N-Cubic Pythagorean Fuzzy Linear Spaces

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Abstract:- This article proposes the notion of N-Cubic Pythagorean Fuzzy Linear Spaces based on the notion of interval valued N- Pythagorean fuzzy linear spaces and N-Pythagorean fuzzy linear spaces. Motivated by the notion of Cubic Pythagorean Fuzzy Linear Spaces, we define P(R)-union and P(R)- (intersection) of N- Cubic Pythagorean Fuzzy Linear Spaces and its properties with a few examples. The notion of external and internal N-Cubic Pythagorean Fuzzy Linear Spaces and their properties are investigated.

Keywords: - N-cubic set, N- Pythagorean fuzzy set, N- Pythagorean fuzzy linear space,

Interval valued N- Pythagorean fuzzy linear space, P(R)-union, P(R)- intersection

1. Introduction

Most of the words that we often use arbitrarily in everyday conversation have vague meanings. Fuzziness is included in the numerical formulas that we employ to describe objects. It is frequently reported that in the year 1965, Zadeh [1] invented the concept of fuzzy sets, which are effective in managing different kinds of uncertainty in many domains. Fuzzy set theory, in contrast with classical set theory, allows for a gradual evaluation of an element's membership in a set. There are many areas in which fuzzy set theory can be applied, including logic, semigroup theory, group theory, analysis, and topology. As an extension of a fuzzy set wherein the members have been assigned to a group of closed subintervals of [0, 1], Zadeh [2] proposed interval-valued fuzzy sets in the year 1975, nearly a decade after his initial contribution. Fuzzy sets have been further developed by Atanassov [3- 4] to include intuitionistic fuzzy sets, which allow for dealing with an element's membership or non-membership. This method of handling ambiguity and uncertainty progressively assumed the role of fuzzy sets.

Cubic set theory is a further development of fuzzy set theory first presented by Jun et. al. [5]. It explored a variety of cubic set attributes, including internal and external cubic sets as well as the P-union, P-intersection, R-union, and R-intersection of these sets. Cubic sets address the beneficial features of numerous physical issues while completely ignoring their negative aspects. A negative valued function was mentioned, and N-structures were created by Jun et. al. [6]. Additionally, they used BCK/BCI algebra and subtraction algebra to apply N structure theory [7]. To encompass the negative part of cubic sets along the codomain $[-1, 0]$, Jun introduced the notion of N-cubic sets, which encompass the adverse aspects of cubic sets accompanied the codomain $[-1, 0]$ by combining N-fuzzy sets with interval-valued N-fuzzy sets.

Fuzzy linear space was introduced by Nanda [8]. Later, Gu Wexiang and Lu [9] provided some basic features and redefined the notions of fuzzy field and fuzzy linear space. The idea was subsequently developed to cubic linear space by Vijaybalaji et. al. [10], integrating interval-valued fuzzy linear space with fuzzy linear space and its properties. The concept of N-cubic linear spaces was first introduced by Kavyasree et. al. [11]. Pythagorean fuzzy sets were introduced by Yagar [12]. Peng and Yang [13] later proposed the idea of interval valued Pythagorean fuzzy sets. T. Senapathi et. al. [14–16] introduced the concept of cubic Pythagorean fuzzy sets. BS Reddy et. al.

proposed the concept of cubic Pythagorean fuzzy linear spaces [17] as well as addressed its properties, such as P-union, P-intersection, R-union, and R-intersection of internal and exterior cubic Pythagorean fuzzy linear spaces. Further, Soujanya et. al. developed cubic Pythagorean hesitant fuzzy linear spaces and their relevance in multi-criteria decision making [18]. Moreover, the same research group proposed the concept of N-Cubic Picture Fuzzy Linear Spaces [19].

This paper focuses on the concept of N-Cubic Pythagorean Fuzzy Linear Spaces. We have attempted to apply the concept of N-cubic structure to Pythagorean Fuzzy Linear Spaces. We describe the effect of P-union (resp. P-intersection) and R-union (resp. R-intersection) on N-Cubic Pythagorean Fuzzy Linear Spaces. We demonstrate that the N-Cubic Pythagorean Fuzzy Linear Spaces exhibit closure about the R-intersection. We contradict the idea that the R-union, P-union, and P-intersection of two N-Cubic Pythagorean Fuzzy Linear Spaces make them into another N-Cubic Pythagorean Fuzzy Linear Space by providing examples. The notion of internal and external N-Cubic Pythagorean fuzzy linear spaces is presented. Furthermore, we demonstrate by alternative examples that internal (resp. external) N-Cubic Pythagorean Fuzzy Linear Spaces are not closed concerning P-union, P-intersection, and R-union.

2. Preliminaries

Definition 2.1

An N interval number is a closed sub-interval of [-1,0] and the collection of all closed sub-intervals of [-1,0] is denoted by D [-1,0]. It is of the form D [-1,0] = { $\hat{\imath} = [i^-, i^+]$: $i^- \le i^+$, $i^-, i^+ \epsilon$ [-1,0]. Notably, the operations " $\ge$ ", " $\le$ ","=","max","min" are defined as follows

i) $\hat{\imath}_1 \ge \hat{\imath}_2$ if and only if $i_1{}^- \ge i_2{}^-$ and $i_1{}^+ \ge i_2{}^+$

ii) $\hat{\imath}_1 \le \hat{\imath}_2$ if and only if $i_1{}^- \le i_2{}^-$ and $i_1{}^+ \le i_2{}^+$

iii) $\hat{\imath}_1 = \hat{\imath}_2$ if and only if $i_1{}^- = i_2{}^-$ and $i_1{}^+ = i_2{}^+$

iv) max $\{\hat{\imath}_1, \hat{\imath}_2\}$ = [max $\{i_1{}^-, i_2{}^-\}$, max $\{i_1{}^+, i_2{}^+\}$]

v) min$\{\hat{\imath}_1, \hat{\imath}_2\}$ = [min $\{i_1{}^-, i_2{}^-\}$, min $\{i_1{}^+, i_2{}^+\}$].

Definition 2.2

For an N interval number $\hat{\imath}_s \epsilon$ D [-1,0], where $s \in \Delta$ we define

$$\inf \hat{\imath}_s = [\inf_{s\in\Delta} \hat{\imath}_s^-, \inf_{s\in\Delta} \hat{\imath}_s^+] \text{ and } \sup \hat{\imath}_s = [\sup_{s\in\Delta} \hat{\imath}_s^-, \sup_{s\in\Delta} \hat{\imath}_s^+].$$

Definition 2.3

An interval valued N fuzzy set is denoted by $Ꚍ^N$ on Y is of the form

$Ꚍ^N = \{< y, Ꚍ^N(y): y \epsilon Y >\}$ where $Ꚍ^N: Y \to$ D [-1,0] an $Ꚍ^N(y)$: $[\hat{\vartheta}^-_{Ꚍ^N}(y), \hat{\vartheta}^+_{Ꚍ^N}(y)]$ for all y $\in Y$. Here $\vartheta^-_{Ꚍ^N}(y): Y \to [-1,0]$ and $\hat{\vartheta}^+_{Ꚍ^N}(y): Y \to [-1,0]$ are fuzzy sets in Y such that $\hat{\vartheta}^-_{Ꚍ^N}(y) \le \hat{\vartheta}^+_{Ꚍ^N}(y)$.

Definition 2.4

Let Y be a fixed set, A N- fuzzy set in Y is defined as

$N^F = \{y, \lambda_{N^F}(y): y \in Y\}$ and $\lambda_{N^F}: Y \to [-1,0]$ a membership function for all $y \in Y$.

Definition 2.5

Let Y be a non-empty set. A N-Cubic set in Y is a structure.

$N^c = \{< y, \hat{\vartheta}_{N^c}(y), \lambda_{N^c}(y) > / y \in Y\}$ is denoted by $N^c =< \hat{\vartheta}_{N^c}, \lambda_{N^c} >$ in that

$\hat{\vartheta}_{N^c} = [\vartheta^-_{N^c}, \vartheta^+_{N^c}]$ is an interval valued fuzzy set and $\lambda_{N^c}: Y \to [-1,0]$ is a fuzzy set in Y.

