(0 \leq \beta < 1)$ if $f \in \sigma$, which were introduced by Brannan and Taha [3] in 1985. Especially the classes $\mathcal{S}_\sigma^*(0) = \mathcal{S}_\sigma$, $\mathcal{K}_\sigma(0) = \mathcal{K}_\sigma$ are Bi-starlike and Bi-convex classes reduced to zero order. See [6] for synopsis of Bi-univalent functions.

Deniz et al.[6] obtained the second hankel determinants for these two classes in 2015 and Chinthamani et al. [5] extended the results to find third hankel determinants in 2017. This work was inspired by those works and is developed on estimating the upper bounds of the Toeplitz determinants for these two notorious bi-univalent classes.

The estimation of the bounds of Hankel matrices has received considerable attention in geometric function theory and have numerous applications. Hankel determinants $H_q(n)$ are more related to Toeplitz determinants. For exploring interesting applications of the Toeplitz determinants, see [1].

The Hankel determinants $H_q(n)$ $(n > 0, \ldots q > 0, \ldots)$ of $f(z)$ are defined in [21] as

$H_q(n) = [a_n\ a_{n+1}\ \ldots\ldots\ldots\ldots\ldots a_{n+q-1}\ a_{n+1}\ a_{n+2}\ \ldots\ldots\ldots\ldots\ldots\ldots a_{n+q}\ \vdots\ a_{n+q-1}\ \vdots\ a_{n+q}\ \vdots\ \ldots\ldots\ldots\ldots. a_{n+2q-2}\ ] \qquad (a_1 = 1)$

The symmetric determinants $T_q(n)$ are introduced by Thomas and Halim in [19] and defined as

$T_q(n) = |a_n\ a_{n+1}\ \ldots\ldots\ldots\ldots a_{n+q-1}\ a_{n+1}\ a_n\ \ldots\ldots\ldots\ldots a_{n+q}\ \vdots\ a_{n+q-1}\ \vdots\ a_{n+q}\ \vdots\ \ldots\ldots\ldots\ldots. a_n\ | \qquad (q \in N\backslash 1,\ n \in N) \qquad (1.2)$

The Toeplitz determinants study has caught the interest of many researchers in recent days (see [1, 15, 19, 20]). The Toeplitz determinants are involving coefficients a_n of the class of functions. The special case of $T_q(n)$ by taking $n = 2, q = 2$, leads to

$$T_2(2) = |a_2\ a_3\ a_3\ a_2\ |$$

and $n = 1, q = 3$, gives, $T_3(1) = |a_1\ a_2\ a_3\ a_2\ a_1\ a_2\ a_3\ a_2\ a_1\ |$ and so on.

The heart of this work is estimating the best bound of the symmetric Toeplitz determinants $T_2(2)$ and $T_3(1)$ for the classes $\mathcal{S}_\sigma^*(\beta)$ and $\mathcal{K}_\sigma(\beta)$ which are not often taken for consideration in the literature.

Let P be the class of functions with positive real part consisting of all analytic functions p mapping unit disk $|z| < 1$ to the complex plane satisfying the conditions $p(0) = 1$ and $Re\, p(z) > 0$.

Lemma 1.1. [14] for the functions $p \epsilon P$, represented by the expression

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (1.3)$$

We have the sharp result $|c_k| \leq 2$ $(k = 1,2, \ldots)$.

Lemma 1.2. [10] for the functions $p \epsilon P$, given by the series (1.3), then

$$2c_2 = c_1^2 + x(4 - c_1^2) \qquad (1.4)$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \qquad (1.5)$$

where $|x| \leq 1$ and $|z| \leq 1$.

2 Main Results

Treatment for Bi-Starlike Class of functions:

Theorem 2.1. If $f(z) \in \mathcal{S}_\sigma{*}(\beta)$, $0 \leq \beta < 1$, then upper bound of Toeplitz determinant $T_2(2)$ is $\frac{(1-\beta)^2}{8}$.

Proof. Let $f \epsilon \mathcal{S}_\sigma{*}(\beta)$ and $g = f^{-1}$. Then,

$$\frac{zf'(z)}{f(z)} = \beta + (1-\beta)p(z) \qquad (2.1)$$

and $\frac{wg'(w)}{g(w)} = \beta + (1-\beta)g(w)$ (2.2)

Where $p(z) = 1 + c_1z + c_2z^2 + \cdots$ $g(w) = 1 + d_1w + d_2w^2 + \cdots$ *in* $\mathcal{P}$.

