Stronger and Weaker Forms of Open Mappings via Pythagorean Fuzzy Nano M-open Sets

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Abstract:- In this paper, we introduce the concept of Pythagorean fuzzy nano M open and Pythagorean fuzzy nano M closed mappings in Pythagorean fuzzy nano topological spaces. Also, we study about Pythagorean fuzzy nano M Homeomorphism, almost Pythagorean fuzzy nano M totally mappings, almost Pythagorean fuzzy nano M totally continuous mappings and super Pythagorean fuzzy nano M clopen continuous functions and their properties in Pythagorean fuzzy nano topological spaces.

.***Keywords***: $\mathcal{PF}\mathfrak{N}Mo$ set, $\mathcal{PF}\mathfrak{N}MO$ map, $\mathcal{PF}\mathfrak{N}MHom$, $\mathcal{APF}\mathfrak{N}M\mathcal{T}$ map, $\mathcal{APF}\mathfrak{N}M\mathcal{T}Cts$ map, $\mathcal{SUPF}\mathfrak{N}MbloCts$ map.

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1. Introduction

The first successful attempt towards containing non-probabilistic uncertainty, i.e. uncertainty which is not incite by randomness of an event, into mathematical modeling was made in 1965 by Zadeh [8] through his significant theory on fuzzy sets. A fuzzy set is a set where each element of the universe belongs to it but with some value or degree of belongingness which lies between 0 and 1 and such values are called membership value of an element in that set. Later on Chang [1] was the first to introduce the concept of fuzzy topology. Rough set theory is introduced by Pawlak [7] as a replacement mathematical tool for representing reasoning and deciding handling vagueness and uncertainty. This theory provides the approximation of sets by means of equivalence relations and is taken into account together of the primary non-statistical approaches in data analysis. A rough set are often described by a pair of definable sets called lower and upper approximations. The lower approximation is that the greatest definable set contained within the given set of objects while the upper approximation is that the smallest definable set that contains the given set. Rough set concept are often defined quite generally by means of topological operations, interior and closure, called approximations.

In 2013, a new topology called Nano topology was introduced by Lellis Thivagar [3] which is an extension of rough set theory. He also introduced Nano topological spaces which were defined in terms of approximations and boundary region of a subset of a universe using an equivalence relation on it. The elements of a Nano topological space are called the Nano open sets and its complements are called the Nano closed sets. Nano means something very small. Nano topology thus literally means the study of very small surface. The fundamental ideas in Nano topology are those of approximations and indiscernibility relation. Furthermore nano δ open sets in nano topological space was studied in [6]. Recently, Lellis Thivagar et. al [4] explored a new concept of neutrosophic nano topology. The notion of M-open sets in topological spaces were introduced by El-Maghrabi and Al-Juhani [2] in 2011 and studied some of their properties, and also M-open sets in a nano topological spaces by Padma et. al [5].

Research Gap: No investigation on some new mappings such as Pythagorean fuzzy nano M open, Pythagorean fuzzy nano M closed mappings, Pythagorean fuzzy nano M Homeomorphism, almost Pythagorean fuzzy nano M totally mappings, almost Pythagorean fuzzy nano M totally continuous mappings and super Pythagorean fuzzy nano M clopen continuous functions on Pythagorean fuzzy nano topological space has been reported in the Pythagorean fuzzy literature.

In this paper we introduce Pythagorean fuzzy nano M open and Pythagorean fuzzy nano M closed mappings in Pythagorean fuzzy nano topological spaces. Also, we study about Pythagorean fuzzy nano M Homeomorphism, almost Pythagorean fuzzy nano M totally mappings, almost Pythagorean fuzzy nano M totally continuous mappings and super Pythagorean fuzzy nano M clopen continuous functions and discuss their properties in $\mathcal{PFN}ts$'s.

2 Preliminaries

Definition 1 *[8] A function λ from X into the unit interval I is called a fuzzy set in X. For every $x \in X, \lambda(x) \in I$ is called the grade of membership of x in λ. Some authors say that λ is a Pythagorean fuzzy subset of X instead of saying that λ is a fuzzy set in X. The class of all fuzzy sets from X into the closed unit interval I will be denoted by I^X.*

Definition 2 *[8] If λ and ξ are any two fuzzy subsets of a set X, then λ is said to be included in ξ or λ is contained in ξ or λ is less than or equal to ξ iff $\lambda(x) \leq \xi(x)$ for all x in X and is denoted by $\lambda \leq \xi$. Equivalently, $\lambda \leq \xi$ iff $\mu_\lambda(x) \leq \mu_\xi(x)$ for all x in X.*

Note that every fuzzy subset is included in itself and empty fuzzy subset is included in every fuzzy subset.

Definition 3 *[8] Two fuzzy subsets λ and μ of a set X are said to be equal, written $\lambda = \mu$, if $\lambda(x) = \mu(x)$ for every x in X.*

Definition 4 *[8] The complement of a fuzzy subset λ in a set X, denoted by $1 - \lambda$, is the fuzzy subset of X defined by $1 - \lambda(x)$ for all x in X. Note that $1 - (1 - \lambda) = \lambda$.*

Definition 5 *[8] The union of two fuzzy subsets λ and μ in a set X, denoted by $\lambda \vee \mu$, is fuzzy subset in X defined by $(\lambda \vee \mu)(x) = max\{\lambda(x), \mu(x)\}$, for all x in X.*

In general, the union of a family of fuzzy subsets $\{\xi_i : i \in I\}$ is a fuzzy subset denoted by $\vee_{i\in I} \xi_i$ and defined by $(\vee_{i\in I} \xi_i)(x) = \sup\{\xi_i(x) : i \in I\}$, for all x in X.

Definition 6 *[8] The intersection of two fuzzy subsets λ and μ in a set X, denoted by $\lambda \wedge \mu$, is fuzzy subset in X defined by $(\lambda \wedge \mu)(x) = min\{\lambda(x), \mu(x)\}$, for all x in X.*

In general, the intersection of a family of fuzzy subsets $\{\xi_i : i \in I\}$ in a Pythagorean fuzzy subset is denoted by $\wedge_{i\in I} \xi_i$ and defined by $(\wedge_{i\in I} \xi_i)(x) = \inf\{\xi_i(x) : i \in I\}$, for all x in X.

Definition 7 *[4] Let U be a non-empty set and R be an equivalence relation on U. Let F be a Pythagorean fuzzy set in U with the membership function μ_F. The Pythagorean fuzzy nano lower, Pythagorean fuzzy nano upper approximation and Pythagorean fuzzy nano boundary approximation of F in (U, R) are denoted by $\underline{\mathcal{PFN}}(F), \overline{\mathcal{PFN}}(F)$ and $B_{\mathcal{PFN}}(F)$ respectively, and defined as follows:*

1. $\underline{\mathcal{PFN}}(F) = \left\{\langle x, \mu_{\underline{R}(A)}(x)\rangle / y \in [x]_R, x \in U\right\}$
2. $\overline{\mathcal{PFN}}(F) = \left\{\langle x, \mu_{\overline{R}(A)}(x)\rangle / y \in [x]_R, x \in U\right\}$
3. $B_{\mathcal{PFN}}(F) = \overline{\mathcal{PFN}}(F) - \underline{\mathcal{PFN}}(F)$

where $\mu_{\underline{R}(A)}(x) = \wedge_{y\in[x]_R} \mu_A(y)$. $\mu_{\overline{R}(A)}(x) = \vee_{y\in[x]_R} \mu_A(y)$.

Definition 8 *[4] Let U be an universe, R be an equivalence relation on U and F be a Pythagorean fuzzy set in U and if the collection $\tau_P(F) = \left\{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFN}}(F), \overline{\mathcal{PFN}}(F), B_{\mathcal{PFN}}(F)\right\}$ forms a topology then it is said to be a*

Pythagorean fuzzy nano topology. We call $(U,\tau_P(F))$ as the Pythagorean fuzzy nano topological space. The elements of $\tau_P(F)$ are called Pythagorean fuzzy nano open (briefly, $\mathcal{PFN}o$) sets.

Remark 1 *[4] $[\tau_P(F)]^c$ is called the dual Pythagorean fuzzy nano topology of $\tau_P(F)$. Elements of $[\tau_P(F)]^c$ are called Pythagorean fuzzy nano closed (briefly, $\mathcal{PFN}c$) sets. Thus, we note that a Pythagorean fuzzy set G of U is Pythagorean fuzzy nano closed in $\tau_P(F)$ if and only if $U-G$ is Pythagorean fuzzy nano open in $\tau_P(F)$.*

Definition 9 *Let $(U,\tau_P(F))$ be a $\mathcal{PFN}ts$ with respect to F where F is a Pythagorean fuzzy subset of U. Then a Pythagorean fuzzy subset S in U is said to be a Pythagorean fuzzy nano*

1. interior of S (briefly, $\mathcal{PFN}int(S)$) is defined by $\mathcal{PFN}int(S)=\cup\{I: I\subseteq S\ \&\ I \text{isa} \mathcal{PFN}o \text{set in} U\}$.

2. closure of S (briefly, $\mathcal{PFN}cl(S)$) is defined by $\mathcal{PFN}cl(S)=\cap\{A: S\subseteq A\ \&\ A \text{isa} \mathcal{PFN}c \text{set in} U\}$.

3. regular open (briefly, $\mathcal{PFN}ro$) set if $S=\mathcal{PFN}int(\mathcal{PFN}cl(S))$.

4. regular closed (briefly, $\mathcal{PFN}rc$) set if $S=\mathcal{PFN}cl(\mathcal{PFN}int(S))$.

5. θ interior of S (briefly, $\mathcal{PFN}\theta int(S)$) is defined by $\mathcal{PFN}\theta int(S)=\cup\{\mathcal{PFN}int(I): I\subseteq S\ \&\ I \text{isa} \mathcal{PFN}c \text{ set in} U\}$.

6. θ closure of S (briefly, $\mathcal{PFN}\theta cl(S)$) is defined by $\mathcal{PFN}\theta cl(S)=\cap\{\mathcal{PFN}cl(A): S\subseteq A\ \&\ A \text{isa} \mathcal{PFN}o \text{set in} U\}$.

7. δ interior of S (briefly, $\mathcal{PFN}\delta int(S)$) is defined by $\mathcal{PFN}\delta int(S)=\cup\{I: I\subseteq S\ \&\ I \text{isa} \mathcal{PFN}ro \text{ set in} U\}$.

8. δ closure of S (briefly, $\mathcal{PFN}\delta cl(S)$) is defined by $\mathcal{PFN}\delta cl(S)=\cap\{A: S\subseteq A\ \&\ A \text{isa} \mathcal{PFN}rc \text{set in } U\}$.

9. θ open (briefly, $\mathcal{PFN}\theta o$) set if $S=\mathcal{PFN}\theta int(S)$.

10. semi open (briefly, $\mathcal{PFNS}o$) set if $S\subseteq\mathcal{PFN}cl(\mathcal{PFN}int(S))$.

11. θ semi open (briefly, $\mathcal{PFN}\theta\mathcal{S}o$) set if $S\subseteq\mathcal{PFN}cl(\mathcal{PFN}\theta int(S))$.

12. pre open (briefly, $\mathcal{PFNP}o$) set if $S\subseteq\mathcal{PFN}int(\mathcal{PFN}cl(S))$.

13. θ pre open (briefly, $\mathcal{PFN}\theta\mathcal{P}o$) set if $S\subseteq\mathcal{PFN}int(\mathcal{PFN}\theta cl(S))$.

14. δ pre open (briefly, $\mathcal{PFN}\delta\mathcal{P}o$) set if $S\subseteq\mathcal{PFN}int(\mathcal{PFN}\delta cl(S))$.

15. θ semi interior of S (briefly, $\mathcal{PFN}\theta\mathcal{S}int(S)$) is defined by $\mathcal{PFN}\theta\mathcal{S}int(S)=\cup\{I: I\subseteq S\ \&\ I \text{isa} \mathcal{PFN}\theta\mathcal{S}o \text{ set in} U\}$.

16. θ semi closure of S (briefly, $\mathcal{PFN}\theta\mathcal{S}cl(S)$) is defined by $\mathcal{PFN}\theta\ \mathcal{S}cl(S)=\cap\{A: S\subseteq A\ \&\ A \text{isa} \mathcal{PFN}\theta\mathcal{S}c \text{set in } U\}$.

17. pre interior of S (briefly, $\mathcal{PFNP}int(S)$) is defined by $\mathcal{PFN}\ \mathcal{P}int(S)=\cup\{I: I\subseteq S\ \&\ I \text{isa} \mathcal{PFNP}o \text{ set in} U\}$.