Definition 2.6

Let Y be a non-empty set. N-Cubic set. A N- Cubic set $N^c = < \hat{\vartheta}_{N^c}, \hat{\lambda}_{N^c} >$ in Y is said to be an internal - N-Cubic set (INCS) if $\vartheta^-_{N^c}(y) \leq \lambda_{N^c} \leq \vartheta^+_{N^c}(y)$ for all y$\in Y$. Similarly, external - N-Cubic set (ENCS) if $\lambda_{N^c} \notin$ if $(\vartheta^-_{N^c}(y), \vartheta^+_{N^c}(y))$.

Definition 2.7

Consider X to be a fixed set, a Pythagorean fuzzy set 'P' in X can be defined as $P_y = \{ (x, \eta_{Py}(x), \theta_{py}(x))/x \in X\}$ where $\eta_{Py}: X \to [0,1]$ represents a degree of membership of an element x $\in X$ and $\theta_{py}(x): X \to [0,1]$ represents the degree of non - non-membership of an element

$x \in X$ satisfying the condition that 0≤ $\eta_{Py}(x) \leq 1$ and 0≤ $\theta_{py}(x) \leq 1$ and

$0 \leq \eta^2_{py}(x) + \theta^2_{py}(x) \leq 1$ for all x$\in X$.

Now the degree of indeterminacy for x to 'P' is given as

$D_p(x) = \sqrt{1 - \eta^2_{py}(x) - \theta^2_{py}(x)}$ *and* $D_p(x)$ satisfies the condition that 0≤ $D_p(x)$ for all $x \in X$.

Definition 2.8

Let X be a fixed set of an interval valued Pythagorean fuzzy set on x is defined as $\tilde{P}_y = \{(x, \tilde{\eta}_{py}(x), \tilde{\theta}_{py}(x)) / x \in X\}$.

Where $\tilde{\eta}_{py}(x) = [\eta^l_{py}(x), \eta^u_{py}(x)] \subset [0,1]$ *and* $\tilde{\theta}_{py}(x) = [\theta^l_{py}(x), \theta^u_{py}(x)] \subset [0,1]$

With $\eta^l_{py}(x) = \inf \eta_{Py}(x)$, $\eta^u_{py}(x) = \sup \eta_{Py}(x)$ like

$\theta^l_{py}(x) = \inf \theta_{py}(x)$, $\theta^u_{py}(x) = \sup \theta_{py}(x)$.

Definition 2.9

Consider X to be a non-empty set. A Cubic Pythagorean fuzzy set of X is a structure of the form

$C_{py} = \{x, \tilde{P}_y(x), P_y(x) \}$ in which $\tilde{P}_y$ is an interval valued Pythagorean fuzzy set in X, P_y is a Pythagorean fuzzy set in X.

Definition 2.10

Consider X to be a fixed set, a N-Pythagorean fuzzy set P^N in X can be defined as

$P^N_y = \{(x, \eta^N_{py}(x), \theta^N_{py}(x)/x \in X\}$

Where $\eta^N_{py}: X \to [-1,0]$ represents degree of membership of an element x $\in X$ and $\theta^N_{py}: X \to [-1,0]$ represents the degree of non-membership of an element x$\in X$ satisfying the condition that $-1 \leq \eta^N_{py}(x) \leq 0$ and $-1 \leq \theta^N_{py}(x) \leq 0$ and $0 \leq \eta^{N^2}_{py}(x) + \theta^{N^2}_{py}(x) \leq 1$ for all x$\in X$.

Now the degree of indeterminacy for x to P is given as $D_{p^N}(x) = \sqrt{1 - \eta^{N2}_{py}(x) - \theta^{N2}_{py}(x)}$ and $D_p(x)$ satisfies the condition that $0 \leq D_{p^N}(x)$ for all $x \in X$.

Definition 2.11

Let X be a fixed set an interval valued N- Pythagorean fuzzy set on X is defined as $\tilde{P}^N_y = \{(x, \tilde{\eta}^N_{py}(x), \tilde{\theta}^N_{py}(x)) / x \in X\}$

Where $\tilde{\eta}^N_{py}(x) = [\eta^{N1}_{py}(x), \eta^{N2}_{py}(x)] \subset [-1,0]$ *and* $\tilde{\theta}^N_{py}(x) = [\theta^{N1}_{py}(x), \theta^{N2}_{py}(x)] \subset [-1,0]$

With $\eta^{N1}_{py}(x)$, $= \inf \tilde{\eta}^N_{py}(x)$, $\eta^{N2}_{py}(x) = \sup \tilde{\eta}^N_{py}(x)$ like

$\theta^{N1}_{py}(x) = \inf \tilde{\theta}^N_{py}(x)$, $\theta^{N2}_{py}(x) = \sup \tilde{\theta}^N_{py}(x)$.

Definition 2.12

Taking X to be a non–empty set. A N-Cubic Pythagorean fuzzy set of X is a structure of the form $C_{py}^{N} = \{x, \tilde{P}_y^N(x), P_y^N(x)/x \in X\}$ $in\ which\ \tilde{P}_y^N$ is an interval valued N-Pythagorean fuzzy set and P_y^N is a N-Pythagorean fuzzy set in X.

Definition 2.13

Let X be a non–empty set. An N-Cubic Pythagorean fuzzy set $C_{py}^{N} = < \tilde{P}_y^N(x), P_y^N(x) >$ in X is said to be an Internal N-Cubic Pythagorean fuzzy set (INCPyFS) if $\tilde{P}_y^{N^-}(x) \le P_y^N(x) \le \tilde{P}_y^{N^+}(x)$ for all $x \in X$.Similarly, external N-Cubic Pythagorean fuzzy set (ENCPyFS) if $P_y^N(x) \notin (\tilde{P}_y^{N^-}(x), \tilde{P}_y^{N^+}(x))$.

Definition 2.14

For any $C_{py_t}^{N} = \{(x, \tilde{P}_{y_t}^N(x), P_{y_t}^N(x))/x \in X\}$ where t∈∧ and ∧ is an index set. Now we define

i) $\bigcup_{t\in\wedge^P} C_{py_i}^N = \{< x, \bigcap_{t\in\wedge} \tilde{P}_{y_t}^N(x), \bigcap_{t\in\wedge} P_{y_t}^N(x) /x \in X >\}$ (P-union)

ii) $\bigcap_{t\in\wedge^P} C_{py_i}^N = \{< x, \bigcup_{t\in\wedge} \tilde{P}_{y_t}^N(x), \bigcup_{t\in\wedge} P_{y_t}^N(x) /x \in X >\}$ (P-intersection)

iii) $\bigcup_{t\in\wedge^R} C_{py_i}^N = \{< x, \bigcap_{t\in\wedge} \tilde{P}_{y_t}^N(x), \bigcup_{t\in\wedge} P_{y_t}^N(x) /x \in X >\}$ (R-union)

iv) $\bigcap_{t\in\wedge^R} C_{py_i}^N = \{< x, \bigcup_{t\in\wedge} \tilde{P}_{y_t}^N(x), \bigcap_{t\in\wedge} P_{y_t}^N(x) /x \in X >\}$ (R-intersection).

3. RESULTS

3.1 N-Cubic Pythagorean Fuzzy Linear Spaces

Definition 3.1

For a non-empty linear space $\mathcal{L}$ over a field F an N-Pythagorean fuzzy set $P_y^N = \{x, \eta_{py}^N(x), \theta_{py}^N(x)/x \in X\}$ is said to be an N-Pythagorean fuzzy linear space $\mathcal{L}_{py}^N = \{\mathcal{L}_N, \eta_{py}^N, \theta_{py}^N\}$ where $\eta_{py}^N: \mathcal{L}_N \to [-1,0]$ and $\theta_{py}^N : \mathcal{L}_N \to [-1,0]$ and also satisfies the following conditions

$\mathcal{L}_{py}^N(\gamma s_1 * \delta s_2) \le \mathcal{L}_{py}^N(s_1) \cup \mathcal{L}_{py}^N(s_2)$ for any $s_1, s_2 \in \mathcal{L}$ and $\gamma, \delta \in F$.