As per the treatment done by chinthamani et al. in [5], we obtain the following relations and coefficients

$$c_1 = -d_1 \qquad (2.3)$$

$$a_2 = (1-\beta)c_1 \qquad (2.4)$$

$$a_3 = (1-\beta)^2c_I^2 + \frac{(1-\beta)}{4}(c_2 - d_2) \qquad (2.5)$$

Since $T_2(2) = (a_3^2 - a_2^2)$, using (2.4) and (2.5), we can establish that

$$|a_3^2 - a_2^2| = |(1-\beta)^4c_1^4 + \frac{(1-\beta)^3}{2}c_1^2(c_2 - d_2) + \frac{(1-\beta)^2}{16}(c_2 - d_2)^2 - (1-\beta)^2c_1^2| \qquad (2.6)$$

By using lemma (1.2) and (2.2), we derive

$$2c_2 = c_1^2 + x(4 - c_1^2) \quad 2d_2 = d_1^2 + x(4 - d_1^2)\,\} \qquad (2.7)$$

implies $\qquad c_2 - d_2 = \frac{(4-c_1^2)}{2}(x - y)$ (2.8)

for some x, y with $|x| \leq 1$, $|y| \leq 1$.

Substituting (2.8) in (2.6), we obtain,

$$|a_3^2 - a_2^2| \leq (1-\beta)^4c_1^4 + \frac{(1-\beta)^3}{4}c_1^2(4 - c_1^2)(|x| + |y|)$$
$$+(1-\beta)^2[\frac{1}{64}(4 - c_1^2)(|x|^2 + |y|^2) + c_1^2] \qquad (2.9)$$

Since $p \epsilon \mathcal{P}$, so $|c_1| \leq 2$. Assuming $c_1 = c$ and $c \epsilon [0,2]$ without restriction. Thus, for $\lambda = |x| \leq 1$ and $\mu = |y| \leq 1$, we get,

$$|a_3^2 - a_2^2| \leq F(\lambda, \mu)$$

where

$$F(\lambda, \mu) = (1-\beta)^4c^4 + (1-\beta)^2c^2 + \frac{(1-\beta)^3}{4}c^2(4 - c^2)(\lambda + \mu)$$

$$+(1-\beta)^2[\frac{1}{64}(4-c^2)(\lambda^2+\mu^2)] \quad (2.10)$$

Let $F(\lambda,\mu) = t_1 + t_2(\lambda+\mu) + t_3(\lambda^2+\mu^2)$ (2.11)

here $t_1 = (1-\beta)^4c^4 + (1-\beta)^2c^2 \geq 0$

$$t_2 = \frac{(1-\beta)^3}{4}c^2(4-c^2) \geq 0$$

$$t_3 = \frac{(1-\beta)^2}{64}(4-c^2) \geq 0$$

Let the closed square S = $\{(\lambda,\mu) : 0 \leq \lambda \leq 1 ;\ 0 \leq \mu \leq 1\}$.

Now, we investigate the maximum of $F(\lambda,\mu)$ in S :

Case (i): For $\lambda = 0$, $0 \leq \mu < 1$, we arrive at

$$F(0,\mu) = U(\mu) = t_1 + t_2\mu + t_3\mu^2$$

In this case, for $0 < \mu < 1$, it is clear that $U'(\mu) = t_2 + 2t_3\mu > 0$ that is U(μ) is an increasing function. Hence the maximum occurs at μ =1.

$$Max\ U(\mu) = U(1) = t_1 + t_2 + t_3$$

Case (ii): Now, for $\lambda = 1$ and $0 \leq \mu < 1$, we obtain

$$F(1,\mu) = V(\mu) = t_1 + t_2 + t_3 + t_2\mu + t_3\mu^2$$

Similarly, it is clear that $V'(\mu) = t_1 + t_2 + t_3 + t_2\mu + 2t_3\mu > 0$. This establishes that it is an increasing function and it attains its maximum at μ =1.

$$max\ V(\mu) = V(1) = t_1 + 2(t_2 + t_3)$$

Since $U(1) \leq V(1)$ for $c \in [0,2], F(\lambda,\mu) = F(1,1)$ is on the boundary of S.

Thus, F attains its maximum at $\lambda = 1$, $\mu = 1$ on S.