18. pre closure of S (briefly, $\mathcal{PFNP}cl(S)$) is defined by $\mathcal{PFNP}\ cl(S)=\cap\{A: S\subseteq A\ \&\ A \text{isa} \mathcal{PFNP}c \text{set in } U\}$.

19. θ pre interior of S (briefly, $\mathcal{PFN}\theta\mathcal{P}int(S)$) is defined by $\mathcal{PFN}\theta\mathcal{P}int(S)=\cup\{I: I\subseteq S\ \&\ I \text{isa} \mathcal{PFN}\theta\mathcal{P}o \text{ set in} U\}$.

20. θ pre closure of S (briefly, $\mathcal{PFN}\theta\mathcal{P}cl(S)$) is defined by $\mathcal{PFN}\theta\ \mathcal{P}cl(S)=\cap\{A: S\subseteq A\ \&\ A \text{isa} \mathcal{PFN}\theta\mathcal{P}c \text{set in } U\}$.

21. δ pre interior of S (briefly, $\mathcal{PFN}\delta\mathcal{P}int(S)$) is defined by $\mathcal{PFN}\delta\mathcal{P}int(S)=\cup\{I: I\subseteq S\ \&\ I \text{isa} \mathcal{PFN}\delta\mathcal{P}o \text{ set in} U\}$.

22. δ pre closure of S (briefly, $\mathcal{PFN}\delta\mathcal{P}cl(S)$) is defined by $\mathcal{PFN}\delta\ \mathcal{P}cl(S)=\cap\{A: S\subseteq A\ \&\ A \text{isa} \mathcal{PFN}\delta\mathcal{P}c \text{set in } U\}$.

The complement of the respective Pythagorean fuzzy nano open sets are called as Pythagorean fuzzy nano closed sets.

Definition 10 *Let $(U, \tau_P(F))$ be a $\mathcal{PF}\mathfrak{N}ts$ with respect to F where F is a Pythagorean fuzzy subset of U. Then a Pythagorean fuzzy subset S in U is said to be a Pythagorean fuzzy nano*

1. M-open (briefly, $\mathcal{PF}\mathfrak{N}Mo$) set if $S \subseteq \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\theta int(S)) \cup \mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}\delta cl(S))$,

2. M-closed (briefly, $\mathcal{PF}\mathfrak{N}Mc$) set if $\mathcal{PF}\mathfrak{N}int(\mathcal{PF}\mathfrak{N}\theta cl(S)) \cap \mathcal{PF}\mathfrak{N}cl(\mathcal{PF}\mathfrak{N}\delta int(S)) \subseteq S$.

3. Pythagorean fuzzy nano M-interior of K is the union of all $\mathcal{PF}\mathfrak{N}Mo$ sets contained in K and denoted by $\mathcal{PF}\mathfrak{N}Mint(K)$.

4. Pythagorean fuzzy nano M-closure of K is the intersection of all $\mathcal{PF}\mathfrak{N}Mc$ sets containing K and denoted by $\mathcal{PF}\mathfrak{N}Mcl(K)$.

The family of all $\mathcal{PF}\mathfrak{N}Mo$ (resp. $\mathcal{PF}\mathfrak{N}Mc$) sets of a space $(U, \tau_P(F))$ will be as always denoted by $\mathcal{PF}\mathfrak{N}MO(U, A)$ (resp. $\mathcal{PF}\mathfrak{N}MC(U, A)$).

Theorem 1 *Let K be a Pythagorean fuzzy subset of a space $(U, \tau_P(F))$ Then*

1. K is a $\mathcal{PF}\mathfrak{N}Mo$ set iff $K = \mathcal{PF}\mathfrak{N}Mint(K)$,

2. K is a $\mathcal{PF}\mathfrak{N}Mc$ set iff $K = \mathcal{PF}\mathfrak{N}Mcl(K)$.

Definition 11 *A function $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is said to be Pythagorean fuzzy nano*

1. continuous (briefly, $\mathcal{PF}\mathfrak{N}Cts$), if for each $\mathcal{PF}\mathfrak{N}o$ set M of U_2, the set $h_P^{-1}(M)$ is $\mathcal{PF}\mathfrak{N}o$ set of U_1.

2. θ continuous (briefly, $\mathcal{PF}\mathfrak{N}\theta Cts$), if for each $\mathcal{PF}\mathfrak{N}o$ set M of U_2, the set $h_P^{-1}(M)$ is $\mathcal{PF}\mathfrak{N}\theta o$ set of U_1.

3. θ semi continuous (briefly, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}Cts$), if for each $\mathcal{PF}\mathfrak{N}o$ set M of U_2, the set $h_P^{-1}(M)$ is $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}o$ set of U_1.

4. δ pre continuous (briefly, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}Cts$), if for each $\mathcal{PF}\mathfrak{N}o$ set M of U_2, the set $h_P^{-1}(M)$ is $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$ set of U_1.

5. M continuous (briefly, $\mathcal{PF}\mathfrak{N}MCts$), if for each $\mathcal{PF}\mathfrak{N}o$ set M of U_2, the set $h_P^{-1}(M)$ is $\mathcal{PF}\mathfrak{N}Mo$ set of U_1.

Theorem 2 *A function $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{PF}\mathfrak{N}MCts$ iff the inverse image of every $\mathcal{PF}\mathfrak{N}c$ set in U_2 is $\mathcal{PF}\mathfrak{N}Mc$ set in U_1.*

Definition 12 *A function $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is called Pythagorean fuzzy nano*

1. irresolute (briefly, $\mathcal{PF}\mathfrak{N}Irr$) function, if for each $\mathcal{PF}\mathfrak{N}\mathcal{S}o$ set M of U_2, the set $h_P^{-1}(M)$ is $\mathcal{PF}\mathfrak{N}\mathcal{S}o$ set of U_1. 2. θ semi irresolute (briefly, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}Irr$) function, if for each $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}o$ set M of U_2, the set $h_P^{-1}(M)$ is $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}o$ set of U_1.

3. δ pre irresolute (briefly, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}Irr$) function, if for each $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$ set M of U_2, the set $h_P^{-1}(M)$ is $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$ set of U_1.

4. M irresolute (briefly, $\mathcal{PF}\mathfrak{N}MIrr$) function, if for each $\mathcal{PF}\mathfrak{N}Mo$ set M of U_2, the set $h_P^{-1}(M)$ is $\mathcal{PF}\mathfrak{N}Mo$ set of U_1.

3 Fuzzy nano $\boldsymbol{M}$-open map and Pytha- gorean fuzzy nano $\boldsymbol{M}$-closed map

In this section, we introduce Pythagorean fuzzy nano M open maps and Pythagorean fuzzy nano M closed maps in $\mathcal{PF}\mathfrak{N}ts$ and obtain certain characterizations of these classes of maps.

Definition 13 *Let $(U_1, \tau_P(F_1))$ and $(U_2, \tau_P(F_2))$ be two $\mathcal{PF}\mathfrak{N}ts$. A function $f: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is said to be Pythagorean fuzzy nano (resp. δ, $\delta\mathcal{S}$, e, θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) open map (briefly, $\mathcal{PF}\mathfrak{N}O$ (resp. $\mathcal{PF}\mathfrak{N}\delta O$,*

$\mathcal{PFN}\delta\mathcal{S}O$, $\mathcal{PFN}eO$, $\mathcal{PFN}\theta O$, $\mathcal{PFN}\theta\mathcal{S}O$, $\mathcal{PFN}\delta\mathcal{P}O$ and $\mathcal{PFN}MO$)) if the image of each $\mathcal{PFN}o$ set in U_1 is $\mathcal{PFN}o$ (resp. $\mathcal{PFN}\delta o$, $\mathcal{PFN}\delta\mathcal{S}o$, $\mathcal{PFN}eo$, $\mathcal{PFN}\theta o$, $\mathcal{PFN}\theta\mathcal{S}o$, $\mathcal{PFN}\delta\mathcal{P}o$ and $\mathcal{PFN}Mo$)-set in U_2.

Definition 14 *Let $(U_1, \tau_P(F_1))$ and $(U_2, \tau_P(F_2))$ be two $\mathcal{PFN}ts$. A function $f: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is said to be Pythagorean fuzzy nano (resp. δ, $\delta\mathcal{S}$, e, θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) closed map (briefly, $\mathcal{PFN}C$ (resp. $\mathcal{PFN}\delta C$, $\mathcal{PFN}\delta\mathcal{S}C$, $\mathcal{PFN}eC$, $\mathcal{PFN}\theta C$, $\mathcal{PFN}\theta\mathcal{S}C$, $\mathcal{PFN}\delta\mathcal{P}C$ and $\mathcal{PFN}MC$)) if the image of each $\mathcal{PFN}c$ set in U_1 is $\mathcal{PFN}c$ (resp. $\mathcal{PFN}\delta c$, $\mathcal{PFN}\delta\mathcal{S}c$, $\mathcal{PFN}ec$, $\mathcal{PFN}\theta c$, $\mathcal{PFN}\theta\mathcal{S}c$, $\mathcal{PFN}\delta\mathcal{P}c$ and $\mathcal{PFN}Mc$)-set in U_2.*

Proposition 1 *Let $(U_1, \tau_P(A_1))$ & $(U_2, \tau_P(A_2))$ be a $\mathcal{PFN}ts$'s. Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be a mapping. Then the following statements are hold for $\mathcal{PFN}ts$, but not conversely.*

1. Every $\mathcal{PFN}\theta O$ is a $\mathcal{PFN}O$.
2. Every $\mathcal{PFN}\theta O$ is a $\mathcal{PFN}\theta\mathcal{S}O$.
3. Every $\mathcal{PFN}\theta\mathcal{S}O$ is a $\mathcal{PFN}MO$.
4. Every $\mathcal{PFN}\delta O$ is a $\mathcal{PFN}O$.
5. Every $\mathcal{PFN}\delta O$ is a $\mathcal{PFN}\delta\mathcal{S}O$.
6. Every $\mathcal{PFN}\delta O$ is a $\mathcal{PFN}\delta\mathcal{P}O$.
7. Every $\mathcal{PFN}\delta\mathcal{S}O$ is a $\mathcal{PFN}eO$.
8. Every $\mathcal{PFN}\delta\mathcal{P}O$ is a $\mathcal{PFN}MO$.
9. Every $\mathcal{PFN}MO$ is a $\mathcal{PFN}eO$.

Proof.

1. Let B be a $\mathcal{PFN}os$ in $(U_2, \tau_P(A_2))$. Since h_P is $\mathcal{PFN}\theta O$, $h_P^{-1}(B)$ is $\mathcal{PFN}\theta os$ in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\theta os$ is a $\mathcal{PFN}os$, $h_P^{-1}(B)$ is a $\mathcal{PFN}os$ in $(U_1, \tau_P(A_1))$. Hence, h_P is a $\mathcal{PFN}O$.

2. Let B be a $\mathcal{PFN}os$ in $(U_2, \tau_P(A_2))$. Since h_P is $\mathcal{PFN}\theta O$, $h_P^{-1}(B)$ is $\mathcal{PFN}\theta os$ in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\theta os$ is a $\mathcal{PFN}\theta\mathcal{S}os$, $h_P^{-1}(B)$ is a $\mathcal{PFN}os$ in $(U_1, \tau_P(A_1))$. Hence, h_P is a $\mathcal{PFN}\theta\mathcal{S}O$.

3. Let B be a $\mathcal{PFN}os$ in $(U_2, \tau_P(A_2))$. Since h_P is $\mathcal{PFN}\theta\mathcal{S}O$, $h_P^{-1}(B)$ is $\mathcal{PFN}\theta\mathcal{S}os$ in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\theta\mathcal{S}os$ is a $\mathcal{PFN}Mos$, $h_P^{-1}(B)$ is a $\mathcal{PFN}Mos$ in $(U_1, \tau_P(A_1))$. Hence, h_P is a $\mathcal{PFN}MO$.

4. Let B be a $\mathcal{PFN}os$ in $(U_2, \tau_P(A_2))$. Since h_P is $\mathcal{PFN}\delta O$, $h_P^{-1}(B)$ is $\mathcal{PFN}\delta os$ in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\delta os$ is a $\mathcal{PFN}os$, $h_P^{-1}(B)$ is a $\mathcal{PFN}os$ in $(U_1, \tau_P(A_1))$. Hence, h_P is a $\mathcal{PFN}O$.

5. Let B be a $\mathcal{PFN}os$ in $(U_2, \tau_P(A_2))$. Since h_P is $\mathcal{PFN}\delta O$, $h_P^{-1}(B)$ is $\mathcal{PFN}\delta os$ in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\delta os$ is a $\mathcal{PFN}\delta\mathcal{S}os$, $h_P^{-1}(B)$ is a $\mathcal{PFN}\delta\mathcal{S}os$ in $(U_1, \tau_P(A_1))$. Hence, h_P is a $\mathcal{PFN}\delta\mathcal{S}O$.