Definition 3.2

An interval valued N-Pythagorean fuzzy set $\tilde{P}_y^N = < \tilde{\eta}_{py}^N, \tilde{\theta}_{py}^N >$ on X is said to be an Interval valued N-Pythagorean fuzzy linear space denoted by $\tilde{\mathcal{L}}_{py}^N$: $\{\mathcal{L}_N, \tilde{\eta}_{py}^N, \tilde{\theta}_{py}^N\}$ over a field F if the following conditions are satisfied $\tilde{\mathcal{L}}_{py}^N(\gamma s_1 * \delta s_2) \le \max\{\tilde{\mathcal{L}}_{py}^N(s_1), \tilde{\mathcal{L}}_{py}^N(s_2)\}$ for any $s_1, s_2 \in \mathcal{L}$ and $\gamma, \delta \in F$.

Definition 3.3

For a linear space $\mathcal{L}$ over a field F a N-Cubic Pythagorean fuzzy set $C_{py}^N = < \tilde{P}_y^N, P_y^N >$ is said to be N-Cubic Pythagorean fuzzy linear space of $\mathcal{L}$ if

i) $\tilde{P}_y^N(\gamma s_1 * \delta s_2) \le \max\{\tilde{P}_y^N(s_1), \tilde{P}_y^N(s_2)\}$

ii) $P_y^N(\gamma s_1 * \delta s_2) \ge \min\{P_y^N((s_1), P_y^N(s_2)\}$

for any $s_1, s_2 \in \mathcal{L}$ and $\gamma, \delta \in F$.

Example 3.1

Let us consider a Numerical example for N-Cubic Pythagorean fuzzy set. Consider Y to be a non -empty Universal set consider the values tabulated below

Table 1 Values of an interval valued N-Pythagorean fuzzy set and N-Pythagorean fuzzy sets

Y	$\tilde{P}_y^N$	P_y^N
y_1	$([-0.6, -0.5], [-0.8, -0.6])$	$[-0.21, -0.09]$

y_2	([−0.5, −0.4], [−0.7, −0.5])	[−0.6, −0.5]
y_3	([−0.8, −0.6], [−0.8, −0.6])	[−0.7, −0.6]

Here $\tilde{P}_y^N$ is an interval valued N- Pythagorean fuzzy linear space and P_y^N is a N- Pythagorean fuzzy linear space of Y over the field GF (2) with the binary operation $s_2 * s_3 = s_1$ and $\gamma = \delta = 1$ from the above conditions, we observed that $[-0.8, -0.6] \leq [-0.7, -0.5]$,

$[-0.6, -0.5] \leq [-0.5, -0.4]$ and $[-0.21, -0.09] \geq [-0.7, -0.6]$ which is sensical. The above example satisfied the condition required for it to be an interval valued N- Pythagorean fuzzy linear space and N- Pythagorean fuzzy linear space.

Therefore, the above example satisfied the conditions required for the N-Cubic Pythagorean fuzzy set to be a N-Cubic Pythagorean fuzzy linear space.

Definition 3.4

Let $\tilde{P}_{y_1}^N = (\tilde{\eta}_{py_1}^N, \tilde{\theta}_{py_1}^N)$ and $\tilde{P}_{y_2}^N = (\tilde{\eta}_{py_2}^N, \tilde{\theta}_{py_2}^N)$ are two interval valued N- Pythagorean fuzzy linear spaces then the union and intersection of two interval valued N- Pythagorean fuzzy linear spaces can be defined as

$$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(x) = \min\{\tilde{P}_{y_1}^N(x), \tilde{P}_{y_2}^N(x)\}$$

$$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(x) = \max\{\tilde{P}_{y_1}^N(x), \tilde{P}_{y_2}^N(x)\}, x \in X.$$

Definition 3.5

Let $P_{y_1}^N = (\eta_{py_1}^N, \theta_{py_1}^N)$, $P_{y_2}^N = (\eta_{py_2}^N, \theta_{py_2}^N)$ be two N- Pythagorean fuzzy linear spaces then their union and intersection can be defined as

$$(P_{y_1}^N \cap P_{y_2}^N)(x) = \min\{P_{y_1}^N(x), P_{y_2}^N(x)\}$$

$$(P_{y_1}^N \cup P_{y_2}^N)(x) = \max\{P_{y_1}^N(x), P_{y_2}^N(x)\}, \ x \in X.$$

Theorem 3.1

Let $C_{py_1}^N = \{\tilde{P}_{y_1}^N, P_{y_1}^N\}$ and $C_{py_2}^N = \{\tilde{P}_{y_2}^N, P_2^N\}$ are two N-Cubic Pythagorean fuzzy linear spaces then their R-intersection $(C_{py_1}^N \cap C_{py_2}^N)_R = (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N, P_{y_1}^N \cap P_{y_2}^N)$ is again an N-Cubic Pythagorean fuzzy linear space.

Proof: We have $(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(x) = \max\{\tilde{P}_{y_1}^N(x), \tilde{P}_{y_2}^N(x)\}$ for all $x \in X$

Consider $(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(\gamma s_1 * \delta s_2) = \max\{\tilde{P}_{y_1}^N(\gamma s_1 * \delta s_2), \tilde{P}_{y_2}^N(\gamma s_1 * \delta s_2)\}$ for any $s_1, s_2 \in \mathcal{L}$ and $\gamma, \delta \in F$.

From the definition 3.3 we have

$$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(\gamma s_1 * \delta s_2) \leq \max\{\max\{\tilde{P}_{y_1}^N(s_1), \tilde{P}_{y_1}^N(s_2)\}, \max\{\tilde{P}_{y_2}^N(s_1), \tilde{P}_{y_2}^N(s_2)\}\}$$

$$= \max\{\max\{\tilde{P}_{y_1}^N(s_1), \tilde{P}_{y_1}^N(s_2)\}, \max\{\tilde{P}_{y_2}^N(s_1), \tilde{P}_{y_2}^N(s_2)\}\}$$

$$= \max\{(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_1), (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_2)\}$$

which imply that

$$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(\gamma s_1 * \delta s_2) \leq \max\{(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_1), (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_2)\}$$

Therefore $\bigcup_{i \in \Lambda} \tilde{P}_{y_i}^N$ is an interval valued N-Pythagorean fuzzy linear space.

Since $(P_{y_1}^N \cap P_{y_2}^N)(x) = \min\{P_{y_1}^N(x), P_{y_2}^N(x)\}$ for all $\in X$

$$(P_{y_1}^N \cap P_{y_2}^N))(\gamma s_1 * \delta s_2) = \min\{P_{y_1}^N(\gamma s_1 * \delta s_2), P_{y_2}^N(\gamma s_1 * \delta s_2)\}$$

From the definition 3.3 we have

$(P_{y_1}^N \cap P_{y_2}^N)(\gamma s_1 * \delta s_2) \geq \min \{\min\{P_{y_1}^N(s_1), P_{y_1}^N(s_2)\}, min\{P_{y_2}^N(s_1), P_{y_2}^N(s_2)\}$

$= \min \{\min\{P_{y_1}^N(s_1), P_{y_1}^N(s_2)\}, \min\{\{P_{y_2}^N(s_1), P_{y_2}^N(s_2)\}$

$= \min \{(P_{y_1}^N \cap P_{y_2}^N)(s_1), (P_{y_1}^N \cap P_{y_2}^N)(s_2)\}$

which imply that

$(P_{y_1}^N \cap P_{y_2}^N)(\gamma s_1 * \delta s_2) \geq \min \{(P_{y_1}^N \cap P_{y_2}^N)(s_1), (P_{y_1}^N \cap P_{y_2}^N)(s_2)\}$

Therefore, $\underset{i\in\Lambda}{\cap} P_{y_i}^N$ is N- Pythagorean fuzzy linear space.