To find Critical points, let $\phi[0,2] \to R$

$\phi(c) = (\lambda,\mu) = F(1,1) = t_1 + 2(t_2 + t_3)$ (2.12)

Inputting t_1, t_2 and t_3 values in $\phi(c)$, yield

$$\phi(c) = \frac{(1-2\beta)}{2}(1-\beta)^3c^4 + (1-\beta)^2\frac{(95-64\beta)}{32}c^2 + \frac{(1-\beta)^2}{8}$$

Assume that $\phi(c)$ reaches its maximum at an interior point $c\ \epsilon$ [0, 2], then by applying first derivative test, we find $\phi'(c) \geq 0$ for c ϵ [0, 2). Hence $\phi'(c) = 0$ implies the critical points $c = 0$ and

$c = \sqrt{\frac{-(95-64\beta)}{16(1-2\beta)(1-\beta)^2}}$.

By applying the second derivative test on the function ϕ(c), we come to know that the function $\phi(c)$ does not attain maximum at the critical point $c = \sqrt{\frac{-(95-64\beta)}{16(1-2\beta)(1-\beta)^2}}$ and hence in the interior of the interval [0, 2] for $\beta\ \epsilon$ [0,1). Since $\phi''(c) < 0$, the maximum points of $\phi(c)$ is $c = 0$. So, we get,

$$\phi(c) = \phi(0) = \frac{(1-\beta)^2}{8}$$

Hence $|T_2(2)| \leq \frac{(1-\beta)^2}{8}$. □

Note. It is evident that the case i) $\mu = 0$, $0 \leq \lambda < 1$ and case ii) $\mu = 1$, $0 \leq \lambda < 1$ can be similarly discussed and obtained the same result.

Remark 2.2. If $f(z)$ is in the class $\mathcal{S}_\sigma$*, we obtain $|T_2(2)| \leq \frac{1}{8}$.

Theorem 2.3. If $f(z) \in \mathcal{S}_\sigma$*$(\beta)$, $0 \leq \beta < 1$, then upper bound of Toeplitz determinant $T_3(1)$ is $1 + \frac{(1-\beta)^2}{2}\frac{[\gamma(\beta)+1518]}{\gamma(\beta)}$ where $\gamma(\beta) = [32(1-\beta)^2 + 1]$.

Proof: To reach the goal, we substitute (2.3) and (2.12) in Toeplitz determinant

$T_3(1) = (1 + 2a_2^2(a_3 - 1) - a_3^2)$, After simplifications as

$$1 + (1-\beta)^4c^4 + 2(1-\beta)^2c^2 + \frac{(1-\beta)^2}{64}(4-c^2)^2(\lambda^2 + \mu^2) = G(\lambda,\mu) \text{ (say)}$$

To check maxima of $F(\lambda,\mu)$: After a careful application of second derivative test, we see $G_{\lambda\lambda}G_{\mu u} - G_{\lambda\mu} > 0\ \forall\, c \in [0,2]$ and $G(\lambda,\mu)$ attains maximum on the boundary of S = $\{(\lambda,\mu) : 0 \leq \lambda \leq 1\,;\ 0 \leq \mu \leq 1\}$. Hence, $\lambda = 1, \mu = 1$ implies

$$G(1,1) = (1-\beta)^2\left[(1-\beta)^2 + \frac{1}{32}\right]c^4 + \frac{7}{4}(1-\beta)^2c^2 + \frac{(1-\beta)^2}{2} + 1 \quad (2.13)$$

$$= \aleph(c) \text{ (say)}$$

Taking $\aleph'(c) = 0 \Rightarrow$ critical point $c_0^2 = \frac{-112}{[32(1-\beta)^2+1]}$ at which $\aleph(c)$ attains maximum. The maximum value of $\aleph(c)$ is

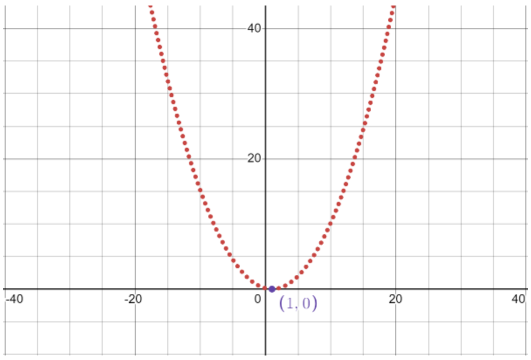
$$\aleph(c_0) = 1 + \frac{(1-\beta)^2}{2} + \frac{49}{8}\frac{(1-\beta)^2}{\left[(1-\beta)^2 + \frac{1}{32}\right]}$$

$$\aleph(c_0) = 1 + \frac{(1-\beta)^2[32(1-\beta)^2 + 393]}{2[32(1-\beta)^2 + 1]}$$

Hence the required result is established. □

Note. It is evident that the case i) $\mu = 0$, $0 \leq \lambda < 1$) and case ii) $\mu = 1$, $0 \leq \lambda < 1$ can be similarly discussed and obtained the same result.