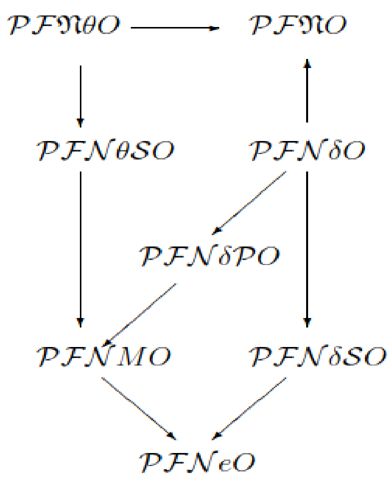
6. Let B be a $\mathcal{PFN}os$ in $(U_2, \tau_P(A_2))$. Since h_P is $\mathcal{PFN}\delta O$, $h_P^{-1}(B)$ is $\mathcal{PFN}\delta os$ in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\delta os$ is a $\mathcal{PFN}\delta\mathcal{P}os$, $h_P^{-1}(B)$ is a $\mathcal{PFN}\delta\mathcal{P}os$ in $(U_1, \tau_P(A_1))$. Hence, h_P is a $\mathcal{PFN}\delta\mathcal{P}O$.

7. Let B be a $\mathcal{PFN}os$ in $(U_2, \tau_P(A_2))$. Since h_P is $\mathcal{PFN}\delta\mathcal{S}O$, $h_P^{-1}(B)$ is $\mathcal{PFN}\delta\mathcal{S}os$ in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\delta\mathcal{S}os$ is a $\mathcal{PFN}eos$, $h_P^{-1}(B)$ is a $\mathcal{PFN}eos$ in $(U_1, \tau_P(A_1))$. Hence, h_P is a $\mathcal{PFN}eO$.

8. Let B be a $\mathcal{PFN}os$ in $(U_2, \tau_P(A_2))$. Since h_P is $\mathcal{PFN}\delta\mathcal{P}O$, $h_P^{-1}(B)$ is $\mathcal{PFN}\delta\mathcal{P}os$ in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}\delta\mathcal{P}os$ is a $\mathcal{PFN}Mos$, $h_P^{-1}(B)$ is a $\mathcal{PFN}Mos$ in $(U_1, \tau_P(A_1))$. Hence, h_P is a $\mathcal{PFN}MO$.

9. Let B be a $\mathcal{PFN}os$ in $(U_2, \tau_P(A_2))$. Since h_P is $\mathcal{PFN}MO$, $h_P^{-1}(B)$ is $\mathcal{PFN}Mos$ in $(U_1, \tau_P(A_1))$. Since every $\mathcal{PFN}Mos$ is a $\mathcal{PFN}eos$, $h_P^{-1}(B)$ is a $\mathcal{PFN}eos$ in $(U_1, \tau_P(A_1))$. Hence, h_P is a $\mathcal{PFN}eO$.

Remark 2 *We obtain the following diagram from the results are discussed above.*



Note: $A \rightarrow B$ denotes A implies B, but not conversely.

Example 1 *Assume* $U_1 = U_2 = U = \{s_1, s_2, s_3, s_4\}$ *be the universe set and the equivalence relation is* $U/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$. *Let* $A = \left\{\left\langle\frac{s_1}{0.3,0.1}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle, \left\langle\frac{s_4}{0.4,0.25}\right\rangle\right\}$ *be a Pythagorean fuzzy subset of* U.

$$\underline{\mathcal{PF}\mathfrak{N}}(A) = \left\{\left\langle\frac{s_1,s_4}{0.3,0.25}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle\right\},$$

$$\overline{\mathcal{PF}\mathfrak{N}}(A) = \left\{\left\langle\frac{s_1,s_4}{0.4,0.1}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle\right\},$$

$$B_{\mathcal{PF}\mathfrak{N}}(A) = \left\{\left\langle\frac{s_1,s_4}{0.25,0.3}\right\rangle, \left\langle\frac{s_2}{0.1,0.5}\right\rangle, \left\langle\frac{s_3}{0.2,0.45}\right\rangle\right\}.$$

Now $\tau_P(A_1) = \tau_P(A_2) = \tau_P(A) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PF}\mathfrak{N}}(A), \overline{\mathcal{PF}\mathfrak{N}}(A), B_{\mathcal{PF}\mathfrak{N}}(A)\}$. Let $h_P: (U, \tau_P(A_2)) \rightarrow (U, \tau_P(A_1))$ be an identity function, Then h_P is $\mathcal{PFN}O$ (resp. $\mathcal{PFN}O$, $\mathcal{PFN}\delta\mathcal{P}O$, $\mathcal{PFN}MO$ and $\mathcal{PFN}eO$) but not $\mathcal{PFN}\theta O$ (resp. $\mathcal{PFN}\delta O$, $\mathcal{PFN}\delta O$, $\mathcal{PFN}\theta\mathcal{S}O$ and $\mathcal{PFN}\delta\mathcal{S}O$). Since, $\overline{\mathcal{PF}\mathfrak{N}}(A)$ is a $\mathcal{PFN}o$ set in U_2 but h_P $(\overline{\mathcal{PF}\mathfrak{N}}(A)) = \overline{\mathcal{PF}\mathfrak{N}}(A)$ is not $\mathcal{PFN}\theta o$ (resp. $\mathcal{PFN}\delta o$, $\mathcal{PFN}\delta o$, $\mathcal{PFN}\theta\mathcal{S}o$ and $\mathcal{PFN}\delta\mathcal{S}o$) set in U_1.

Example 2 *Let* $U_1 = \{s_1, s_2, s_3, s_4\} = U_2 = \{t_1, t_2, t_3, t_4\}$ *are the universe sets and the equivalence relations are* $U_1/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$ *and* $U_2/R = \{\{t_1, t_4\}, \{t_2\}, \{t_3\}\}$. *Let* $A_1 = \left\{\left\langle\frac{s_1}{0.3,0.7}\right\rangle, \left\langle\frac{s_2}{0.1,0.6}\right\rangle, \left\langle\frac{s_3}{0.4,0.4}\right\rangle, \left\langle\frac{s_4}{0.4,0.6}\right\rangle\right\}$ *and*

$A_2 = \left\{\left\langle\frac{t_1}{0.6,0.4}\right\rangle, \left\langle\frac{t_2}{0.6,0.2}\right\rangle, \left\langle\frac{t_3}{0.6,0.4}\right\rangle, \left\langle\frac{t_4}{0.6,0.4}\right\rangle\right\}$ be a Pythagorean fuzzy subsets of U_1 and U_2 respectively.

$$\underline{\mathcal{PF}\mathfrak{N}}(A_1) = \left\{\left\langle\frac{s_1,s_4}{0.3,0.7}\right\rangle, \left\langle\frac{s_2}{0.1,0.6}\right\rangle, \left\langle\frac{s_3}{0.4,0.4}\right\rangle\right\},$$

$$\overline{\mathcal{PF}\mathfrak{N}}(A_1) = \left\{\left\langle\frac{s_1,s_4}{0.4,0.6}\right\rangle, \left\langle\frac{s_2}{0.1,0.6}\right\rangle, \left\langle\frac{s_3}{0.4,0.4}\right\rangle\right\},$$

$$B_{\mathcal{PF}\mathfrak{N}}(A_1) = \left\{\left\langle\frac{s_1,s_4}{0.4,0.6}\right\rangle, \left\langle\frac{s_2}{0.1,0.6}\right\rangle, \left\langle\frac{s_3}{0.4,0.4}\right\rangle\right\},$$

$$\underline{\mathcal{PF}\mathfrak{N}}(A_2) = \left\{\left\langle\frac{t_1,t_4}{0.6,0.4}\right\rangle, \left\langle\frac{t_2}{0.6,0.2}\right\rangle, \left\langle\frac{t_3}{0.6,0.4}\right\rangle\right\},$$

$$\overline{\mathcal{PF}\mathfrak{N}}(A_2) = \left\{\left\langle\frac{t_1,t_4}{0.6,0.4}\right\rangle, \left\langle\frac{t_2}{0.6,0.2}\right\rangle, \left\langle\frac{t_3}{0.6,0.4}\right\rangle\right\},$$

$$B_{\mathcal{PF}\mathfrak{N}}(A_2)=\left\{\left\langle\frac{t_1,t_4}{0.4,0.6}\right\rangle,\left\langle\frac{t_2}{0.2,0.6}\right\rangle,\left\langle\frac{t_3}{0.4,0.6}\right\rangle\right\}.$$

Now $\tau_P(A_1)=\{0_{\mathcal{P}},1_{\mathcal{P}},\underline{\mathcal{PF}\mathfrak{N}}(A_1),\overline{\mathcal{PF}\mathfrak{N}}(A_1)=B_{\mathcal{PF}\mathfrak{N}}(A_1)\}$, $\tau_P(A_2)=\{0_{\mathcal{P}},1_{\mathcal{P}},\underline{\mathcal{PF}\mathfrak{N}}(A_2)=\overline{\mathcal{PF}\mathfrak{N}}(A_2),B_{\mathcal{PF}\mathfrak{N}}(A_2)\}$. Let $h_P:(U_2,\tau_P(A_2))\to(U_1,\tau_P(A_1))$ be an identity function, then h_P is $\mathcal{PFN}\delta\mathcal{S}O$ but not $\mathcal{PFN}\delta O$. Since, $B_{\mathcal{PF}\mathfrak{N}}(A_2)$ is a $\mathcal{PFN}\delta\mathcal{S}o$ set in U_2 but $h_P(B_{\mathcal{PF}\mathfrak{N}}(A_2))=B_{\mathcal{PF}\mathfrak{N}}(A_2)$ is not $\mathcal{PFN}\delta o$ set in U_1.

Example 3 *Let* $U_1=\{s_1,s_2,s_3,s_4\}=U_2=\{t_1,t_2,t_3,t_4\}$ *are the universe sets and the equivalence relations are* $U_1/R=\{\{s_1,s_4\},\{s_2\},\{s_3\}\}$ *and* $U_2/R=\{\{t_1,t_4\},\{t_2\},\{t_3\}\}$. *Let* $A_1=\left\{\left\langle\frac{s_1}{0.7,0.4}\right\rangle,\left\langle\frac{s_2}{0.6,0.6}\right\rangle,\left\langle\frac{s_3}{0.7,0.7}\right\rangle,\left\langle\frac{s_4}{0.7,0.6}\right\rangle\right\}$ *and* $A_2=\left\{\left\langle\frac{t_1}{0.7,0.6}\right\rangle,\left\langle\frac{t_2}{0.5,0.5}\right\rangle,\left\langle\frac{t_3}{0.6,0.6}\right\rangle,\left\langle\frac{t_4}{0.6,0.7}\right\rangle\right\}$ be a Pythagorean fuzzy subsets of U_1 and U_2 respectively.

$$\underline{\mathcal{PF}\mathfrak{N}}(A_1)=\left\{\left\langle\frac{s_1,s_4}{0.7,0.6}\right\rangle,\left\langle\frac{s_2}{0.6,0.6}\right\rangle,\left\langle\frac{s_3}{0.7,0.7}\right\rangle\right\},$$

$$\overline{\mathcal{PF}\mathfrak{N}}(A_1)=\left\{\left\langle\frac{s_1,s_4}{0.7,0.4}\right\rangle,\left\langle\frac{s_2}{0.6,0.6}\right\rangle,\left\langle\frac{s_3}{0.7,0.7}\right\rangle\right\},$$

$$B_{\mathcal{PF}\mathfrak{N}}(A_1)=\left\{\left\langle\frac{s_1,s_4}{0.6,0.7}\right\rangle,\left\langle\frac{s_2}{0.6,0.6}\right\rangle,\left\langle\frac{s_3}{0.7,0.7}\right\rangle\right\},$$

$$\underline{\mathcal{PF}\mathfrak{N}}(A_2)=\left\{\left\langle\frac{t_1,t_4}{0.6,0.7}\right\rangle,\left\langle\frac{t_2}{0.5,0.5}\right\rangle,\left\langle\frac{t_3}{0.6,0.6}\right\rangle\right\},$$

$$\overline{\mathcal{PF}\mathfrak{N}}(A_2)=\left\{\left\langle\frac{t_1,t_4}{0.7,0.6}\right\rangle,\left\langle\frac{t_2}{0.5,0.5}\right\rangle,\left\langle\frac{t_3}{0.6,0.6}\right\rangle\right\},$$

$$B_{\mathcal{PF}\mathfrak{N}}(A_2)=\left\{\left\langle\frac{t_1,t_4}{0.7,0.6}\right\rangle,\left\langle\frac{t_2}{0.5,0.5}\right\rangle,\left\langle\frac{t_3}{0.6,0.6}\right\rangle\right\}.$$

Now $\tau_P(A_1)=\{0_{\mathcal{P}},1_{\mathcal{P}},\underline{\mathcal{PF}\mathfrak{N}}(A_1),\overline{\mathcal{PF}\mathfrak{N}}(A_1),B_{\mathcal{PF}\mathfrak{N}}(A_1)\}$, $\tau_P\ (A_2)=\{0_{\mathcal{P}},1_{\mathcal{P}},\underline{\mathcal{PF}\mathfrak{N}}(A_2),\overline{\mathcal{PF}\mathfrak{N}}(A_2)=B_{\mathcal{PF}\mathfrak{N}}(A_2)\}$. Let $h_P:(U_2,\tau_P(A_2))\to(U_1,\tau_P(A_1))$ be an identity function, then h_P is $\mathcal{PFN}\theta\mathcal{S}O$ but not $\mathcal{PFN}\theta O$. Since, $B_{\mathcal{PF}\mathfrak{N}}(A_2)$ is a $\mathcal{PFN}o$ set in U_2 but $h_P(B_{\mathcal{PF}\mathfrak{N}}(A_2))=B_{\mathcal{PF}\mathfrak{N}}(A_2)$ is not $\mathcal{PFN}\theta o$ set in U_1.