From the above conditions required for the R-intersection to be a N-Cubic Pythagorean fuzzy linear space are satisfied.

Remark

By taking an example we prove that the intersection of two interval valued N – Pythagorean fuzzy linear space doesn't satisfy the first condition of N – Cubic Pythagorean fuzzy linear space.

Remark 3.1

In this remark, we try to prove with the example that the intersection of two interval-valued N- Pythagorean fuzzy linear spaces need not be an interval-valued N- Pythagorean fuzzy linear space.

Example 3.2

Let us consider two an interval valued N-Pythagorean fuzzy sets and N-Pythagorean fuzzy sets tabulated below

Table 2 Values of an interval valued N-Pythagorean fuzzy set and N-Pythagorean fuzzy set

$\mathcal{L}$	$\tilde{P}_y^N$	P_y^N
s_1	$([-0.5, -0.3], [-0.62, -0.4])$	$[-0.78, -0.55]$
s_2	$([-0.3, -0.1], [-0.40, -0.23])$	$[-0.86, -0.79]$
s_3	$([-0.8, -0.6], [-0.75, -0.65])$	$[-0.75, -0.65]$

Table 3 Values of an interval valued N-Pythagorean fuzzy set and N-Pythagorean fuzzy set

$\mathcal{L}$	$\tilde{P}_y^N$	P_y^N
s_1	$([-0.6, -0.5], [-0.52, -0.32])$	$[-0.80, -0.60]$
s_2	$([-0.7, -0.6], [-0.63, -0.31])$	$[-0.65, -0.43]$
s_3	$([-0.35, -0.25], [-0.68, -0.48])$	$[-0.62, -0.3]$

We note that $\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N$ is an interval valued N-Pythagorean fuzzy set in $\mathcal{L}$ for $\gamma = 1, \delta = 1$ in definition 3.3 we have

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_2 * s_3) \leq \max \{\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N(s_2), \tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N(s_3)\}$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) \leq \max \{\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N(s_2), \tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N(s_3)\}$

$\leq \max \{[-0.7, -0.6]\ [-0.63, -0.31], [-0.8, -0.6][-0.75, 0.65]\}$

$\leq \{[-0.7, -0.6], [-0.63, -0.31]\}$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) = \{[-0.6, -0.5]\ [-0.62, -0.4]\} \leq \{[-0.7, -0.6], [-0.63, -0.31]\}$

Since $[-0.6, -0.5] \leq [-0.7, -0.6]$

$[-0.62, -0.4] \leq [-0.63, -0.31]$ which is non-sensical.

Therefore, we conclude that the intersection of two interval valued N-Pythagorean fuzzy linear space need not be an interval valued N-Pythagorean fuzzy linear space.

Remark 3.2

Similarly, in this remark, we provide an example to show that the union of N-Pythagorean fuzzy linear space need not satisfy the second condition of N-Cubic Pythagorean fuzzy linear space as in definition 3.3.

Example 3.3

We note that $(P_{y_1}^N \cup P_{y_2}^N)$ is N-Pythagorean fuzzy set in $\mathcal{L}$

From definition 3.3 we have

$(P_{y_1}^N \cup P_{y_2}^N)(s_2 * s_3) \geq \min \{(P_{y_1}^N \cup P_{y_2}^N)(s_2), (P_{y_1}^N \cup P_{y_2}^N)(s_3)\}$

$(P_{y_1}^N \cup P_{y_2}^N)(s_1) \geq \min \{(P_{y_1}^N \cup P_{y_2}^N)(s_2), (P_{y_1}^N \cup P_{y_2}^N)(s_3)\}$

$\geq \min \{[-0.65, -0.43], [-0.62, -0.3]\}$

$(P_{y_1}^N \cup P_{y_2}^N)(s_1) = [-0.78, -0.58] \geq [-0.65, -0.43]$ which is incorrect.

Therefore, from the above example, it is clear that the union of N-Pythagorean fuzzy linear space need not be an N-Pythagorean fuzzy linear space.

Lemma 3.1

From the above theorem and examples, the following statements can be provided

i) Let $C_{py_1}^N = \{ \tilde{P}_{y_1}^N, P_{y_1}^N \}$ and $C_{py_2}^N = \{ \tilde{P}_{y_2}^N, P_2^N \}$ be two N-Cubic Pythagorean fuzzy linear spaces. Then their R-union $(C_{py_1}^N \cup C_{py_2}^N)_R = (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N, P_{y_1}^N \cup P_{y_2}^N)$ need not be a N-Cubic Pythagorean fuzzy linear space.

ii) Let $C_{py_1}^N = \{ \tilde{P}_{y_1}^N, P_{y_1}^N \}$ and $C_{py_2}^N = \{ \tilde{P}_{y_2}^N, P_2^N \}$ be two N-Cubic Pythagorean fuzzy linear spaces. Then their P-union $(C_{py_1}^N \cup C_{py_2}^N)_P = (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N, P_{y_1}^N \cap P_{y_2}^N)$ need not be a N-Cubic Pythagorean fuzzy linear space.

iii) Let $C_{py_1}^N = \{ \tilde{P}_{y_1}^N, P_{y_1}^N \}$ and $C_{py_2}^N = \{ \tilde{P}_{y_2}^N, P_2^N \}$ be two N-Cubic Pythagorean fuzzy linear spaces. Then their P-intersection $(C_{py_1}^N \cup C_{py_2}^N)_P = (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N, P_{y_1}^N \cup P_{y_2}^N)$ need not be a N-Cubic Pythagorean fuzzy linear space.

Proof:

i) Example 3.2 and 3.3 we can conclude that the R-union of two N-Cubic Pythagorean fuzzy linear spaces need not again be a N-Cubic Pythagorean fuzzy linear space.

ii) Consider the N-Pythagorean fuzzy linear spaces $P_{y_1}^N$ and P_2^N as in the table 2 and table 3.

Now let's consider the intersection of the N-Pythagorean fuzzy linear spaces we have

$(P_{y_1}^N \cap P_{y_2}^N)(s_1) = [-0.80, -0.6]$

$(P_{y_1}^N \cap P_{y_2}^N)(s_2) = [-0.86, -0.79]$

$(P_{y_1}^N \cap P_{y_2}^N)(s_3) = [-0.75, -0.65]$

For $\gamma = 1, \delta = 1$ in definition 3.3 we have

$(P_{y_1}^N \cap P_{y_2}^N)(s_2 * s_3) \geq \min\{ P_{y_1}^N \cap P_{y_2}^N(s_2), P_{y_1}^N \cap P_{y_2}^N(s_3)\}$

$(P_{y_1}^N \cap P_{y_2}^N)(s_1) \geq \min\{ P_{y_1}^N \cap P_{y_2}^N(s_2), P_{y_1}^N \cap P_{y_2}^N(s_3)\}$

$\geq \min\{ [-0.86, -0.79], [-0.75, -0.65]\}$

$(P_{y_1}^N \cap P_{y_2}^N)(s_1) = [-0.80, -0.6] \geq [-0.86, -0.79]$

which is appropriate and satisfies the second condition of N-Cubic Pythagorean fuzzy linear space. But from the example 3.2 ($\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N$) is not an interval valued N-Pythagorean fuzzy linear space.

Therefore, the P-union is not a N-Cubic Pythagorean fuzzy linear space.

iii) Again, consider the interval -valued N-Pythagorean fuzzy linear space $\tilde{P}_{y_1}^N$ *and* $\tilde{P}_{y_2}^N$ as in Table 2 and Table 3.

Now let us consider the union of interval valued N-Pythagorean fuzzy linear spaces we have

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_1) = ([-0.5, -0.3], [-0.52, -0.32])$

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_2) = ([-0.3, -0.1], [-0.40, -0.23])$

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_3) = ([-0.35, -0.25], [-0.68, -0.48])$

For $\gamma = 1, \delta = 1$ in definition 3.3 we have

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_2 * s_3) \leq \max\{\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_2), (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_3)\}$

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_1) \leq \max\{\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_2), (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_3)\}$

$\leq \max\{[[-0.3, -0.1]\ [-0.40, -0.23], [-0.35, -0.25]\ [-0.68, -0.48]\}$

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_1) = [-0.5, -0.3]\ [-0.52, -0.32] \leq \{[-0.3, -0.1], [-0.40, -0.23]\}$

$[-0.5, -0.3] \leq [-0.3, -0.1]$

$[-0.52, -0.32] \leq [-0.40, -0.23]$

which is appropriate and satisfies the first condition of N-Cubic Pythagorean fuzzy linear space.