Remark 2.4. If $f(z)$ is in the class $\mathcal{S}_\sigma$*, then $|T_3(1)| \leq \frac{131}{93}$, $0 \leq \beta < 1$.



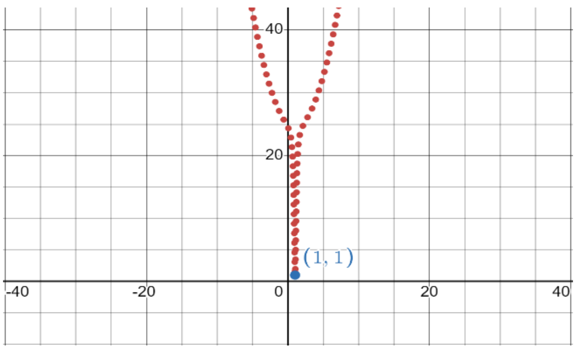


Fig 1 shows max of $T_2(2)$, max of $T_3(1)$ for Bi-Starlike functions respectively

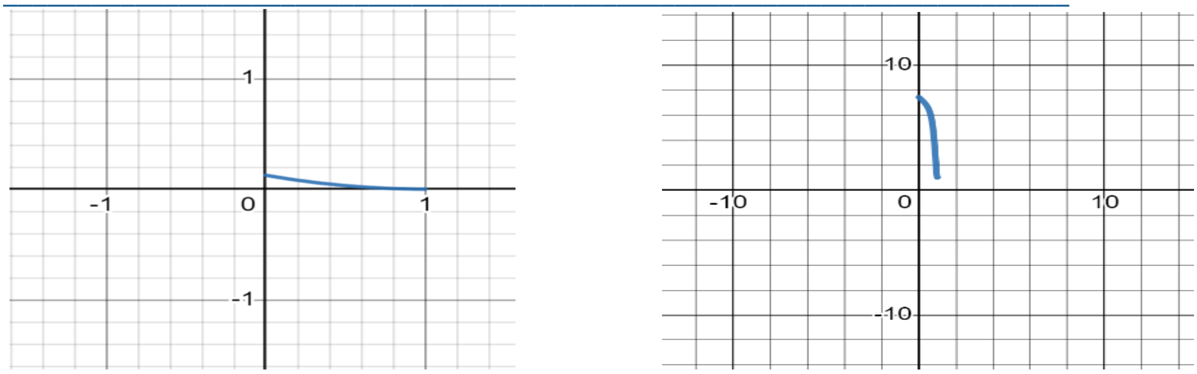


Fig 2 shows max of $T_2(2)$ and $T_3(1)$ for $\mathcal{S}_\sigma*(\beta)$ respectively with the range restriction $0 \leq \beta < 1$.

Treatment for Bi-Convex class of functions

Theorem 2.5. If $f(z) \in \mathcal{K}_\sigma(\beta)$, $0 \leq \beta < 1$. Then upper bound of Toeplitz determinant $T_2(2)$ is $\frac{(1-\beta)^2}{18}(18\beta^2 - 39\beta + 22)$.

Proof: Given $f \in \mathcal{K}_\sigma(\beta)$ and $g = f^{-1}$ then we make use of the treatment done by chinthamani et al. in [5], we obtain the following relations and coefficients

$$a_2 = \frac{(1-\beta)}{2} c_1 \tag{2.14}$$

$$a_3 = \frac{(1-\beta)^2}{4} c_I^2 + \frac{(1-\beta)}{12}(c_2 - d_2) \tag{2.15}$$

Substituting (2.14), (2.15) and (2.8) in $T_2(2) = (a_3^2 - a_2^2)$, and applying triangle inequality subsequently, we obtain,

$$|a_3^2 - a_2^2| \leq \frac{(1-\beta)^4}{16} c_1^2 + \frac{(1-\beta)^3}{16} c_1^2 (4 - c_1^2)(|x| + |y|)$$

$$+(1-\beta)^2 \frac{1}{576}(4 - c_1^2)^2(|x|^2 + |y|^2) + \frac{(1-\beta)^2}{4} c_1^2 \tag{2.16}$$

Since $p \in \mathcal{P}$, so $|c_1| \leq 2$. Assuming $c_1 = c$ and $c \in [0, 2]$. Thus, for $\lambda = |x| \leq 1$ and $\mu = |y| \leq 1$, we get