Example 4 *Let* $U_1=\{s_1,s_2,s_3,s_4\}=U_2=\{t_1,t_2,t_3,t_4\}$ *are the universe sets and the equivalence relations are* $U_1/R=\{\{s_1,s_4\},\{s_2\},\{s_3\}\}$ *and* $U_2/R=\{\{t_1,t_4\},\{t_2\},\{t_3\}\}$. *Let* $A_1=\left\{\left\langle\frac{s_1}{0.8,0.4}\right\rangle,\left\langle\frac{s_2}{0.6,0.6}\right\rangle,\left\langle\frac{s_3}{0.7,0.7}\right\rangle,\left\langle\frac{s_4}{0.5,0.7}\right\rangle\right\}$ *and*

$A_2=\left\{\left\langle\frac{t_1}{0.6,0.3}\right\rangle,\left\langle\frac{t_2}{0.6,0.6}\right\rangle,\left\langle\frac{t_3}{0.7,0.7}\right\rangle,\left\langle\frac{t_4}{0.6,0.3}\right\rangle\right\}$ be a Pythagorean fuzzy subsets of U_1 and U_2 respectively.

$$\underline{\mathcal{PF}\mathfrak{N}}(A_1)=\left\{\left\langle\frac{s_1,s_4}{0.5,0.7}\right\rangle,\left\langle\frac{s_2}{0.6,0.6}\right\rangle,\left\langle\frac{s_3}{0.7,0.7}\right\rangle\right\},$$

$$\overline{\mathcal{PF}\mathfrak{N}}(A_1)=\left\{\left\langle\frac{s_1,s_4}{0.8,0.4}\right\rangle,\left\langle\frac{s_2}{0.6,0.6}\right\rangle,\left\langle\frac{s_3}{0.7,0.7}\right\rangle\right\},$$

$$B_{\mathcal{PF}\mathfrak{N}}(A_1)=\left\{\left\langle\frac{s_1,s_4}{0.7,0.5}\right\rangle,\left\langle\frac{s_2}{0.6,0.6}\right\rangle,\left\langle\frac{s_3}{0.7,0.7}\right\rangle\right\},$$

$$\underline{\mathcal{PF}\mathfrak{N}}(A_2)=\left\{\left\langle\frac{t_1,t_4}{0.6,0.3}\right\rangle,\left\langle\frac{t_2}{0.6,0.6}\right\rangle,\left\langle\frac{t_3}{0.7,0.7}\right\rangle\right\},$$

$$\overline{\mathcal{PF}\mathfrak{N}}(A_2)=\left\{\left\langle\frac{t_1,t_4}{0.6,0.3}\right\rangle,\left\langle\frac{t_2}{0.6,0.6}\right\rangle,\left\langle\frac{t_3}{0.7,0.7}\right\rangle\right\},$$

$$B_{\mathcal{PF}\mathfrak{N}}(A_2)=\left\{\left\langle\frac{t_1,t_4}{0.3,0.6}\right\rangle,\left\langle\frac{t_2}{0.6,0.6}\right\rangle,\left\langle\frac{t_3}{0.7,0.7}\right\rangle\right\}.$$

Now $\tau_P(A_1)=\{0_{\mathcal{P}},1_{\mathcal{P}},\underline{\mathcal{PF}\mathfrak{N}}(A_1),\overline{\mathcal{PF}\mathfrak{N}}(A_1),B_{\mathcal{PF}\mathfrak{N}}(A_1)\}$, $\tau_P\ (A_2)=\{0_{\mathcal{P}},1_{\mathcal{P}},\underline{\mathcal{PF}\mathfrak{N}}(A_2)=\overline{\mathcal{PF}\mathfrak{N}}(A_2),B_{\mathcal{PF}\mathfrak{N}}(A_2)\}$. Let $h_P:(U_2,\tau_P(A_2))\to(U_1,\tau_P(A_1))$ be an identity function, then h_P is $\mathcal{PFN}MO$ but not $\mathcal{PFN}\delta\mathcal{P}O$. Since, $B_{\mathcal{PF}\mathfrak{N}}(A_2)$ is a $\mathcal{PFN}o$ set in U_2 but $h_P(B_{\mathcal{PF}\mathfrak{N}}(A_2))=B_{\mathcal{PF}\mathfrak{N}}(A_2)$ is not $\mathcal{PFN}\delta\mathcal{P}o$ set in U_1.

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Example 5 *Let $U_1 = \{s_1, s_2, s_3, s_4\} = U_2 = \{t_1, t_2, t_3, t_4\}$ are the universe sets and the equivalence relations are $U_1/R = \{\{s_1, s_3\}, \{s_2, s_4\}\}$ and $U_2/R = \{\{t_1, t_3\}, \{t_2, t_4\}\}$.* Let $A_1 = \left\{\left\langle\frac{s_1}{0.1,0.8}\right\rangle, \left\langle\frac{s_2}{0.3,0.7}\right\rangle, \left\langle\frac{s_3}{0.2,0.9}\right\rangle, \left\langle\frac{s_4}{0.4,0.6}\right\rangle\right\}$ and $A_2 = \left\{\left\langle\frac{t_1}{0.4,0.8}\right\rangle, \left\langle\frac{t_2}{0.5,0.4}\right\rangle, \left\langle\frac{t_3}{0.6,0.6}\right\rangle, \left\langle\frac{t_4}{0.7,0.6}\right\rangle\right\}$ be a Pythagorean fuzzy subsets of U_1 and U_2 respectively.

$$\underline{\mathcal{PFN}}(A_1) = \left\{\left\langle\frac{s_1,s_3}{0.1,0.9}\right\rangle, \left\langle\frac{s_2,s_4}{0.3,0.7}\right\rangle\right\},$$

$$\overline{\mathcal{PFN}}(A_1) = \left\{\left\langle\frac{s_1,s_3}{0.2,0.8}\right\rangle, \left\langle\frac{s_2,s_4}{0.4,0.6}\right\rangle\right\},$$

$$B_{\mathcal{PFN}}(A_1) = \left\{\left\langle\frac{s_1,s_3}{0.2,0.8}\right\rangle, \left\langle\frac{s_2,s_4}{0.4,0.6}\right\rangle\right\},$$

$$\underline{\mathcal{PFN}}(A_2) = \left\{\left\langle\frac{t_1,t_3}{0.4,0.8}\right\rangle, \left\langle\frac{t_2,t_4}{0.5,0.6}\right\rangle\right\},$$

$$\overline{\mathcal{PFN}}(A_2) = \left\{\left\langle\frac{t_1,t_3}{0.6,0.6}\right\rangle, \left\langle\frac{t_2,t_4}{0.7,0.4}\right\rangle\right\},$$

$$B_{\mathcal{PFN}}(A_2) = \left\{\left\langle\frac{t_1,t_3}{0.6,0.6}\right\rangle, \left\langle\frac{t_2,t_4}{0.6,0.5}\right\rangle\right\}.$$

Now $\tau_P(A_1) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFN}}(A_1), \overline{\mathcal{PFN}}(A_1) = B_{\mathcal{PFN}}(A_1)\}$,$\tau_P(A_2) = 0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PFN}}(A_2), \overline{\mathcal{PFN}}(A_2), B_{\mathcal{PFN}}(A_2)\}$. Let $h_P: (U_2,\ \tau_P(A_2)) \to (U_1, \tau_P(A_1))$ be an identity function, then h_P is $\mathcal{PFN}eO$ but not $\mathcal{PFN}MO$. Since, $B_{\mathcal{PFN}}(A_2)$ is a $\mathcal{PFN}o$ set in U_2 but $h_P(B_{\mathcal{PFN}}(A_2)) = B_{\mathcal{PFN}}(A_2)$ is not $\mathcal{PFN}Mo$ set in U_1.

Theorem 3 *A function $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{PFN}MC$ mapping if and only if $\mathcal{PFN}Mcl(h_P(A)) \subseteq h_P(\mathcal{PFN}\ cl(A))$ for every Pythagorean fuzzy set A of U_1.*

Proof. Suppose $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is a $\mathcal{PFN}MC$ function and A is any Pythagorean fuzzy set in U_1. Then $\mathcal{PFN}cl(A)$ is a $\mathcal{PFN}c$ set in U_1. Since h_P is $\mathcal{PFN}MC$, $h_P(\mathcal{PFN}cl(A))$ is a $\mathcal{PFN}Mc$ set in U_2. Then by Theorem 1 (ii), $\mathcal{PFN}Mcl(h_P(\mathcal{PFN}\ \ cl(A))) = h_P(\mathcal{PFN}cl(A))$. Therefore $\mathcal{PFN}Mclh_P(A)) \subseteq \mathcal{PFN}Mcl$ $(h_P(\mathcal{PFN}cl(A))) = h_P(\mathcal{PFN}cl(A))$. Hence $\mathcal{PFN}Mcl(h_P(A)) \subseteq h_P(\mathcal{PFN}cl(A))$.

Conversely, Let A be a $\mathcal{PFN}c$ set in U_1. Then $\mathcal{PFN}cl(A) = A$ and so $h_P(A) = h_P(\mathcal{PFN}cl(A))$. By our assumption $\mathcal{PFN}Mcl\ (h_P(A)) \subseteq h_P(A)$. But $h_P(A) \subseteq \mathcal{PFN}Mcl(h_P(A))$. Hence $\mathcal{PFN}\ Mcl(h_P(A)) = h_P(A)$ and therefore by Theorem 1 (ii), $h_P(A)$ is $\mathcal{PFN}Mc$ in U_2. Thus h_P is a $\mathcal{PFN}MC$ map.

Theorem 4 *A map $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{PFN}MC$ mapping if and only if for each Pythagorean fuzzy set S of U_2 and for each $\mathcal{PFN}o$ set U of U_1 containing $h_P^{-1}(S)$ there exists a $\mathcal{PFN}Mo$ set V of U_2 such that $S \subseteq V$ and $h_P^{-1}(V) \subseteq U$.*

Proof. Suppose h_P is a $\mathcal{PFN}MC$ map. Let S be any Pythagor- ean fuzzy set in U_2 and U be a $\mathcal{PFN}Mo$ set of U_1 such that $h_P^{-1}(S) \subseteq U$. Then $V = (h_P(U^c))^c$ is $\mathcal{PFN}Mo$ set containing S such that $h_P^{-1}(V) \subseteq U$. Conversely, Let S be a $\mathcal{PFN}c$ set of U_1. Then $h_P^{-1}((h_P(S))^c) \subseteq S^c$ and S^c is $\mathcal{PFN}o$ in U_1.

By assumption, there exists a $\mathcal{PFN}Mo$ set V of U_2 such that $(h_P(S))^c \subseteq V$ and $h_P^{-1}(V) \subseteq S^c$ and so $S \subseteq (h_P^{-1}(V))^c$. Hence $V^c \subseteq h_P(S) \subseteq h_P((h_P^{-1}(V))^c) \subseteq V^c$, which implies $h_P(S) = V^c$. Since V^c is $\mathcal{PFN}Mc$, $h_P(S)$ is $\mathcal{PFN}Mc$ and h_P is $\mathcal{PFN}MC$ map.