But from example 3.3, $P_{y_1}^N \cup P_{y_2}^N$ is not a N-Pythagorean fuzzy linear space.

Hence, P-intersection is not a N-Cubic Pythagorean fuzzy linear space.

4.1 Internal and External N-Cubic Pythagorean Fuzzy linear spaces

Definition 4.1

A N-Cubic Pythagorean fuzzy sets $C_{py}^N = \{\tilde{P}_y^N, P_y^N\}$ in a linear space $\mathcal{L}$ over a field F is said to be an internal N-Cubic Pythagorean fuzzy linear space (INCPyFLS) if

$\tilde{P}_y^{N^-}(\gamma s_1 * \delta s_2) \leq P_y^N(\gamma s_1 * \delta s_2) \leq \tilde{P}_y^{N^+}(\gamma s_1 * \delta s_2)$ for all $s_1, s_2 \in \mathcal{L}$ and $\gamma, \delta \in F$.

Example 4.1

Let us consider the values of an interval valued N-Pythagorean fuzzy set and N-Pythagorean fuzzy set as in Table 3. Now for $\gamma = \delta = 1$ and $s_1 * s_3 = s_2$ in definition 4.1 we have

$\tilde{P}_y^{N^-}(s_1 * s_3) \leq P_y^N(s_1 * s_3) \leq \tilde{P}_y^{N^+}(s_1 * s_3)$

$\tilde{P}_y^{N^-}(s_2) \leq P_y^N(s_2) \leq \tilde{P}_y^{N^+}(s_2)$

$-0.65 \in [-0.7, -0.6], -0.43 \in [-0.63, -0.31]$

Hence, $C_{py}^N = \{\tilde{P}_y^N, P_y^N\}$ is an internal N-Cubic Pythagorean fuzzy linear space.

Definition 4.2

A N-Cubic Pythagorean fuzzy sets $C_{py}^N = \{\tilde{P}_y^N, P_y^N\}$ in a linear space $\mathcal{L}$ over a field, F is said to be an External N-Cubic Pythagorean fuzzy linear space (ENCPyFLS) if

$P_y^N(\gamma s_1 * \delta s_2) \notin ((\tilde{P}_y^{N^-}(\gamma s_1 * \delta s_2), \tilde{P}_y^{N^+}(\gamma s_1 * \delta s_2))$ for all $s_1, s_2 \in \mathcal{L}$ and $\gamma, \delta \in F$.

Example 4.2

Let us consider the values of an interval valued N-Pythagorean fuzzy set and N-Pythagorean fuzzy set as in Table 2. Now for $\gamma = \delta = 1$ and $s_1 * s_3 = s_2$ in definition 4.2 we have

$P_y^N(s_1 * s_3) \notin ((\tilde{P}_y^{N^-}(s_1 * s_3), \tilde{P}_y^{N^+}(s_1 * s_3))$

$P_y^N(s_2) \notin ((\tilde{P}_y^{N^-}(s_2), \tilde{P}_y^{N^+}(s_2))$

$-0.86 \notin [-0.3, -0.1], -0.79 \notin [-0.40, -0.23]$

Hence $C_{py}^N = \{\tilde{P}_y^N, P_y^N\}$ is an External N-Cubic Pythagorean Fuzzy linear space.

Proposition 4.1

Let $\mathrm{C}^{\mathrm{N}}_{\mathrm{py_1 I}} = \{\tilde{P}_{y_1}^N, P_{y_1}^N\}$ and $\mathrm{C}^{\mathrm{N}}_{\mathrm{py_2 I}} = \{\tilde{P}_{y_2}^N, P_{y_2}^N\}$ be two INCPyFLS then their R-intersection

$(\mathrm{C}^{\mathrm{N}}_{\mathrm{py_1 I}} \cap \mathrm{C}^{\mathrm{N}}_{\mathrm{py_2 I}})_{\mathrm{R}} = (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N, P_{y_1}^N \cap P_{y_2}^N)$ is an INCPyFLS.

Proof: Since $\mathrm{C}^{\mathrm{N}}_{\mathrm{py_1 I}}$ and $\mathrm{C}^{\mathrm{N}}_{\mathrm{py_2 I}}$ are INCPyFLS in $\mathcal{L}$, we have

$\tilde{P}_{y_1}^{N^-}(\gamma s_1 * \delta s_2) \leq P_{y_1}^N(\gamma s_1 * \delta s_2) \leq \tilde{P}_{y_1}^{N^+}(\gamma s_1 * \delta s_2)$

$\tilde{P}_{y_2}^{N^-}(\gamma s_1 * \delta s_2) \leq P_{y_2}^N(\gamma s_1 * \delta s_2) \leq \tilde{P}_{y_2}^{N^+}(\gamma s_1 * \delta s_2)$

For all $s_1, s_2 \in \mathcal{L}$ and $\gamma, \delta \in F$.

Now, from theorem 3.1, the union of interval-valued N-Pythagorean fuzzy linear space is again an interval-valued N-Pythagorean fuzzy linear space, and the intersection of N-Pythagorean fuzzy linear space is again N-Pythagorean fuzzy linear space.

We have

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)^- (\gamma s_1 * \delta s_2) \leq (P_{y_1}^N \cap P_{y_2}^N)(\gamma s_1 * \delta s_2) \leq (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)^+ (\gamma s_1 * \delta s_2)$

Hence $(\mathrm{C}^{\mathrm{N}}_{\mathrm{py_1 I}} \cap \mathrm{C}^{\mathrm{N}}_{\mathrm{py_2 I}})_{\mathrm{R}} = (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N, P_{y_1}^N \cap P_{y_2}^N)$ is an INCPyFLS.

Proposition 4.2

Let $\mathrm{C}^{\mathrm{N}}_{\mathrm{py_1 E}} = \{\tilde{P}_{y_1}^N, P_{y_1}^N\}$ and $\mathrm{C}^{\mathrm{N}}_{\mathrm{py_2 E}} = \{\tilde{P}_{y_2}^N, P_2^N\}$ be two ENCPyFLS then their R-intersection

$(\mathrm{C}^{\mathrm{N}}_{\mathrm{py_1 E}} \cap \mathrm{C}^{\mathrm{N}}_{\mathrm{py_2 E}})_{\mathrm{R}} = (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N, P_{y_1}^N \cap P_{y_2}^N)$ is an ENCPyFLS.

Proof:

Since $\mathrm{C}^{\mathrm{N}}_{\mathrm{py_1 E}}$ and $\mathrm{C}^{\mathrm{N}}_{\mathrm{py_2 E}}$ are ENCPyFLS in $\mathcal{L}$ we have

$$P_{y_1}^N(\gamma s_1 * \delta s_2) \notin ((\tilde{P}_{y_1}^{N^-}(\gamma s_1 * \delta s_2), \tilde{P}_{y_1}^{N^+}(\gamma s_1 * \delta s_2))$$

$$P_{y_2}^N(\gamma s_1 * \delta s_2) \notin ((\tilde{P}_{y_2}^{N^-}(\gamma s_1 * \delta s_2), \tilde{P}_{y_2}^{N^+}(\gamma s_1 * \delta s_2))$$

Now since the union of interval valued N-Pythagorean fuzzy linear space is again an interval valued N-Pythagorean fuzzy linear space and the intersection of N-Pythagorean fuzzy linear space is again N-Pythagorean fuzzy linear space. We have

$(P_{y_1}^N \cap P_{y_2}^N)(\gamma s_1 * \delta s_2) \notin ((\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)^- (\gamma s_1 * \delta s_2), (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)^+ (\gamma s_1 * \delta s_2))$

$(\mathrm{C}^{\mathrm{N}}_{\mathrm{py_1 E}} \cap \mathrm{C}^{\mathrm{N}}_{\mathrm{py_2 E}})_{\mathrm{R}} = (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N, P_{y_1}^N \cap P_{y_2}^N)$ is an ENCPyFLS.