$|a_3^2 - a_2^2| \leq H(\lambda, \mu)$ where

$$H(\lambda, \mu) = \frac{(1-\beta)^4}{16} c^4 + \frac{(1-\beta)^2}{4} c^2 + \frac{(1-\beta)^3}{16} c^2 (4 - c^2)(\lambda + \mu)$$

$$+(1-\beta)^2 [\frac{1}{576}(4 - c^2)^2(\lambda^2 + \mu^2)] \tag{2.17}$$

$$\text{Let } H(\lambda, \mu) = m_1 + m_2(\lambda + \mu) + m_3(\lambda^2 + \mu^2) \tag{2.18}$$

here $m_1 = \frac{(1-\beta)^4}{16} c^4 + \frac{(1-\beta)^2}{4} c^2 \geq 0$

$m_2 = \frac{(1-\beta)^3}{16} c^2 (4 - c^2) \geq 0$

$m_3 = \frac{(1-\beta)^2}{576}(4 - c^2)^2 \geq 0$

To Maximize $H(\lambda, \mu)$:

we investigate the maximum of $H(\lambda, \mu)$ in S = $\{(\lambda, \mu) : 0 \leq \lambda \leq 1 ; 0 \leq \mu \leq 1\}$:

Case (i): For $\lambda = 0$, $0 \leq \mu < 1$, we arrive at

$$H(0,\mu) = U(\mu) = m_1 + m_2\mu + m_3\mu^2$$

In this case, for $0 \leq \mu < 1$, it is clear that $U'(\mu) = m_2 + 2m_3\mu > 0$

This implies that the maximum occurs for the increasing function U(μ) at μ =1.

$$Max\ U(\mu) = U(1) = m_1 + m_2 + m_3$$

Case (ii): Now, for $\lambda = 1$ and $0 \leq \mu < 1$, we obtain

$$H(1,\mu) = V(\mu) = m_1 + m_2 + m_3 + m_2\mu + m_3\mu^2$$

Similarly, it is clear that $V'(\mu) = m_1 + m_2 + m_3 + m_2\mu + 2m_3\mu > 0$

It is obvious to have the maximum value at μ =1 as it is an increasing function.

$max\ V(\mu) = V(1) = m_1 + 2(m_2 + m_3)$

Since $U(1) \leq V(1)$ for $c \in [0,2], H(\lambda,\mu) = H(1,1)$ is on the boundary of S.

Thus, the maximum of F occurs at $\lambda = 1$, $\mu = 1$ on S.

CRITICAL POINTS: Let $\tau : [0,2] \to R$, then

$$\tau(c) = max\, H(\lambda,\mu) = H(1,1) = m_1 + 2(m_2 + m_3) \quad (2.19)$$

Inputting T_1, T_2 and T_3 values in τ (c) yield

$$\tau(c) = \frac{(1-\beta)^4}{16}c^4 + \frac{(1-\beta)^2}{4}c^2 + \frac{(1-\beta)^3}{24}c^2(4-c^2) + (1-\beta)^2\frac{1}{288}(4-c^2)^2 \quad (2.20)$$

A detailed investigation on maximum values for T_1, T_2, T_3 separately yields

$$\tau(c) = \frac{(1-\beta)^2}{18}(18\beta^2 - 39\beta + 22)$$ □

Note. It is obvious that the case i) $\mu = 0$, $0 \leq \lambda < 1$) and case ii) $\mu = 1$, $0 \leq \lambda < 1$ can also be similarly discussed and obtained the same result.

Remark 2.6. If $f(z)$ is in the class $\mathcal{K}_\sigma$, then $|T_2(2)| \leq \frac{11}{9}$.

Theorem 2.7. If $f(z) \in$ class $\mathcal{K}_\sigma(\beta)$, $0 \leq \beta < 1$, then upper bound of Toeplitz determinant $|T_3(1)|$ is $1 + \frac{(1-\beta)^2(\delta(\beta)+1734)}{18\,\delta(\beta)}$ where $\delta(\beta) = [36(1-\beta)^2 + 1]$.