Remark 3 *The composition of two $\mathcal{PFN}MO$ maps need not be a $\mathcal{PFN}MO$ map, which is shown in the following example.*

Example 6 *Let $U_1 = \{s_1, s_2, s_3, s_4\} = U_2 = \{t_1, t_2, t_3, t_4\} = U_3 = \{u_1, u_2, u_3, u_4\}$ are the universe sets and the equivalence relations are $U_1/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$, $U_2/R = \{\{t_1, t_4\}, \{t_2\}, \{t_3\}\}$ and $U_3/R = \{\{u_1, u_4\}, \{u_2\}, \{u_3\}\}$.*

Let $A_1 = \left\{\left\langle\frac{s_1}{0.4,0.3}\right\rangle, \left\langle\frac{s_2}{0.4,0.2}\right\rangle, \left\langle\frac{s_3}{0.5,0.3}\right\rangle, \left\langle\frac{s_4}{0.5,0.2}\right\rangle\right\}$,

$A_2 = \left\{\left\langle\frac{t_1}{0.9,0.3}\right\rangle, \left\langle\frac{t_2}{0.5,0.7}\right\rangle, \left\langle\frac{t_3}{0.3,0.5}\right\rangle, \left\langle\frac{t_4}{0.6,0.7}\right\rangle\right\}$ and

$A_3 = \left\{\left\langle\frac{u_1}{0.3,0.25}\right\rangle, \left\langle\frac{u_2}{0.1,0.5}\right\rangle, \left\langle\frac{u_3}{0.2,0.45}\right\rangle, \left\langle\frac{u_4}{0.4,0.25}\right\rangle\right\}$ be a Pythagorean fuzzy subsets of U_1, U_2 and U_3 respectively.

$$\underline{\mathcal{PF}\mathfrak{N}}(A_1) = \left\{\left\langle\frac{s_1,s_4}{0.4,0.3}\right\rangle, \left\langle\frac{s_2}{0.4,0.2}\right\rangle, \left\langle\frac{s_3}{0.5,0.3}\right\rangle\right\},$$

$$\overline{\mathcal{PF}\mathfrak{N}}(A_1) = \left\{\left\langle\frac{s_1,s_4}{0.5,0.2}\right\rangle, \left\langle\frac{s_2}{0.4,0.2}\right\rangle, \left\langle\frac{s_3}{0.5,0.3}\right\rangle\right\},$$

$$B_{\mathcal{PF}\mathfrak{N}}(A_1) = \left\{\left\langle\frac{s_1,s_4}{0.3,0.4}\right\rangle, \left\langle\frac{s_2}{0.2,0.4}\right\rangle, \left\langle\frac{s_3}{0.3,0.5}\right\rangle\right\},$$

$$\underline{\mathcal{PF}\mathfrak{N}}(A_2) = \left\{\left\langle\frac{t_1,t_4}{0.6,0.7}\right\rangle, \left\langle\frac{t_2}{0.5,0.7}\right\rangle, \left\langle\frac{t_3}{0.3,0.5}\right\rangle\right\},$$

$$\overline{\mathcal{PF}\mathfrak{N}}(A_2) = \left\{\left\langle\frac{t_1,t_4}{0.9,0.3}\right\rangle, \left\langle\frac{t_2}{0.5,0.7}\right\rangle, \left\langle\frac{t_3}{0.3,0.5}\right\rangle\right\},$$

$$B_{\mathcal{PF}\mathfrak{N}}(A_2) = \left\{\left\langle\frac{t_1,t_4}{0.7,0.6}\right\rangle, \left\langle\frac{t_2}{0.5,0.7}\right\rangle, \left\langle\frac{t_3}{0.3,0.5}\right\rangle\right\},$$

$$\underline{\mathcal{PF}\mathfrak{N}}(A_3) = \left\{\left\langle\frac{u_1,u_4}{0.3,0.25}\right\rangle, \left\langle\frac{u_2}{0.1,0.5}\right\rangle, \left\langle\frac{u_3}{0.2,0.45}\right\rangle\right\},$$

$$\overline{\mathcal{PF}\mathfrak{N}}(A_3) = \left\{\left\langle\frac{u_1,u_4}{0.4,0.1}\right\rangle, \left\langle\frac{u_2}{0.1,0.5}\right\rangle, \left\langle\frac{u_3}{0.2,0.45}\right\rangle\right\},$$

$$B_{\mathcal{PF}\mathfrak{N}}(A_3) = \left\{\left\langle\frac{u_1,u_4}{0.25,0.3}\right\rangle, \left\langle\frac{u_2}{0.1,0.5}\right\rangle, \left\langle\frac{u_3}{0.2,0.45}\right\rangle\right\}.$$

Now $\tau_P(A_1) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PF}\mathfrak{N}}(A_1), \overline{\mathcal{PF}\mathfrak{N}}(A_1), B_{\mathcal{PF}\mathfrak{N}}(A_1)\}$, $\tau_P(A_2) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PF}\mathfrak{N}}(A_2), \overline{\mathcal{PF}\mathfrak{N}}(A_2), B_{\mathcal{PF}\mathfrak{N}}(A_2)\}$, $\tau_P(A_3) = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \underline{\mathcal{PF}\mathfrak{N}}(A_3) = \overline{\mathcal{PF}\mathfrak{N}}(A_3), B_{\mathcal{PF}\mathfrak{N}}(A_3)\}$. Let $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ and $g_P: (U_2, \tau_P(A_2)) \to (U_3, \tau_P(A_3))$ be an identity function, then h_P and g_P are $\mathcal{PFNMO}$ but $(h_P \circ g_P)$ is not $\mathcal{PFNMO}$. Since, $B_{\mathcal{PF}\mathfrak{N}}(A_1)$ is a $\mathcal{PFN}o$ set in U_1 but $(h_P \circ g_P)(B_{\mathcal{PF}\mathfrak{N}}(A_1)) = B_{\mathcal{PF}\mathfrak{N}}(A_1)$ is not $\mathcal{PFNM}o$ set in U_3.

Theorem 5 *Let $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ be a $\mathcal{PF}\mathfrak{N}C$ map and $g_P: (U_2, \tau_P(F_2)) \to (U_3, \tau_P(F_3))$ be a $\mathcal{PF}\mathfrak{N}MC$ map. Then their composition $g_P \circ h_P: (U_1, \tau_P(F_1)) \to (U_3, \tau_P(F_3))$ is $\mathcal{PF}\mathfrak{N}MC$.*

Proof. Let F be a $\mathcal{PF}\mathfrak{N}c$ set in U_1. Since h_P is $\mathcal{PF}\mathfrak{N}C$, $h_P(F)$ is $\mathcal{PF}\mathfrak{N}c$ in U_2. Since g_P is $\mathcal{PF}\mathfrak{N}MC$, $g_P(h_P(F)) = (g_P \circ h_P)(F)$ is $\mathcal{PF}\mathfrak{N}Mc$ in U_3. Hence $g_P \circ h_P$ is a $\mathcal{PF}\mathfrak{N}MC$ map.

Theorem 6 *Let $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ and $g_P: (U_2, \tau_P(F_2)) \to (U_3, \tau_P(F_3))$ be two mappings such that their composition $g_P \circ h_P: (U_1, \tau_P(F_1)) \to (U_3, \tau_P(F_3))$ is $\mathcal{PF}\mathfrak{N}MC$ map. Then the followings are true.*

1. If h_P is $\mathcal{PF}\mathfrak{N}Cts$ and surjective, then g_P is $\mathcal{PF}\mathfrak{N}MC$ map.
2. If g_P is $\mathcal{PF}\mathfrak{N}MIrr$ and injective, then h_P is $\mathcal{PF}\mathfrak{N}MC$ map.

Proof. (i) Let A be a $\mathcal{PF}\mathfrak{N}c$ set of U_2. Since h_P is $\mathcal{PF}\mathfrak{N}Cts$ map, $h_P^{-1}(A)$ is $\mathcal{PF}\mathfrak{N}cs$ in U_1. Since $g_P \circ h_P$ is $\mathcal{PF}\mathfrak{N}MC$ map, $(g_P \circ h_P)(h_P^{-1}(A))$ is $\mathcal{PF}\mathfrak{N}Mcs$ in M. Since h_P is surjective, $g_P(A)$ is $\mathcal{PF}\mathfrak{N}Mcs$ in U_3. Hence g_P is $\mathcal{PF}\mathfrak{N}MC$ map.

(ii) Let B be any $\mathcal{PF}\mathfrak{N}c$ set of U_1. Since $g_P \circ h_P$ is $\mathcal{PF}\mathfrak{N}MC$ map, $(g_P \circ h_P)(B)$ is $\mathcal{PF}\mathfrak{N}Mcs$ in U_3. Since g_P is $\mathcal{PF}\mathfrak{N}MIrr$, $g_P^{-1}(g_P \circ h_P(B))$ is $\mathcal{PF}\mathfrak{N}Mcs$ in U_2. Since g_P is injective, $h_P(B)$ is $\mathcal{PF}\mathfrak{N}Mcs$ in U_2. Hence h_P is $\mathcal{PF}\mathfrak{N}MC$ map.

Theorem 7 *Let $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ be $\mathcal{PF}\mathfrak{N}MC$ map.*

1. If A is $\mathcal{PF}\mathfrak{N}c$ set of U_1, then the restriction $h_P: (U_A, \tau_P(F_A)) \to (U_2, \tau_P(F_2))$ is $\mathcal{PF}\mathfrak{N}MC$ map.

2. If $A = h_P^{-1}(B)$ for some $\mathcal{PF}\mathfrak{N}c$ set B of U_2, then the restriction $h_P: (U_A, \tau_P(F_A)) \to (U_2, \tau_P(F_2))$ is $\mathcal{PF}\mathfrak{N}MC$ map.

Proof. (i) Let B be any $\mathcal{PF}\mathfrak{N}c$ set of A. Then $B = A \cap L$ for some $\mathcal{PF}\mathfrak{N}c$ set L of U_1 and so B is $\mathcal{PF}\mathfrak{N}cs$ in U_1. By hypothesis, $h_P(B)$ is $\mathcal{PF}\mathfrak{N}Mcs$ in U_2. But $h_P(B) = h_P(B)$, therefore h_P is a $\mathcal{PF}\mathfrak{N}MC$ map.

(ii) Let D be a $\mathcal{PF}\mathfrak{N}c$ set of A. Then $D = A \cap H$, for some $\mathcal{PF}\mathfrak{N}c$ set H in U_1. Now, $h_P(D) = h_P(A \cap H) = h_P(h_P^{-1}(B) \cap H) = B \cap h_P(H)$. Since h_P is $\mathcal{PF}\mathfrak{N}MC$, $h_P(H)$ is $\mathcal{PF}\mathfrak{N}Mcs$ in U_2. Hence h_P is a $\mathcal{PF}\mathfrak{N}MC$ map.

Theorem 8 *A function $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{PF}\mathfrak{N}MO$ map if and only if $h_P(\mathcal{PF}\mathfrak{N}int(A)) \subseteq \mathcal{PF}\mathfrak{N}Mint(h_P(A))$, for every Pythagorean fuzzy set A of U_1.*

Proof. Suppose $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is a $\mathcal{PF}\mathfrak{N}MO$ function and A is any Pythagorean fuzzy set in U_1. Then $\mathcal{PF}\mathfrak{N}int(A)$ is a $\mathcal{PF}\mathfrak{N}o$ set in U_1. Since h_P is $\mathcal{PF}\mathfrak{N}MO$, $h_P(\mathcal{PF}\mathfrak{N}int(A))$ is a $\mathcal{PF}\mathfrak{N}Mo$ set. Since $\mathcal{PF}\mathfrak{N}Mint(h_P(\mathcal{PF}\mathfrak{N}int(A))) \subseteq \mathcal{PF}\mathfrak{N}Mint\ (h_P(A))$, $h_P(\mathcal{PF}\mathfrak{N}int(A)) \subseteq \mathcal{PF}\mathfrak{N}Mint(h_P(A))$.

Conversely, $h_P(\mathcal{PF}\mathfrak{N}int(A)) \subseteq \mathcal{PF}\mathfrak{N}Mint(h_P(A))$ for every Pythagorean fuzzy set A in U_1. Let U be a $\mathcal{PF}\mathfrak{N}o$ set in U_1. Then $\mathcal{PF}\mathfrak{N}int(U) = U$ and by hypothesis, $h_P(U) \subseteq \mathcal{PF}\mathfrak{N}Mint(h_P(U))$. But $\mathcal{PF}\mathfrak{N}Mint(h_P(U)) \subseteq h_P(U)$. Therefore, $h_P(U) = \mathcal{PF}\mathfrak{N}Mint\ (h_P(U))$. Then by Theorem 1 (i), $h_P(U)$ is $\mathcal{PF}\mathfrak{N}Mo$. Hence h_P is a $\mathcal{PF}\mathfrak{N}MO$ map.