Proposition 4.3

Let $C^N_{py_1 I} = \{ \tilde{P}^N_{y_1}, P^N_{y_1} \}$ and $C^N_{py_2 I} = \{ \tilde{P}^N_{y_2}, P^N_{y_2} \}$ be two INCPyFLS then their P-intersection

$(C^N_{py_1 I} \cap C^N_{py_2 I})_P = (\tilde{P}^N_{y_1} \cup \tilde{P}^N_{y_2}, P^N_{y_1} \cup P^N_{y_2})$ need not be an INCPyFLS.

Proof:

Let us consider the values of interval valued N-Pythagorean fuzzy linear space and N-Pythagorean fuzzy linear space as shown in table 4 and table 5.

Table 4 Values of interval valued N-Pythagorean fuzzy sets and N-Pythagorean fuzzy sets

$\mathcal{L}$	$\tilde{P}^N_y$	P^N_y
s_1	([−0.7, −0.5], [−0.65, −0.50])	[−0.68, −0.55]
s_2	([−0.8, −0.52], [−0.75, −0.55])	[−0.72, −0.55]
s_3	([−0.51, −0.39], [−0.55, −0.35])	[−0.49, −0.39]

Table 5 Values of interval valued N-Pythagorean fuzzy sets and N- Pythagorean fuzzy sets

$\mathcal{L}$	$\tilde{P}^N_y$	P^N_y
s_1	([−0.65, −0.4], [−0.51, −0.29])	[−0.65, −0.49]
s_2	([−0.75, −0.5], [−0.63, −0.43])	[−0.53, −0.43]
s_3	([−0.65, −0.50], [−0.45, −0.20])	[−0.6, −0.45]

$(\tilde{P}^N_{y_1} \cup \tilde{P}^N_{y_2})(s_1) = ([-0.65, -0.4][-0.51, -0.29])$

$(\tilde{P}^N_{y_1} \cup \tilde{P}^N_{y_2})(s_2) = ([-0.7, -0.5][-0.63, -0.43])$

$(\tilde{P}^N_{y_1} \cup \tilde{P}^N_{y_2})(s_3) = ([-0.51, -0.39][-0.45, -0.20])$

$\gamma = \delta = 1$ in 3.3 we have

$(\tilde{P}^N_{y_1} \cup \tilde{P}^N_{y_2})(s_2 * s_3) \leq \max \{ (\tilde{P}^N_{y_1} \cup \tilde{P}^N_{y_2})(s_2), (\tilde{P}^N_{y_1} \cup \tilde{P}^N_{y_2})(s_3)\}$

$(\tilde{P}^N_{y_1} \cup \tilde{P}^N_{y_2})(s_1) \leq \max \{ (\tilde{P}^N_{y_1} \cup \tilde{P}^N_{y_2})(s_2), (\tilde{P}^N_{y_1} \cup \tilde{P}^N_{y_2})(s_3)\}$

$(\tilde{P}^N_{y_1} \cup \tilde{P}^N_{y_2})(s_1) \leq \max \{[-0.7, -0.5][-0.63, -0.43], [-0.51, -0.39][-0.45, -0.20]\}$

$\leq \{ [-0.51, -0.39], [-0.45, -0.20]\}$

$(\tilde{P}^N_{y_1} \cup \tilde{P}^N_{y_2})(s_1)$ = [-0.65, -0.4] [-0.51, -0.29] ≤ $\{ [-0.51, -0.39], [-0.45, -0.20]\}$

[-0.65, -0.4] ≤ $[-0.51, -0.39]$

[-0.51, -0.29] ≤ $[-0.45, -0.20]$ which is correct.

$(P^N_{y_1} \cup P^N_{y_2})(s_1) = [-0.65, -0.49]$

$(P^N_{y_1} \cup P^N_{y_2})(s_2) = [-0.53, -0.43]$

$(P^N_{y_1} \cup P^N_{y_2})(s_3) = [-0.49, -0.39]$

for $\gamma = \delta = 1$ in 3.3 we have

$(P^N_{y_1} \cup P^N_{y_2})(s_2 * s_3) \geq \min \{(P^N_{y_1} \cup P^N_{y_2})(s_2), (P^N_{y_1} \cup P^N_{y_2})(s_3)\}$

$(P^N_{y_1} \cup P^N_{y_2})(s_1) \geq \min \{(P^N_{y_1} \cup P^N_{y_2})(s_2), (P^N_{y_1} \cup P^N_{y_2})(s_3)\}$

$\geq \min \{[-0.53, -0.43], [-0.49, -0.39]\}$

$\geq [-0.53, -0.43]$

$(P^N_{y_1} \cup P^N_{y_2})(s_1) = [-0.65, -0.49] \geq [-0.53, -0.43]$ which is incorrect.

Hence, the P-intersection of two INCPyFLS need not be an INCPyFLS.

Proposition 4.4

Let $C^N_{py_1 I} = \{ \tilde{P}^N_{y_1}, P^N_{y_1} \}$ and $C^N_{py_2 I} = \{ \tilde{P}^N_{y_2}, P^N_{y_2} \}$ be two INCPyFLS then their P-union

$(C^N_{py_1 I} \cup C^N_{py_2 I})_P = (\tilde{P}^N_{y_1} \cap \tilde{P}^N_{y_2}, P^N_{y_1} \cap P^N_{y_2})$ need not be an INCPyFLS.

Let us consider the values of interval valued N-Pythagorean fuzzy linear space and N-Pythagorean fuzzy linear space as shown in Table **4** and also values tabulated below

Table 6 Values of interval valued N-Pythagorean fuzzy sets and N- Pythagorean fuzzy sets

$\mathcal{L}$	$\tilde{P}^N_y$	P^N_y
s_1	([−0.80, −0.5], [−0.62, −0.40])	[−0.8, −0.60]
s_2	([−0.6, −0.5], [−0.55, −0.30])	[−0.55, −0.42]
s_3	([−0.42, −0.32], [−0.35, −0.25])	[−0.4, −0.28]

$(\tilde{P}^N_{y_1} \cap \tilde{P}^N_{y_2})(s_1) = ([-0.80, -0.5] [-0.65, -0.50])$

$(\tilde{P}^N_{y_1} \cap \tilde{P}^N_{y_2})(s_2) = ([-0.8, -0.52] [-0.75, -0.65])$

$(\tilde{P}^N_{y_1} \cap \tilde{P}^N_{y_2})(s_3) = ([-0.51, -0.39] [-0.55, -0.35])$

for $\gamma = \delta = 1$ in 3.3 we have

$(\tilde{P}^N_{y_1} \cap \tilde{P}^N_{y_2})(s_2 * s_3) \leq \max \{ (\tilde{P}^N_{y_1} \cap \tilde{P}^N_{y_2})(s_2), (\tilde{P}^N_{y_1} \cap \tilde{P}^N_{y_2})(s_3)\}$

$(\tilde{P}^N_{y_1} \cap \tilde{P}^N_{y_2})(s_1) \leq \max \{ (\tilde{P}^N_{y_1} \cap \tilde{P}^N_{y_2})(s_2), (\tilde{P}^N_{y_1} \cap \tilde{P}^N_{y_2})(s_3)\}$

$(\tilde{P}^N_{y_1} \cap \tilde{P}^N_{y_2})(s_1) \leq \max\{[-0.8, -0.52] [-0.75, -0.65], [-0.51, -0.39] [-0.55, -0.35])$

$(\tilde{P}^N_{y_1} \cap \tilde{P}^N_{y_2})(s_1) \leq \{[-0.51, -0.39] [-0.55, -0.35]\}$

$(\tilde{P}^N_{y_1} \cap \tilde{P}^N_{y_2})(s_1) = [-0.80, -0.5] [-0.65, -0.50]) \leq \{[-0.51, -0.39] [-0.55, -0.35]\}$

$[-0.80, -0.5] \leq [-0.51, -0.39]$

$[-0.65, -0.50] \leq [-0.55, -0.35]$ which is correct.