Proof : With the knowledge of $c_1 = -d_1, a_2 = \frac{1}{2}(1-\beta)c_1$,

$a_3 = \frac{1}{4}(1-\beta)^2{c_1}^2 + \frac{1}{12}(1-\beta)(c_2 - d_2)$ as obtained by the authors in [28] for Bi-convex functions of order β, and applying in

$T_3(1) = 1 + 2a_2^2(a_3 - 1) - a_3^2)$, we get

$$1 + 2a_2^2(a_3-1) - a_3^2) = 1 + \frac{(1-\beta)^2}{2}c_1^2[(\frac{1}{4}(1-\beta)^2c_1^2 + \frac{1}{12}(1-\beta)(c_2-d_2) - 1]$$

$$-\left[\frac{1}{4}(1-\beta)^2c_1^2 + \frac{1}{12}(1-\beta)(c_2-d_2)\right]^2 \quad (2.21)$$

Using (2.8) in above equation, we obtain

$$1 + 2a_2^2(a_3-1) - a_3^2)$$

$$= 1 - \frac{(1-\beta)^4}{8}c^4 - \frac{1}{2}(1-\beta)^2c^2 - \frac{1}{144}(1-\beta)^2\left[\frac{(4-c_1^2)^2}{4}(x-y)^2\right] \quad (2.22)$$

For some x, y with $|x| \leq 1, |y| \leq 1$. Taking triangle inequality implies

$$|T_3(1)| \leq 1 + \frac{(1-\beta)^4}{8}c^4 + \frac{1}{2}(1-\beta)^2c^2 + \frac{1}{144}(1-\beta)^2(4-c^2)^2(|x|^2+|y|^2)$$

Since $p \in \mathcal{P}$, so $|c_1| \leq 2$. Assuming $c_1 = c$ and $c \in [0, 2]$ without restriction.

Thus, for $\lambda = |x| \leq 1$ and $\mu = |y| \leq 1$, we get

$|T_3(1)| \leq J(\lambda, \mu)$ where

$$J(\lambda,\mu) = 1 + \frac{(1-\beta)^4}{8}c^4 + \frac{(1-\beta)^2}{2}c^2 + \frac{(1-\beta)^2}{576}(4-c^2)^2(\lambda^2+\mu^2) \quad (2.23)$$

After a careful application of second derivative test as followed in theorem 2.5, we see that $J(\lambda,\mu)$ attains maximum at the boundary of S = $\{(\lambda,\mu) : 0 \leq \lambda \leq 1\,;\ 0 \leq \mu \leq 1\}$. Hence, $\lambda = 1, \mu = 1$ implies $J(1,1) = \vartheta(c)$

$$\vartheta(c) = 1 + \frac{(1-\beta)^4}{8}c^4 + \frac{1}{2}(1-\beta)^2c^2 + \frac{1}{288}(1-\beta)^2(4-c^2)^2 \quad (2.24)$$

$$\vartheta(c) = 1 + \left(\frac{(1-\beta)^4}{8} + \frac{(1-\beta)^2}{288}\right)c^4 + \left(\frac{(1-\beta)^2}{2} - \frac{(1-\beta)^2}{36}\right)c^2 + \frac{16(1-\beta)^2}{288}$$

$$\vartheta(c) = 1 + \left[(1-\beta)^2 + \frac{1}{36}\right]\frac{(1-\beta)^2}{8}c^4 + \frac{17}{36}(1-\beta)^2c^2 + \frac{(1-\beta)^2}{288}$$

To find the maxima of $\vartheta(c)$, we apply the derivative test to $\vartheta(c)$, we arrive

$$\vartheta'(c) = 0 \Rightarrow c = 0 \ or\ c_0^2 = \frac{-17}{9\left[(1-\beta)^2 + \frac{1}{36}\right]} = \frac{-68}{[36(1-\beta)^2+1]}$$

These are the critical points. At c_0, we get

$$\vartheta''(c_0) = -\frac{36}{18}(1-\beta)^2 < 0$$

Hence $\vartheta(c)$ attains maximum at this critical point.

$$\vartheta(c_0) = 1 + \frac{289}{3}\frac{(1-\beta)^2}{[36(1-\beta)^2+1]} + \frac{(1-\beta)^2}{18}$$

$$\Rightarrow \vartheta(c_0) = 1 + \frac{(1-\beta)^2[36(1-\beta)^2+1735]}{18\,[36(1-\beta)^2+1]}$$

This establishes the result. □

Remark 2.8. If $f(z)$ is in the class $\mathcal{K}_\sigma$, then $|T_3(1)| \leq 3.659159159\ldots$