Definition 15 *Let A and B be any two Pythagorean fuzzy subsets of a $\mathcal{PF}\mathfrak{N}ts$'s. Then A is Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) q-neighbourhood (briefly, $\mathcal{PF}\mathfrak{N}q$-nbhd (resp. $\mathcal{PF}\mathfrak{N}\theta q$-nbhd, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}q$-nbhd, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}q$-nbhd & $\mathcal{PF}\mathfrak{N}Mq$-nbhd)) with B if there exists a $\mathcal{PF}\mathfrak{N}o$ (resp. $\mathcal{PF}\mathfrak{N}\theta o$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}o$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$ and $\mathcal{PF}\mathfrak{N}Mo$) set O with $AqO \subseteq B$.*

Theorem 9 *Let $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ be a mapping. Then the following statements are equivalent.*

1. h_P is a $\mathcal{PF}\mathfrak{N}MO$ mapping,

2. For a subset A of U_1, $h_P(\mathcal{PF}\mathfrak{N}int(A)) \subseteq \mathcal{PF}\mathfrak{N}Mint(f(A))$.

3. For each $x_\alpha \in U_1$ and for each $\mathcal{PF}\mathfrak{N}q$-nbhd U of x_α in U_1, there exists A $\mathcal{PF}\mathfrak{N}Mq$-nbhd W of $h_P(x_\alpha)$ in U_2 such that $W \subseteq h_P(U)$.

Proof. (i) ⇒ (ii): Suppose $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is a $\mathcal{PF}\mathfrak{N}MO$ function and $A \subseteq U_1$. Then $\mathcal{PF}\mathfrak{N}int(A)$ is a $\mathcal{PF}\mathfrak{N}o$ set in U_1. Since h_P is $\mathcal{PF}\mathfrak{N}MO$ map, $h_P(\mathcal{PF}\mathfrak{N}int(A))$ is a $\mathcal{PF}\mathfrak{N}Mo$ set. Since $\mathcal{PF}\mathfrak{N}Mint(h_P(\mathcal{PF}\mathfrak{N}int(A))) \subseteq \mathcal{PF}\mathfrak{N}Mint(h_P(A))$, $h_P\ (\mathcal{PF}\mathfrak{N}int(A)) \subseteq \mathcal{PF}\mathfrak{N}Mint(h_P(A))$. This proves (ii).

(ii) ⇒ (iii): Let $x_\alpha \in U_1$ and U be any arbitrary $\mathcal{PF}\mathfrak{N}q$-nbhd of x_α in U_1. Then there exists a $\mathcal{PF}\mathfrak{N}o$ set G such that $x_\alpha \in G \subseteq U$. By (ii), $h_P(G) = h_P(\mathcal{PF}\mathfrak{N}int(G)) \subseteq \mathcal{PF}\mathfrak{N}Mint(h_P(G))$. But, $\mathcal{PF}\mathfrak{N}Mint(h_P(G)) \subseteq h_P(G)$. Therefore, $\mathcal{PF}\mathfrak{N}Mint(h_P(G)) = h_P(G)$ and hence $h_P(G)$ is $\mathcal{PF}\mathfrak{N}Mos$ in U_2. Since $x_\alpha \in G \subseteq U$, $h_P(x_\alpha) \in h_P(G) \subseteq h_P(U)$ and so (iii) holds, by taking $W = h_P(G)$.

(iii) ⇒ (i): Let U be any $\mathcal{PF}\mathfrak{N}o$ set in U_1. Let $x_\alpha \in U$ and $h_P(x_\alpha) = y_\beta$. Then for each $x_\alpha \in U$, $y \in h_P(U)$, by assumption there exists a $\mathcal{PF}\mathfrak{N}qM$-nbhd $W(y_\beta)$ of y_β in U_2 such that $W(y_\beta) \subseteq h_P(U)$. Since $W(y_\beta)$ is a $\mathcal{PF}\mathfrak{N}qM$-nbhd of y_β, there exists a $\mathcal{PF}\mathfrak{N}Mo$ set $V(y_\beta)$ in U_2 such that $y_\beta \in V(y_\beta) \subseteq W(y_\beta)$. Therefore, $h_P(U) = \cup \{V(y_\beta) | y_\beta \in h_P(U)\}$. Since the union of $\mathcal{PF}\mathfrak{N}Mo$ sets is $\mathcal{PF}\mathfrak{N}Mo$, $h_P(U)$ is a $\mathcal{PF}\mathfrak{N}Mo$ set in U_2. Thus, h_P is a $\mathcal{PF}\mathfrak{N}MO$ map.

Theorem 10 *For any bijective map $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ the following statements are equivalent:*

1. $h_P^{-1}: (U_2, \tau_P(F_2)) \to (U_1, \tau_P(F_1))$ is $\mathcal{PF}\mathfrak{N}MCts$.

2. h_P is $\mathcal{PF}\mathfrak{N}MO$ map.

3. h_P is $\mathcal{PF}\mathfrak{N}MC$ map.

Proof. (i) → (ii): Let U be a $\mathcal{PF}\mathfrak{N}o$ set in U_1. By assumption, $(h_P^{-1})^{-1}(U) = h_P(U)$ is $\mathcal{PF}\mathfrak{N}Mos$ in U_2 and so h_P is $\mathcal{PF}\mathfrak{N}MO$ map.

(ii) → (iii): Let F be a $\mathcal{PF}\mathfrak{N}c$ set of U_1. Then F^c is a $\mathcal{PF}\mathfrak{N}o$ set in U_1. By assumption $h_P(F^c)$ is $\mathcal{PF}\mathfrak{N}Mo$ set in U_2. But $h_P(F^c) = (h_P(F))^c$. Therefore $h_P(F)$ is $\mathcal{PF}\mathfrak{N}Mc$ set in U_2. Hence, h_P is $\mathcal{PF}\mathfrak{N}MC$ map.

(iii) ⇒ (i): Let F be a $\mathcal{PF}\mathfrak{N}Mc$ set of U_1. By assumption, $h_P(F)$ is $\mathcal{PF}\mathfrak{N}Mc$ set in U_2. But $h_P(F) = (h_P^{-1})^{-1}(F)$ and therefore by Theorem 2, h_P^{-1} is $\mathcal{PF}\mathfrak{N}MCts$.

Remark 4 *Theorems 3 to 10 and Remark 3 are holds for $\mathcal{PF}\mathfrak{N}o$, $\mathcal{PF}\mathfrak{N}\theta o$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}o$ & $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$ sets.*

4 Pythagorean Fuzzy nano *M* homeomorphism

The purpose of this section is to introduces the idea of Pythagorean fuzzy nano M homeomorphism in $\mathcal{PF}\mathfrak{N}ts$ and establish some of their attributes.

Definition 16 *Let $(U_1, \tau_P(F_1))$ and $(U_2, \tau_P(F_2))$ be $\mathcal{PF}\mathfrak{N}ts$. A mapping $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is said to be a Pythag- orean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) homeomorphism (briefly, $\mathcal{PF}\mathfrak{N}Hom$ (resp. $\mathcal{PF}\mathfrak{N}\theta Hom$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}Hom$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}Hom$ and $\mathcal{PF}\mathfrak{N}MHom$)) if h_P is bijective, $\mathcal{PF}\mathfrak{N}O$ (resp. $\mathcal{PF}\mathfrak{N}\theta O$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}O$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}O$ and $\mathcal{PF}\mathfrak{N}MO$) function and $\mathcal{PF}\mathfrak{N}O$ (resp. $\mathcal{PF}\mathfrak{N}\theta O$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}O$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}O$ and $\mathcal{PF}\mathfrak{N}MO$) mapping.*

Example 7 *In example 6, $h_P: (U_1, \tau_P(A_1)) \to (U_2, \tau_P(A_2))$ be an identity function, then h_P is $\mathcal{PFN}MO$ and h_P^{-1} is $\mathcal{PFN}MO$. Then h_P is $\mathcal{PF}\mathfrak{N}MHom$.*

Theorem 11 *Let $(U_1, \tau_P(F_1))$ and $(U_2, \tau_P(F_2))$ be two $\mathcal{PF}\mathfrak{N}ts$ and $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ be a bijective function. Then h_P is a $\mathcal{PF}\mathfrak{N}MHom$ if and only if h_P is a $\mathcal{PF}\mathfrak{N}MCts$ function and $\mathcal{PF}\mathfrak{N}MC$ mapping.*

Proof. Let h_P be a $\mathcal{PF}\mathfrak{N}MHom$ homeomorphism. From Definition 16 h_P is a $\mathcal{PF}\mathfrak{N}MCts$ function. From Theorem 10, we have h_P^{-1} is a $\mathcal{PF}\mathfrak{N}MC$ function. So, $(h_P^{-1})^{-1} = h_P$ is a $\mathcal{PF}\mathfrak{N}MC$ function.

Theorem 12 *Let $g_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ be a bijective mapping. If g_P is $\mathcal{PF}\mathfrak{N}MCts$, then the following statements are equivalent: [(a)]*

1. g_P is a $\mathcal{PF}\mathfrak{N}MC$ mapping.
2. g_P is a $\mathcal{PF}\mathfrak{N}MO$ mapping.
3. g_P^{-1} is a $\mathcal{PF}\mathfrak{N}MHom$.

Proof. (a) ⇒ (b) Let us assume that g_P is a bijective mapping and a $\mathcal{PF}\mathfrak{N}MC$ mapping. Hence, g_P^{-1} is a $\mathcal{PF}\mathfrak{N}MCts$ mapping. Since each $\mathcal{PF}\mathfrak{N}o$ set is a $\mathcal{PF}\mathfrak{N}Mo$ set, g_P is a $\mathcal{PF}\mathfrak{N}MO$ mapping.

(b) ⇒ (c) Let g_P be a bijective and $\mathcal{PF}\mathfrak{N}MO$ mapping. Furthermore, g_P^{-1} is a $\mathcal{PF}\mathfrak{N}MCts$ mapping. Hence, g_P and g_P^{-1} are $\mathcal{PF}\mathfrak{N}MCts$. Therefore, g_P is a $\mathcal{PF}\mathfrak{N}MHom$.

(c) ⇒ (a) Let g_P be a $\mathcal{PF}\mathfrak{N}MHom$. Then g_P and g_P^{-1} are $\mathcal{PF}\mathfrak{N}MCts$. Since each $\mathcal{PF}\mathfrak{N}c$ set in U_1 is a $\mathcal{PF}\mathfrak{N}Mc$ set in U_2, hence g_P is a $\mathcal{PF}\mathfrak{N}MC$ mapping.

Remark 5 *Theorems 11 and 12 are holds for $\mathcal{PF}\mathfrak{N}o$, $\mathcal{PF}\mathfrak{N}\theta o$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}o$ & $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$ sets.*

5 Almost Pythagorean fuzzy nano *M* totally mappings

In this section, we introduce almost Pythagorean fuzzy nano M totally mappings and we discuss some basic properties.

Definition 17 *A function $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is said to be*

1. Almost Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) open map (briefly, $\mathcal{APF}\mathfrak{N}O$ (resp. $\mathcal{APF}\mathfrak{N}\theta O$, $\mathcal{APF}\mathfrak{N}\theta\mathcal{S}O$, $\mathcal{APF}\mathfrak{N}\delta\mathcal{P}O$ and $\mathcal{APF}\mathfrak{N}MO$)) if the image of each $\mathcal{PF}\mathfrak{N}ro$ set in U_1 is $\mathcal{PF}\mathfrak{N}o$ (resp. $\mathcal{PF}\mathfrak{N}\theta o$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}o$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$ and $\mathcal{PF}\mathfrak{N}Mo$)-set in U_2.

2. Almost Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) closed map (briefly, $\mathcal{APFN}C$ (resp. $\mathcal{APFN}\theta C$, $\mathcal{APFN}\theta\mathcal{S}C$, $\mathcal{APFN}\delta\mathcal{P}C$ and $\mathcal{APFN}MC$)) if the image of each $\mathcal{PFN}rc$ set in U_1 is $\mathcal{PFN}c$ (resp. $\mathcal{PFN}\theta c$, $\mathcal{PFN}\theta\mathcal{S}c$, $\mathcal{PFN}\delta\mathcal{P}c$ and $\mathcal{PFN}Mc$)-set in U_2.

3. Almost Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) clopen map (briefly, $\mathcal{APFN}clO$ (resp. $\mathcal{APFN}\theta clO$, $\mathcal{APFN}\theta\ \mathcal{S}clO$, $\mathcal{APFN}\delta\mathcal{P}clO$ and $\mathcal{APFN}MclO$)) if the image of each $\mathcal{PFN}rclo$ set in U_1 is $\mathcal{PFN}clo$ (resp. $\mathcal{PFN}\theta clo$, $\mathcal{PFN}\theta\mathcal{S}clo$, $\mathcal{PFN}\delta\mathcal{P}clo$ and $\mathcal{PFN}Mclo$)-set in U_2.

4. Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) totally open map (briefly, $\mathcal{PFNT}O$ (resp. $\mathcal{PFN}\theta\mathcal{T}O$, $\mathcal{PFN}\theta\mathcal{ST}O$, $\mathcal{PFN}\delta\mathcal{PT}O$ and $\mathcal{PFN}M\mathcal{T}O$)) if the image of each $\mathcal{PFN}o$ (resp. $\mathcal{PFN}\theta o$, $\mathcal{PFN}\theta\mathcal{S}o$, $\mathcal{PFN}\delta\mathcal{P}o$ and $\mathcal{PFN}Mo$) set in U_1 is $\mathcal{PFN}clo$ (resp. $\mathcal{PFN}\theta clo$, $\mathcal{PFN}\theta\mathcal{S}clo$, $\mathcal{PFN}\delta\mathcal{P}clo$ and $\mathcal{PFN}Mclo$)-set in U_2.

5. Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) totally closed map (briefly, $\mathcal{PFNT}C$ (resp. $\mathcal{PFN}\theta\mathcal{T}C$, $\mathcal{PFN}\theta\mathcal{ST}C$, $\mathcal{PFN}\delta\mathcal{PT}C$ and $\mathcal{PFN}M\mathcal{T}C$)) if the image of each $\mathcal{PFN}c$ (resp. $\mathcal{PFN}\theta c$, $\mathcal{PFN}\theta\mathcal{S}c$, $\mathcal{PFN}\delta\mathcal{P}c$ and $\mathcal{PFN}Mc$) set in U_1 is $\mathcal{PFN}clo$ (resp. $\mathcal{PFN}\theta clo$, $\mathcal{PFN}\theta\mathcal{S}clo$, $\mathcal{PFN}\delta\mathcal{P}clo$ and $\mathcal{PFN}Mclo$)-set in U_2.

6. Almost Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) totally open map (briefly, $\mathcal{APFNT}O$ (resp. $\mathcal{APFN}\theta\mathcal{T}O$, $\mathcal{APFN}\theta\mathcal{ST}O$, $\mathcal{APFN}\delta\mathcal{PT}O$ and $\mathcal{APFN}M\mathcal{T}O$)) if the image of each $\mathcal{PFN}ro$ set in U_1 is $\mathcal{PFN}clo$ (resp. $\mathcal{PFN}\theta clo$, $\mathcal{PFN}\theta\mathcal{S}clo$, $\mathcal{PFN}\delta\mathcal{P}clo$ and $\mathcal{PFN}Mclo$)-set in U_2.

7. Almost Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) totally closed map (briefly, $\mathcal{APFNT}C$ (resp. $\mathcal{APFN}\theta\mathcal{T}C$, $\mathcal{APFN}\theta\mathcal{ST}C$, $\mathcal{APFN}\delta\mathcal{PT}C$ and $\mathcal{APFN}M\mathcal{T}C$)) if the image of each $\mathcal{PFN}rc$ set in U_1 is $\mathcal{PFN}clo$ (resp. $\mathcal{PFN}\theta clo$, $\mathcal{PFN}\theta\mathcal{S}clo$, $\mathcal{PFN}\delta\mathcal{P}clo$ and $\mathcal{PFN}Mclo$)-set in U_2.

8. Almost Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) totally clopen map (briefly, $\mathcal{APFNT}clO$ (resp. $\mathcal{APFN}\theta\mathcal{T}clO$, $\mathcal{APFN}\theta\mathcal{ST}clO$, $\mathcal{APFN}\delta\mathcal{PT}clO$ and $\mathcal{APFN}M\mathcal{T}clO$)) if the image of each $\mathcal{PFN}rclo$ set in U_1 is $\mathcal{PFN}clo$ (resp. $\mathcal{PFN}\theta clo$, $\mathcal{PFN}\theta\mathcal{S}clo$, $\mathcal{PFN}\delta\mathcal{P}clo$ and $\mathcal{PFN}Mclo$)-set in U_2.

Theorem 13 *Every $\mathcal{APFN}M\mathcal{T}C$ map is $\mathcal{APFN}MC$.*

Proof. Let U_1 and U_2 be $\mathcal{PFN}ts$. Let $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ be an $\mathcal{APFN}M\mathcal{T}C$ mapping. To prove h_P is $\mathcal{APFN}\ MC$, let H be any $\mathcal{PFN}rc$ subset of U_1. Since h_P is $\mathcal{APFN}M\mathcal{T}C$ mapping, $h_P(H)$ is $\mathcal{APFN}Mclo$ in U_2. This implies that $h_P(H)$ is $\mathcal{PFN}c$ in U_2. Therefore h_P is $\mathcal{APFN}MC$.

Corollary 1 *Every $\mathcal{APFN}M\mathcal{T}O$ map is $\mathcal{APFN}MO$.*

Theorem 14 *If a bijective function $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{APFN}M\mathcal{T}O$, then the image of each $\mathcal{PFN}rc$ set in U_1 is $\mathcal{APFN}Mclo$ set in U_2.*

Proof. Let F be a $\mathcal{PFN}rc$ set in U_1. Then F^c is $\mathcal{PFN}ro$ in U_1. Since h_P is $\mathcal{APFN}M\mathcal{T}O$, $h_P^c = [f(F)]^c$ is $\mathcal{APFN}Mclo$ in U_2. This implies that $h_P(F)$ is $\mathcal{APFN}Mclo$ set in U_2.

Theorem 15 *A surjective function $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{APFN}M\mathcal{T}O$ if and only if for each subset B of U_2 and for each $\mathcal{PFN}ro$ set U containing $h_P^{-1}(B)$, there is a $\mathcal{PFN}Mclo$ set V of U_2 such that $B \subseteq V$ and $h_P^{-1}(V) \subseteq U$.*

Proof. Suppose $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is a surjective and $\mathcal{APFN}M\mathcal{T}O$ function and $B \subseteq V$. Let U be $\mathcal{PFN}ro$ set of U_1 such that $h_P^{-1}(B) \subseteq U$. Since h_P is $\mathcal{APFN}M\mathcal{T}O$ function, $h_P(U) = [h_P(U^c)]^c$ is $\mathcal{PFN}Mclo$ set. Then $V = [h_P(U^c)]^c$ is $\mathcal{PFN}Mclo$ set of U_2 containing B such that $h_P^{-1}(V) \subseteq U$.

Theorem 16 *A map $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{APFN}M\mathcal{T}O$ if and only if for each subset A of U_2 and each $\mathcal{PFN}rc$ set U containing $h_P^{-1}(A)$ there is a $\mathcal{PFN}Mclo$ set V of U_2 such that $A \subseteq V$ and $h_P^{-1}(V) \subseteq U$.*

Proof. Suppose h_P is $\mathcal{APFN}M\mathcal{T}O$. Let $A \subseteq Y$ and U be a $\mathcal{PFN}rc$ set of U_1 such that $h_P^{-1}(A) \subseteq U$. Now U^c is $\mathcal{PFN}ro$ and h_P is $\mathcal{APFN}M\mathcal{T}O$, $h_P(U^c)$ is $\mathcal{PFN}Mclo$ set in U_2. Then $V = (h_P(U^c))^c$ is a $\mathcal{PFN}Mclo$ set in U_2. Note that $h_P^{-1}(A) \subseteq U$ implies $A \subseteq V$ and $h_P^{-1}(V) = (h_P^{-1}(h_P(U^c)))^c \subseteq (U^c)^c = U$. That is $h_P^{-1}(V) \subseteq U$.

Conversely, let F be a $\mathcal{PF}\mathfrak{N}ro$ set of U_1. Then $h_P^{-1}(h_P(F)^c) \subseteq F^c$ and F^c is $\mathcal{PF}\mathfrak{N}rc$ set in U_1. By hypothesis, there exist a $\mathcal{PF}\mathfrak{N}Mclo$ set V in U_2 such that $h_P(F^c) \subseteq V$ and $V^c \subseteq h_P(F)$ and so $F \subseteq (h_P^{-1}(V))^c$. Hence $h_P(F) \subseteq h_P((h_P^{-1}(V))^c)$ which implies $h_P(F) \subseteq V^c$. Since V^c is $\mathcal{PF}\mathfrak{N}Mclo$, $h_P(F)$ is $\mathcal{PF}\mathfrak{N}Mclo$. That is $h_P(F)$ is $\mathcal{PF}\mathfrak{N}Mclo$ in U_2. Therefore h_P is $\mathcal{APF}\mathfrak{N}M\mathcal{T}O$.

Corollary 2 *A map $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{APF}\mathfrak{N}M\mathcal{T}C$ if and only if for each subset A of U_2 and each $\mathcal{PF}\mathfrak{N}ro$ set U containing $h_P^{-1}(A)$, there is a $\mathcal{PF}\mathfrak{N}Mclo$ set V of U_2 such that $A \subseteq V$ and $h_P^{-1}(V) \subseteq U$.*

Theorem 17 *If $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{APF}\mathfrak{N}M\ \mathcal{T}C$ and A is $\mathcal{PF}\mathfrak{N}rc$ subset of U_1 then $h_P: (U_A, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{APF}\mathfrak{N}M\mathcal{T}C$.*

Proof. Consider the function $h_P: (U_A, \tau_P(F_A)) \to (U_2, \tau_P(F_2))$ and let V be any $\mathcal{PF}\mathfrak{N}Mclo$ set in U_2. Since h_P is $\mathcal{APF}\mathfrak{N}M\mathcal{T}C$, $h_P^{-1}(V)$ is $\mathcal{PF}\mathfrak{N}rc$ subset of U_1. Since A is $\mathcal{PF}\mathfrak{N}rc$ susset of U_1 and $h_P^{-1}(V) = A \cap h_P^{-1}(V)$ is $\mathcal{PF}\mathfrak{N}rcs$ in A, it follows $h_P^{-1}(V)$ is $\mathcal{PF}\mathfrak{N}rcs$ in A. Hence h_P is $\mathcal{APF}\mathfrak{N}M\mathcal{T}C$.

Remark 6 *$\mathcal{APF}\mathfrak{N}M\mathcal{T}clO$ mapping is $\mathcal{APF}\mathfrak{N}M\mathcal{T}O$ and $\mathcal{APF}\mathfrak{N}M\mathcal{T}C$ map.*

Remark 7 *Theorems 13 to 17, Corollaries 1 and 2 and Remark 6 are holds for $\mathcal{PF}\mathfrak{N}o$, $\mathcal{PF}\mathfrak{N}\theta o$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}o$ & $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$ sets.*

6 Almost Pythagorean fuzzy nano ***M*** totally continuous functions

In this section, some new continuous functions are introduced and discussed their characterizations.

Definition 18 *A map $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is said to be*

1. Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) totally continuous (briefly, $\mathcal{PF}\mathfrak{N}\mathcal{T}Cts$ (resp. $\mathcal{PF}\mathfrak{N}\theta\mathcal{T}Cts$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}\mathcal{T}Cts$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}\mathcal{T}Cts$ and $\mathcal{PF}\mathfrak{N}M\mathcal{T}Cts$)) if $h_P^{-1}(V)$ is $\mathcal{PF}\mathfrak{N}clo$ (resp. $\mathcal{PF}\mathfrak{N}\theta clo$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}clo$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}clo$ and $\mathcal{PF}\mathfrak{N}Mclo$) set in U_1 for each $\mathcal{PF}\mathfrak{N}o$ (resp. $\mathcal{PF}\mathfrak{N}\theta o$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}o$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$ and $\mathcal{PF}\mathfrak{N}Mo$) set V in U_2.

2. Almost Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) totally continuous (briefly, $\mathcal{APF}\mathfrak{N}\mathcal{T}Cts$ (resp. $\mathcal{APF}\mathfrak{N}\theta\mathcal{T}Cts$, $\mathcal{APF}\mathfrak{N}\theta\mathcal{S}\mathcal{T}Cts$, $\mathcal{APF}\mathfrak{N}\delta\mathcal{P}\mathcal{T}Cts$ and $\mathcal{APF}\mathfrak{N}M\mathcal{T}Cts$)) if $h_P^{-1}(V)$ is $\mathcal{PF}\mathfrak{N}clo$ (resp. $\mathcal{PF}\mathfrak{N}\theta clo$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}clo$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}clo$ and $\mathcal{PF}\mathfrak{N}Mclo$) set in U_1 for each $\mathcal{PF}\mathfrak{N}ro$ set V in U_2.