$(P^N_{y_1} \cap P^N_{y_2})(s_1) = [-0.80, -0.6]$

$(P^N_{y_1} \cap P^N_{y_2})(s_2) = [-0.72, -0.55]$

$(P^N_{y_1} \cap P^N_{y_2})(s_3) = [-0.49, -0.39]$

for $\gamma = \delta = 1$ in 3.3 we have

$(P^N_{y_1} \cap P^N_{y_2})(s_2 * s_3) \geq \min\{ P^N_{y_1} \cap P^N_{y_2}(s_2), P^N_{y_1} \cap P^N_{y_2}(s_3)\}$

$(P^N_{y_1} \cap P^N_{y_2})(s_1) \geq \min\{ P^N_{y_1} \cap P^N_{y_2}(s_2), P^N_{y_1} \cap P^N_{y_2}(s_3)\}$

$\geq \min\{[-0.72,-0.55],[-0.49,-0.39]\}$

$\geq [-0.72,-0.55]$

$(P_{y_1}^N \cap P_{y_2}^N)(s_1) = [-0.80,-0.6] \geq [-0.72,-0.55]$ which is incorrect.

Therefore, P-union of two INCPyFLS need not be INCPyFLS.

Proposition 4.5

Let $C_{py_1 I}^N = \{\tilde{P}_{y_1}^N, P_{y_1}^N\}$ and $C_{py_2 I}^N = \{\tilde{P}_{y_2}^N, P_{y_2}^N\}$ be two INCPyFLS then their R-union

$(C_{py_1 I}^N \cup C_{py_2 I}^N)_R = (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N, P_{y_1}^N \cup P_{y_2}^N)$ need not be an INCPyFLS.

Let us consider the values of interval valued N-Pythagorean fuzzy linear space and N-Pythagorean fuzzy linear space as shown in Table 5 and also values tabulated below

Table 7

Values of interval valued N-Pythagorean fuzzy sets and N-Pythagorean fuzzy sets

$\mathcal{L}$	$\tilde{P}_y^N$	P_y^N
s_1	([−0.56,−0.35], [−0.32,−0.10])	[−0.4, −0.20]
s_2	([−0.68,−0.42], [−0.54,−0.40])	[−0.66, −0.47]
s_3	([−0.82,−0.52], [−0.56,−0.31])	[−0.76, −0.56]

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) = ([-0.65,-0.4]\ [-0.51,-0.29])$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_2) = ([-0.75,-0.50]\ [-0.63,-0.43])$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_3) = ([-0.82,-0.52]\ [-0.56,-0.31])$

for $\gamma = \delta = 1$ in 3.3 we have

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_2 * s_3) \leq \max\{(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_2), (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_3)\}$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) \leq \max\{(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_2), (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_3)\}$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) \leq \max\{[-0.75,-0.50]\ [-0.63,-0.43],[-0.82,-0.52]\ [-0.56,-0.31])$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) \leq ([-0.75,-0.50],[-0.56,-0.31])$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) = ([-0.65,-0.4]\ [-0.51,-0.29]) \leq ([-0.75,-0.50],[-0.56,-0.31])$

$[-0.65,-0.4] \leq [-0.75,-0.50],$

$[-0.51,-0.29]) \leq [-0.56,-0.31]$ which is incorrect

$(P_{y_1}^N \cup P_{y_2}^N)(s_1) = [-0.4,-0.2]$

$(P_{y_1}^N \cup P_{y_2}^N)(s_2) = [-0.53,-0.43]$

$(P_{y_1}^N \cup P_{y_2}^N)(s_3) = [-0.60,-0.45]$

for $\gamma = \delta = 1$ in 3.3 we have

$(P_{y_1}^N \cup P_{y_2}^N)(s_2 * s_3) \geq \min\{(P_{y_1}^N \cup P_{y_2}^N)(s_2), (P_{y_1}^N \cup P_{y_2}^N)(s_3)\}$

$(P_{y_1}^N \cup P_{y_2}^N)(s_1) \geq \min\{(P_{y_1}^N \cup P_{y_2}^N)(s_2), (P_{y_1}^N \cup P_{y_2}^N)(s_3)\}$

$\geq \min\{[-0.53, -0.43], [-0.60, -0.45]\}$

$\geq [-0.60, -0.45]$

$(P_{y_1}^N \cup P_{y_2}^N)(s_1) = [-0.4, -0.2] \geq [-0.60, -0.45]$ which is correct

Hence, the R-union of two INCPyFLS need not be an INCPyFLS.

Proposition 4.6

Let $C_{py_1E}^N = \{\tilde{P}_{y_1}^N, P_{y_1}^N\}$ and $C_{py_2E}^N = \{\tilde{P}_{y_2}^N, P_2^N\}$ be two ENCPyFLS then their P-intersection

$(C_{py_1E}^N \cap C_{py_2E}^N)_P = (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N, P_{y_1}^N \cup P_{y_2}^N)$ need not be ENCPyFLS.

Proof: Let us consider the values of interval valued N-Pythagorean fuzzy linear space and N-Pythagorean fuzzy linear space as shown in Table **6** and Table **8.**

Table 8 Values of interval valued N-Pythagorean fuzzy sets and N- Pythagorean fuzzy sets

$\mathcal{L}$	$\tilde{P}_y^N$	P_y^N
s_1	([−0.67, −0.55], [−0.64, −0.43])	[−0.70, −0.65]
s_2	([−0.78, −0.5], [−0.52, −0.35])	[−0.49, −0.30]
s_3	([−0.66, −0.44], [−0.49, −0.29])	[−0.72, −0.52]

Table 9 Values of interval valued N-Pythagorean fuzzy sets and N-Pythagorean fuzzy sets

$\mathcal{L}$	$\tilde{P}_y^N$	P_y^N
s_1	([−0.56, −0.3], [−0.42, −0.30])	[−0.61, −0.47]
s_2	([−0.74, −0.47], [−0.31, −0.12])	[−0.40, −0.20]
s_3	([−0.8, −0.49], [−0.68, −0.56])	[−0.37, −0.1]

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_1) = ([-0.56, -0.3][-0.42, -0.3])$

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_2) = ([-0.74, -0.47][-0.31, -0.12])$

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_3) = ([-0.66, -0.44][-0.49, -0.29])$

$\gamma = \delta = 1$ in 3.3 we have

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_2 * s_3) \leq \max\{(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_2), (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_3)\}$

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_1) \leq \max\{(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_2), (\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_3)\}$

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_1) \leq \max\{[-0.74, -0.47][-0.31, -0.12], [-0.66, -0.44][-0.49, -0.29]\}$

$\leq \{[-0.66, -0.44], [-0.31, -0.12]\}$

$(\tilde{P}_{y_1}^N \cup \tilde{P}_{y_2}^N)(s_1) = ([-0.56, -0.3][-0.42, -0.3]) \leq \{[-0.66, -0.44], [-0.31, -0.12]\}$

$[-0.56, -0.3] \leq [-0.66, -0.44],$

$[-0.42, -0.3]) \leq [-0.31, -0.12]$ which is incorrect.

$(P_{y_1}^N \cup P_{y_2}^N)(s_1) = [-0.61, -0.47]$

$(P_{y_1}^N \cup P_{y_2}^N)(s_2) = [-0.40, -0.20]$

$(P_{y_1}^N \cup P_{y_2}^N)(s_3) = [-0.37, -0.1]$

for $\gamma = \delta = 1$ in 3.3 we have

$(P_{y_1}^N \cup P_{y_2}^N)(s_2 * s_3) \geq \min \{(P_{y_1}^N \cup P_{y_2}^N)(s_2), (P_{y_1}^N \cup P_{y_2}^N)(s_3)\}$

$(P_{y_1}^N \cup P_{y_2}^N)(s_1) \geq \min \{(P_{y_1}^N \cup P_{y_2}^N)(s_2), (P_{y_1}^N \cup P_{y_2}^N)(s_3)\}$

$\geq \min \{[-0.40, -0.20], [-0.37, -0.1]\}$

$\geq [-0.40, -0.20]$

$(P_{y_1}^N \cup P_{y_2}^N)(s_1) = [-0.61, -0.47] \geq [-0.40, -0.20]$ which is incorrect.