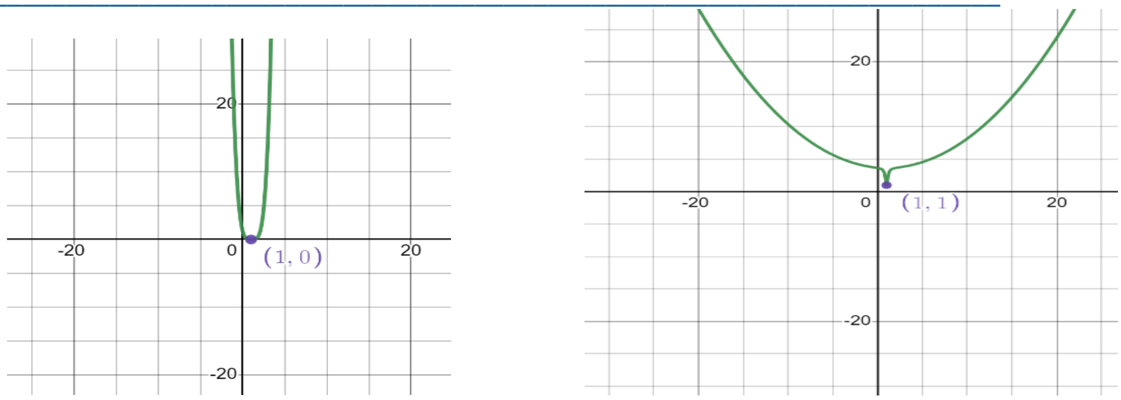


Fig 3 shows max of $T_2(2)$, max of $T_3(1)$ for Bi-Convex functions respectively

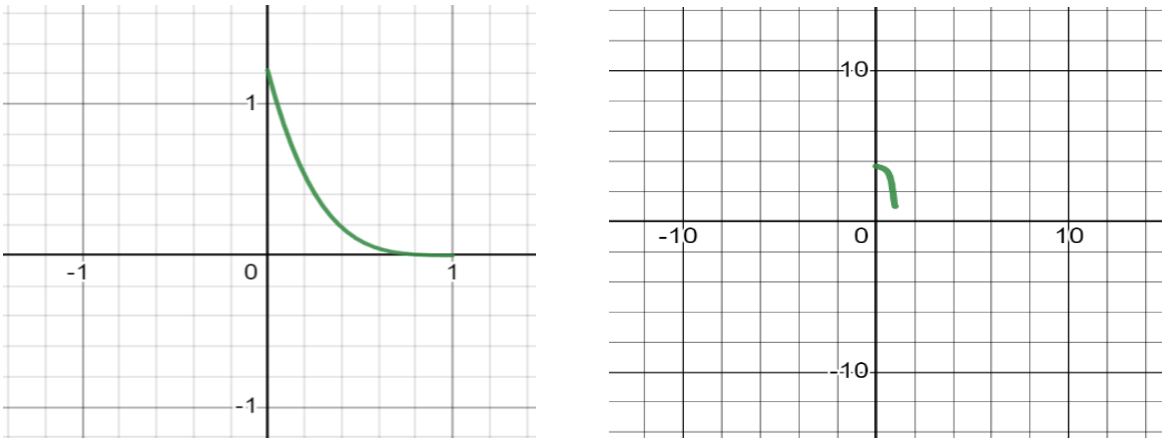


Fig 4 shows max of $T_2(2)$ and $T_3(1)$ for $\mathcal{K}_\sigma(\beta)$ respectively with the range restriction $0 \leq \beta < 1$.

3. Future work

In this work, the upper bounds of $T_2(2)$ and $T_3(1)$ are arrived. Similar extension to any higher order can be done. Since non-sharp bounds are obtained here, the problem remains open for further exploration by emerging researchers to unlock the sharper bounds.

References

[1] Ayinla, Rasheed O Bello and A. Risikat,Toeplitz Determinants for a Subclass of Analytic Functions, Journal of Progressive Research in Mathematics, Vol 18,1 - 2021 — E-ISSN: 2395-0218. URL: http://scitecresearch.com/journals/index.php/jprm/article/view/2034

[2] K.O.Babalola, On $H_3(1)$ Hankel determinant for some classes of univalent functions, Inequality theory and Applications, Nova Science Publishers, 6 (2010), 1-7. URL:

https://uilspace.unilorin.edu.ng/bitstream/handle/20.500.12484/1886/Paper%2026.pdf?sequence=1

[3] D.A.Brannan and T.S. Taha, On some classes of bi-univalent functions in S.M. Mazhar, A. Hamoui, N.S. Faour (Eds.) Math Anal and Appl. Kuwait; February 18-21, 1985 in : KFAS Proceedings Series Vol. 3, Pergamon Press, Elsevier Science Limited, Oxford, 1988, pp. 53{60. See also studia Univ. Babes Bolyai Math. 31(2) (1986), 70-77.