3. Almost Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) totally clopen continuous (briefly, $\mathcal{APF}\mathfrak{N}\mathcal{T}cloCts$ (resp. $\mathcal{APF}\mathfrak{N}\ \theta\mathcal{T}cloCts$, $\mathcal{APF}\mathfrak{N}\theta\mathcal{S}\mathcal{T}cloCts$, $\mathcal{APF}\mathfrak{N}\delta\mathcal{P}\mathcal{T}cloCts$ and $\mathcal{APF}\mathfrak{N}\ M\mathcal{T}cloCts$)) if $h_P^{-1}(V)$ is $\mathcal{PF}\mathfrak{N}clo$ (resp. $\mathcal{PF}\mathfrak{N}\theta clo$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}\ clo$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}clo$ and $\mathcal{PF}\mathfrak{N}Mclo$) set in U_1 for each $\mathcal{PF}\mathfrak{N}rclo$ set V in U_2.

Theorem 18 *A function $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{APF}\mathfrak{N}M\mathcal{T}Cts$ function if the inverse image of every $\mathcal{PF}\mathfrak{N}rc$ set of U_2 is $\mathcal{PF}\mathfrak{N}Mclo$ set in U_1.*

Proof. Let $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ be $\mathcal{APF}\mathfrak{N}M\mathcal{T}Cts$ and F be any $\mathcal{PF}\mathfrak{N}rc$ set in U_2. Then F^c is $\mathcal{PF}\mathfrak{N}ro$ set in U_2. Since h_P is $\mathcal{APF}\mathfrak{N}M\mathcal{T}Cts$, $h_P^{-1}(F^c)$ is $\mathcal{PF}\mathfrak{N}Mclo$ set in U_1. That is $(h_P^{-1}(F))^c$ is $\mathcal{PF}\mathfrak{N}Mclo$ set in U_1. This implies that $h_P^{-1}(F)$ is $\mathcal{PF}\mathfrak{N}Mclo$ set in U_1.

Theorem 19 *A function $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{APF}\mathfrak{N}M\mathcal{T}Cts$ is an $\mathcal{APF}\mathfrak{N}MCts$ function.*

Proof. Suppose $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{APF}\mathfrak{N}M\mathcal{T}\ Cts$ and U is any $\mathcal{PF}\mathfrak{N}ro$ subset of U_2. Since h_P is $\mathcal{APF}\mathfrak{N}M\mathcal{T}Cts$, $h_P^{-1}(U)$ is $\mathcal{PF}\mathfrak{N}Mclo$ in U_1. This implies that $h_P^{-1}(U)$ is $\mathcal{PF}\mathfrak{N}Mo$ in U_1. Therefore the function h_P is $\mathcal{APF}\mathfrak{N}MCts$.

Theorem 20 *For any bijective map $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ the following statements are equivalent:*

1. $h_P^{-1}: (U_2, \tau_P(F_2)) \to (U_1, \tau_P(F_1))$ is $\mathcal{APF}\mathfrak{N}M\mathcal{T}Cts$.
2. h_P is $\mathcal{APF}\mathfrak{N}M\mathcal{T}O$.
3. h_P is $\mathcal{APF}\mathfrak{N}M\mathcal{T}C$.

Proof. (i) → (ii): Let U be a $\mathcal{PF}\mathfrak{N}ro$ set of U_1. By assumption, $(h_P^{-1})^{-1}(U) = h_P(U)$ is $\mathcal{PF}\mathfrak{N}Mclo$ in U_2 and so h_P is $\mathcal{APF}\mathfrak{N}M\mathcal{TO}$.

(ii) → (iii): Let F be a $\mathcal{PF}\mathfrak{N}rc$ set of U_1. Then F^c is $\mathcal{PF}\mathfrak{N}ro$ set in U_1. By assumption $h_P(F^c)$ is $\mathcal{PF}\mathfrak{N}Mclo$ set in U_2. Hence h_P is $\mathcal{APF}\mathfrak{N}M\mathcal{TC}$.

(iii) → (i): Let F be a $\mathcal{PF}\mathfrak{N}rc$ set of U_1. By assumption, $h_P(F)$ is $\mathcal{PF}\mathfrak{N}Mclo$ set in U_2. But $h_P(F) = (h_P^{-1})^{-1}(F)$ and therefore h_P^{-1} is $\mathcal{APF}\mathfrak{N}M\mathcal{TC}ts$.

Remark 8 *Theorems 18 to 20 are holds for $\mathcal{PF}\mathfrak{N}o$, $\mathcal{PF}\mathfrak{N}\theta o$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}o$ & $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$ sets.*

7 Super Pythagorean fuzzy nano *M* clopen continuous functions

In this section, we introduce the concept of super $\mathcal{PF}\mathfrak{N}Mclo$ continuous in $\mathcal{PF}\mathfrak{N}ts$.

Definition 19 *A map $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is said to be super Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) clopen continuous (briefly, $\mathcal{SUPF}\mathfrak{N}cloCts$ (resp. $\mathcal{SUPF}\mathfrak{N}\theta cloCts$, $\mathcal{SUPF}\mathfrak{N}\theta\mathcal{S}cloCts$, $\mathcal{SUPF}\mathfrak{N}\delta\mathcal{P}cloCts$ and $\mathcal{SUPF}\mathfrak{N}McloCts$)) if for each $x_\alpha \in U_1$ and for each $\mathcal{PF}\mathfrak{N}clo$ (resp. $\mathcal{PF}\mathfrak{N}\theta clo$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}clo$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}clo$ and $\mathcal{PF}\mathfrak{N}Mclo$) set V containing $h_P(x_\alpha)$ in U_2, there exist a $\mathcal{PF}\mathfrak{N}ro$ set U containing x_α such that $h_P(U) \subseteq V$.*

Theorem 21 *Let $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ be $\mathcal{APF}\mathfrak{N}\ M\mathcal{TO}$. Then h_P is $\mathcal{SUPF}\mathfrak{N}McloCts$ if $h_P(x_\alpha)$ is $\mathcal{PF}\mathfrak{N}Mclo$ in U_2.*

Proof. Let G be $\mathcal{PF}\mathfrak{N}Mclo$ set in U_2. Now $h_P^{-1}(G)$ is $\mathcal{PF}\mathfrak{N}ros$ in U_1. Since the intersection of $\mathcal{PF}\mathfrak{N}Mclo$ set is $\mathcal{PF}\mathfrak{N}Mclo$ set in U_2, $h_P(h_P^{-1}(G)) = G \wedge h_P(x_\alpha)$ is $\mathcal{PF}\mathfrak{N}Mclo$ in U_2. Therefore, $h_P^{-1}(G)$ is $\mathcal{PF}\mathfrak{N}ro$ in U_1. Hence h_P is $\mathcal{SUPF}\mathfrak{N}McloCts$ function.

Theorem 22 *If $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is surjective and $\mathcal{APF}\mathfrak{N}M\mathcal{TO}$, then h_P is $\mathcal{SUPF}\mathfrak{N}MclocTs$.*

Proof. Let G be $\mathcal{PF}\mathfrak{N}Mclo$ set in U_2. Take $A = h_P^{-1}(G)$. Since $h_P(A) = G$ is $\mathcal{PF}\mathfrak{N}Mclo$ set in U_2, by the Theorem 21, A is $\mathcal{PF}\mathfrak{N}ro$ set in U_1. Therefore h_P is $\mathcal{SUPF}\mathfrak{N}MclocTs$.

Definition 20 *A map $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is said to be Pythagorean fuzzy nano (resp. θ, $\theta\mathcal{S}$, $\delta\mathcal{P}$ and M) clopen irresolute function (briefly, $\mathcal{PF}\mathfrak{N}cloIrr$ (resp. $\mathcal{PF}\mathfrak{N}\theta cloIrr$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}cloIrr$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}cloIrr$ and $\mathcal{PF}\mathfrak{N}MCloIrr$)) if $h_P^{-1}(V)$ is $\mathcal{PF}\mathfrak{N}clo$ (resp. $\mathcal{PF}\mathfrak{N}\theta clo$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}clo$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}clo$ and $\mathcal{PF}\mathfrak{N}Mclo$) set in U_1 for each $\mathcal{PF}\mathfrak{N}clo$ (resp. $\mathcal{PF}\mathfrak{N}\theta clo$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}clo$, $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}clo$ and $\mathcal{PF}\mathfrak{N}Mclo$) set V in U_2.*

Theorem 23 *Let $(U_1, \tau_P(F_1))$, $(U_2, \tau_P(F_2))$ and $(U_3, \tau_P(F_3))$ be $\mathcal{PF}\mathfrak{N}ts$. Then the composition $g_P \circ h_P: (U_1, \tau_P(F_1)) \to (U_3, \tau_P\ \ (F_3))$ is $\mathcal{SUPF}\mathfrak{N}McloCts$ function where $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ is $\mathcal{SUPF}\mathfrak{N}McloCts$ function and $g_P: (U_2, \tau_P(F_2)) \to (U_3, \tau_P(F_3))$ is $\mathcal{PF}\mathfrak{N}MCloIrr$ function.*

Proof. Let A be a $\mathcal{PF}\mathfrak{N}rc$ set of U_1. Since h_P is $\mathcal{SUPF}\mathfrak{N}Mclo\ Cts$, $h_P(A)$ is $\mathcal{PF}\mathfrak{N}Mclo$ set in U_2. Then by hypothesis, $h_P(A)$ is $\mathcal{PF}\mathfrak{N}Mclo$ set. Since g_P is $\mathcal{PF}\mathfrak{N}MCloIrr$, $g_P(h_P(A)) = (g_P \circ h_P)(A)$. Therefore $g_P \circ h_P$ is $\mathcal{SUPF}\mathfrak{N}MCloCts$.

Theorem 24 *If $h_P: (U_1, \tau_P(F_1)) \to (U_2, \tau_P(F_2))$ and $g_P: (U_2, \tau_P(F_2)) \to (U_3, \tau_P(F_3))$ are two mappings such that their composition $g_P \circ h_P: (U_1, \tau_P(F_1)) \to (U_3, \tau_P(F_3))$ is $\mathcal{APF}\mathfrak{N}M\mathcal{TC}$ mapping then the following statements are true.*

1. If h_P is $\mathcal{SUPF}\mathfrak{N}MCloCts$ and surjective, then g_P is a $\mathcal{PF}\mathfrak{N}M\ cloIrr$ function.

2. If g_P is $\mathcal{PF}\mathfrak{N}MCloIrr$ function and injective, then h_P is an $\mathcal{APF}\mathfrak{N}M\mathcal{TC}$ function.

Proof. (i) Let A be a $\mathcal{PF}\mathfrak{N}Mclo$ set of U_2. Since h_P is $\mathcal{SUPF}\mathfrak{N}\ MCloCts$, $h_P^{-1}(A)$ is $\mathcal{PF}\mathfrak{N}rcs$ in U_1. Since $(g_P \circ h_P)(h_P^{-1}(A))$ is $\mathcal{PF}\mathfrak{N}Mclo$ set in M. Since h_P is surjective, $g(A)$ is $\mathcal{PF}\mathfrak{N}Mclo$ set in U_3. Therefore g_P is $\mathcal{PF}\mathfrak{N}MCloIrr$ function.

(ii) Let B be $\mathcal{PF}\mathfrak{N}rc$ set of U_1. Since $g_P \circ h_P$ is $\mathcal{APF}\mathfrak{N}M\mathcal{TC}$, $g_P \circ h_P(B)$ is $\mathcal{PF}\mathfrak{N}Mclo$ set in U_3. Since g_P is a $\mathcal{PF}\mathfrak{N}McloIrr$ function, $g_P^{-1}((g_P \circ h_P)(B))$ is $\mathcal{PF}\mathfrak{N}Mclo$ set in U_2. That is $h_P(B)$ is $\mathcal{PF}\mathfrak{N}Mclo$ set in U_2. Since h_P is injective, h_P is an $\mathcal{APF}\mathfrak{N}M\mathcal{TC}$ function.

Remark 9 *Theorems 21 to 24 are holds for $\mathcal{PF}\mathfrak{N}o$, $\mathcal{PF}\mathfrak{N}\theta o$, $\mathcal{PF}\mathfrak{N}\theta\mathcal{S}o$ & $\mathcal{PF}\mathfrak{N}\delta\mathcal{P}o$ sets.*

Conclusion

In this paper, we have continued to study the properties of Pythag- orean fuzzy nano M open and Pythagorean fuzzy nano M closed mappings in Pythagorean fuzzy nano topological spaces. Also, we study about Pythagorean fuzzy nano M Homeomorphism, almost Pythagorean fuzzy nano M totally mappings, almost Pythagorean fuzzy nano M totally continuous mappings and super Pythagorean fuzzy nano M clopen continuous functions and established the relations between them we obtain some new characterizations of these mappings in Pythagorean fuzzy nano topological spaces.

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