Hence, the P-intersection of two ENCPyFLS need not be an ENCPyFLS.

Proposition 4.7

Let $C_{py_1 E}^N = \{ \tilde{P}_{y_1}^N, P_{y_1}^N \}$ and $C_{py_2 E}^N = \{ \tilde{P}_{y_2}^N, P_2^N \}$ be two ENCPyFLS then their P-union

$(C_{py_1 E}^N \cup C_{py_2 E}^N)_P = (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N, P_{y_1}^N \cap P_{y_2}^N)$ need not be ENCPyFLS.

Proof: Let us consider the values of interval valued N-Pythagorean fuzzy linear spaces and N-Pythagorean fuzzy linear spaces as shown in Table 8 and values tabulated below

Table 10

Values of interval valued N-Pythagorean fuzzy sets and N-Pythagorean fuzzy sets

$\mathcal{L}$	$\tilde{P}_y^N$	P_y^N
s_1	([−0.79, −0.6], [−0.58, −0.43])	[−0.79, −0.67]
s_2	([−0.64, −0.42], [−0.41, −0.21])	[−0.66, −0.43]
s_3	([−0.75, −0.45], [−0.62, −0.46])	[−0.44, −0.22]

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) = ([-0.79, -0.6]\ [-0.64, -0.43])$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_2) = ([-0.78, -0.50]\ [-0.52, -0.35])$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_3) = ([-0.75, -0.45]\ [-0.62, -0.46])$

for $\gamma = \delta = 1$ in 3.3 we have

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_2 * s_3) \leq \max \{ (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_2), (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_3)\}$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) \leq \max \{ (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_2), (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_3)\}$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) \leq \max\{[-0.78, -0.50]\ [-0.52, -0.35], [-0.75, -0.45]\ [-0.62, -0.46])$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) \leq ([-0.75, -0.45], [-0.52, -0.35])$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) = ([-0.79, -0.6]\ [-0.64, -0.43]) \leq ([-0.75, -0.45], [-0.52, -0.35])$

$[-0.79, -0.6] \leq [-0.75, -0.45],$

$[-0.64, -0.43] \leq [-0.52, -0.3]$ which is correct.

$(P_{y_1}^N \cap P_{y_2}^N)(s_1) = [-0.79, -0.67]$

$(P_{y_1}^N \cap P_{y_2}^N)(s_2) = [-0.66, -0.43]$

$(P_{y_1}^N \cap P_{y_2}^N)(s_3) = [-0.72, -0.52]$

for $\gamma = \delta = 1$ in 3.3 we have

$(P_{y_1}^N \cap P_{y_2}^N)(s_2 * s_3) \geq \min\{ P_{y_1}^N \cap P_{y_2}^N(s_2), P_{y_1}^N \cap P_{y_2}^N(s_3)\}$

$(P_{y_1}^N \cap P_{y_2}^N)(s_1) \geq \min\{ P_{y_1}^N \cap P_{y_2}^N(s_2), P_{y_1}^N \cap P_{y_2}^N(s_3)\}$

$\geq \min\{[-0.66, -0.43], [-0.72, -0.52]\}$

$\geq [-0.72, -0.55]$

$(P_{y_1}^N \cap P_{y_2}^N)(s_1) = [-0.79, -0.67] \geq [-0.72, -0.55]$ which is incorrect.

Therefore, a P-union of two ENCPyFLS need not be ENCPyFLS.

Proposition 4.8

Let $C^N_{py_1E} = \{ \tilde{P}_{y_1}^N, P_{y_1}^N \}$ and $C^N_{py_2E} = \{ \tilde{P}_{y_2}^N, P_2^N \}$ be two ENCPyFLS then their R-union

$(C^N_{py_1E} \cup C^N_{py_2E})_R = (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N, P_{y_1}^N \cup P_{y_2}^N)$ need not be ENCPyFLS.

Proof: Let us consider the values of interval valued N-Pythagorean fuzzy linear space and N-Pythagorean fuzzy linear space as shown in table 9 and table 11.

Table 11

Values of interval valued N-Pythagorean fuzzy sets and N-Pythagorean fuzzy sets

$\mathcal{L}$	$\tilde{P}_y^N$	P_y^N
s_1	([−0.77, −0.64], [−0.55, −0.44])	[−0.80, −0.6]
s_2	([−0.61, −0.41], [−0.52, −0.38])	[−0.35, −0.10]
s_3	([−0.47, −0.23], [−0.57, −0.45])	[−0.2, −0.05]

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) = ([-0.77, -0.64][-0.55, -0.44])$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_2) = ([-0.74, -0.47][-0.52, -0.38])$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_3) = ([-0.8, -0.49][-0.68, -0.56])$

$\gamma = \delta = 1$ in 3.3 we have

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_2 * s_3) \leq \max\{ (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_2), (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_3)\}$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) \leq \max\{ (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_2), (\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_3)\}$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) \leq \max\{[-0.74, -0.47][-0.52, -0.38], [-0.8, -0.49][-0.68, -0.56]\}$

$\leq \{ [-0.74, -0.47], [-0.52, -0.38]\}$

$(\tilde{P}_{y_1}^N \cap \tilde{P}_{y_2}^N)(s_1) = ([-0.77, -0.64][-0.55, -0.44]) \leq \{ [-0.74, -0.47], [-0.52, -0.38]\}$

$[-0.77, -0.64] \leq [-0.74, -0.47]$,

$[-0.55, -0.44] \leq, [-0.52, -0.38]$ which is correct.

$(P_{y_1}^N \cup P_{y_2}^N)(s_1) = [-0.61, -0.47]$

$(P_{y_1}^N \cup P_{y_2}^N)(s_2) = [-0.35, -0.1]$

$(P_{y_1}^N \cup P_{y_2}^N)(s_3) = [-0.2, -0.05]$

for $\gamma = \delta = 1$ in 3.3 we have

$(P_{y_1}^N \cup P_{y_2}^N)(s_2 * s_3) \geq \min\{(P_{y_1}^N \cup P_{y_2}^N)(s_2), (P_{y_1}^N \cup P_{y_2}^N)(s_3)\}$

$(P_{y_1}^N \cup P_{y_2}^N)(s_1) \geq \min\{(P_{y_1}^N \cup P_{y_2}^N)(s_2), (P_{y_1}^N \cup P_{y_2}^N)(s_3)\}$

$\geq \min\{[-0.35, -0.1], [-0.2, -0.05]\}$

$(P_{y_1}^N \cup P_{y_2}^N)(s_1) \geq [-0.35, -0.1]$

$(P_{y_1}^N \cup P_{y_2}^N)(s_1) = [-0.61, -0.47] \geq [-0.35, -0.1]$ which is incorrect.

Hence, the R-union of two ENCPyFLS need not be an ENCPyFLS.

5. Conclusion

In order to extend the idea of N-cubic linear spaces, we introduced the notion of N-Cubic Pythagorean Fuzzy Linear Spaces which also handles the negative features of certain things. The main rationale of this paper is to extend the idea of N-cubic sets to Pythagorean Fuzzy Linear Spaces and discuss in detail two types of N-Cubic Pythagorean Fuzzy Linear Spaces called ENCPyFLS and INCPyFLS with examples. We also discuss the basic operations like P-union (resp. intersection) and R-union (resp. intersection) of N-Cubic Pythagorean Fuzzy Linear Spaces, ENCPyFLS and INCPyFLS. Sooner, different aggregation operators can be dealt with N-Cubic Pythagorean Fuzzy Linear Spaces.

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