[4] D.G.Cantor, Power series with integral coefficients, Bull. Amer. Math. Soc., 69 (1963), 362-366. URL: https://www.ams.org/journals/bull/1963-69-03/S0002-9904-1963-10923-4/

[5] S. Chinthamani A. Gangadharan and R.Muthucumarasamy, Third Hankel Determinant for Bi-starlike and Bi-convex functions of order β, Journal of Computational and Theoretical Nanoscience, Vol 14(4), 2052-2058, 2017, DOI: 10.1166/jctn.2017.6541

[6] E. Deniz, M. Caglar and H. Orhan, Second Hankel Determinant for Bi-Starlike and Bi-Convex functions of order β ,Applied Mathematics and computation. Sep,2015. URL: https://www.researchgate.net/publication/270572059_Second_Hankel_determinant_for_bi-starlike_and_bi-convex_functions_of_order_b

[7] P.L. Duren, Univalent functions, (Springer- Verlag), 259, Springer, New York, 1983. URL: https://books.google.co.in/books?hl=en&lr=&id=8mVIaZmsy0gC&oi=fnd&pg=PA1&ots=mlklehPevy&sig=YQFCICaSjKwdbUuoMC9e0UDPlE8&redir_esc=y#v=onepage&q&f=false

[8] M. Fekete and G. Szego, Eine Bemerkung uber ungerade Schlicht Functionen, J. London Math. Soc., 8 (1933), 85-89. https://doi.org/10.1112/jlms/s1-8.2.85

[9] A.W. Goodman, Univalent functions, Vol. I, Mariner Publishing Co. Inc., 1983. https://worldcat.org/en/title/924796664

[10] U. Grenander and G. Szegom Toeplitz forms and their applications, Second edition, Chelsea Publishing Co., New York, (1984). https://books.google.co.in/books?id=5SFsiYoLY14C&lpg=PP1&pg=PP1#v=onepage&q&f=false

[11] F.R. Koegh and E.P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc., 20 (1969), 8-12. https://doi.org/10.1090/S0002-9939-1969-0232926-9

[12] S.S. Kumar, V. Kumar and V. Ravichandran, Estimates for the initial coefficicient of bi-univalent functions, Tamsui Oxford Journal of Information and Mathematical Sciences, 29(4)(2013),487-504.ResearchGate: https://www.researchgate.net/publication/303725991_Estimates_for_the_Initial_Coefficients_of_Bi-univalent_Functions

[13] M. Lewin, On the coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc., 18 (1967). https://www.ams.org/journals/proc/1967-018-01/S0002-9939-1967-0206255-1/S0002-9939-1967-0206255-1.pdf

[14] Ch.Pommerenke, Univalent functions, Vandenhoeck and Rupercht, Gottingen, 1975. https://worldcat.org/en/title/605236424

[15] C.Ramachandran and D.Kavitha, January 2017, Toeplitz determinant for Some Subclasses of Analytic Functions, Global Journal of Pure and Applied Mathematics 13(2):785-793. https://www.researchgate.net/publication/315656152_Toeplitz_determinant_for_Some_Subclasses_of_Analytic_Functions

[16] S. Sivasubramanian, R. Sivakumar and T. Bulboaca, T.N. Shanmugam, On the class of bi-univalent functions, C.R. Acad. Sci. Ser. I, 352 (2014), 895-900. http://www.numdam.org/item/10.1016/j.crma.2014.09.015.pdf

[17] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math.Lett. 23 (2010), no. 10, 1188–1192. DOI: 10.1016/j.aml.2010.05.009

[18] T.S. Taha, Topics in Univalent Function theory, Ph.D thesis, University of London, 1981.

[20] D.K. Thomas and A.A. Halim, Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions, Bull. Malays. Math. Sci. Soc. 40 (2017), no. 4, 1781–1790. DOI: 10.1007/s40840-016-0385-4

[21] Ye. Ke, and Lek-Heng Lim, Every matrix is a product of Toeplitz matrices, Foundations of Computational Mathematics., 16.3 (2016): 577–598. URL: https://www.stat.uchicago.edu/~lekheng/work/toeplitz.pdf

[22] J.W. Noonan and D.K. Thomas, On the second Hankel Determinant of areally mean p-valent functions, Trans. Amer. Math. Soc., 223(2) (1976), 337-346. https://www.jstor.org/stable/1997533