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Simultaneous Inverse Approximation To A Laguerre Polynomials and its Derivative by Pál-Type Interpolation

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Abstract:- Using the zeros of the polynomials, the aim of this work is to create an interpolatory polynomial with Laguerre conditions, where $L_n^{(k)}(x)$ is the Laguerre polynomial of degree *n* and whose derivative $L_n^{(k)'}(x)$ is of degree $n-1$. A unified study of a modified inverse Pál-type interpolation issue is conducted. We have proved the explicit interpolation equations and demonstrate the problem's regularity. Additionally, if the inner nodal points are the roots of the interpolatory polynomials and yield an estimate over the whole real number line, the polynomial's existence and uniqueness are demonstrated.

Keywords: *Lagrange interpolation, Laguerre abscissas, Pál-type interpolation, Explicit form, Order of convergence.*

1. Introduction

The Lagrange or Hermite-Fejér interpolation based on the zeros of Laguerre polynomials has been considered in the literature by Szegö [1] and Szabados [2], who studied the uniform convergence of this interpolation process under proper hypothesis on the function.

M. G. de Bruin [4] introduced 'incomplete' Pál type interpolation problem where he omitted one or two real nodes from sets of non-uniformly distributed nodes. In the sequel, a number of problems will be treated where one or two of the zeros are omitted from the set of interpolation points. This type of problem is quite different from the problems where one or more zeros are added to the set of points that are used in the interpolation to the derivative. In his paper the regularity of nine Pál-type interpolation problems is proved where the nodes form a subset of the sets of zeros. He has also studied the regularity of some interpolation problems on non-uniformly distributed nodes on the unit circle.

The study on Lacunary Polynomial Interpolation started with the evolution of Birkhoff interpolation. It is finely honed theory on real nodes. Lacunary Polynomial Interpolation at special nodes received attention after the investigations of P. Turán and his associates. A revolution in theory of polynomial interpolation at special nodes was due to L. G. Pál [3]. He introduced a new type of Lacunary Polynomial Interpolation on zeros of two different polynomials, referred as Pál-type interpolation. L. G Pál has introduced a modification of the Hermite-Fejér interpolation, in which the function values are interpolated at the zeros of the polynomial $w(x)$ in Pál type interpolation, whereas the first derivative values are interpolated at the roots of $w'(x)$. The sole difference between $w(x)$ and $w'(x)$ in inverse Pál-type interpolation is that the derivative values are interpolated at the roots of $w(x)$ and the function values at the roots of $w'(x)$. That is, the derivative of the function at the roots of $w'(x)$ or $w(x)$ is interpolated using the derivative of the interpolational polynomial. This interpolation feature highlighted the topic of what circumstances allow a simultaneous approximation to a differentiable function and its derivative to be obtained using the Pál -type interpolation.

Following Pál's idea many authors [5], [8], [9], [10] researched this kind of interpolation. In 2004, Lénárd [6] investigated the Pál-type interpolation problem on the nodes of Laguerre abscissas. Pál demonstrated that there is no distinct polynomial of degree $\le 2n-2$ when function values are dictated on one set of *n* points and derivatives values on another set of $n-1$ points, but note that there is a unique polynomial having the degree $\le 2n-1$ when function value is defined at one more point that does not belong to the previous collection of *n* points. Srivastava [7] studied special problem of mixed type weighted interpolation on the mixed zeros of Hermite polynomial and its derivative. She has proved the existence, uniqueness and convergence of the theorem.

In this paper we study the following interpolation problem: On the infinite interval $[0,\infty)$ let $\{x_i\}_{i=0}^{n}$ and $\{y_i\}_{i=1}^{n}$ be the arbitrary two sets of inter scaled nodal points:

$$0 \le x_0 < y_1 < x_1 < \ldots < x_{n-1} < y_n < x_n < +\infty . \tag{1}$$

For any fixed integer $k \ge 1$, obtain a least degree polynomial $B_m(x)$ satisfying the $(0;1)$ interpolation conditions:

$$B_m(y_i) = z_i, \quad (i = 1, \ldots, n) \tag{2}$$

$$B_m'(x_i) = z_i' \quad (i = 1, \ldots, n-1) \tag{3}$$

with Hermite-type boundary conditions

$$B_m^{(j)}(x_0) = z_0^{(j)}, \quad (j = 0, \ldots, k), \tag{4}$$

Where z_i, z_i' and $z_0^{(j)}$ are arbitrary real numbers.

Here we prove that, if $\{x_i\}_{i=0}^{n}$ and $\{y_i\}_{i=1}^{n}$ are the zeros of the Laguerre polynomial $L_n^{(k)}(x)$ and its derivative $L_n^{(k)'}(x)$, respectively, and $x_0 = 0$, then there exists a unique polynomial $B_m(x)$ of degree $2n+k-1$ that satisfies the above conditions. The interpolational polynomial $B_m(x)$ is a modified Pál-type interpolational polynomial with $w_{n+k}(x) = x^k L_n^{(k)}(x)$. Here we prove existence, uniqueness and explicit representation of the fundamental polynomials with weight function $w(x) = e^{-x} x^{-k}$.

2. Preliminaries

We have used some well known results of the Laguerre polynomial $L_n^{(k)}(x)$ which are as follows:
The differential equation of the Laguerre polynomial is given by

$$xD^2 L_n^{(k)}(x) + (1+k-x) D L_n^{(k)}(x) + n L_n^{(k)}(x) = 0, \tag{5}$$

where *n* is a positive integer and $k > -1$. For the roots of $L_n^{(k)}(x)$ we have

$$\sqrt{x_j} = \frac{1}{2\sqrt{n}} [\, j\pi + O(1)] \tag{6}$$

$$\left| L_n^{(k)'}(x_j) \right| \sim j^{-k-\frac{3}{2}} n^{k+1}, \quad (0 < x_j \le \Omega, n = 1,2,3,\ldots) \tag{7}$$

$$\left| L_n^{(k)}(x) \right| = \{ x^{-\frac{k}{2}-\frac{1}{4}} O(n^{\frac{k}{2}-\frac{1}{4}}), \quad cn^{-1} \le x \le \Omega$$

$$= \{ O(n^k), \qquad 0 \le x \le cn^{-1} \tag{8}$$

$$O(l_j(x)) = O(l_j^*(x)) = 1, \tag{9}$$

Now we also have some properties of fundamental polynomials of the Lagrange interpolation which are given in Szegö [1] as:

$$l_j(x) = \frac{L_n^{(k)}(x)}{L_n^{(k)'}(x_j)(x-x_j)}, \tag{10}$$

$$l_j^*(x) = \frac{L_n^{(k)'}(x)}{L_n^{(k)''}(y_j)(x-y_j)} \tag{11}$$

$$l_j(x_i) = \delta_{ij}, \ l_j^*(y_i) = \delta_{ij} \tag{12}$$

$$\left|\int_0^x l_j(t)dt\right| = \left|\int_0^x l_j^*(t)dt\right| = O(n^{-1}) \tag{13}$$

$$L_n^{(k)}(x) = L_n^{(k+1)}(x) - L_{n-1}^{(k+1)}(x) \tag{14}$$

Here the degree of the polynomial $l_j(x)$ is $n-1$ and the degree of the polynomial $l_j^*(x)$ is $n-2$.

3. Explicit Representation of Interpolatory Polynomial

Let the inter scaled nodal points be given by (1), where $\{x_i\}_{i=0}^n$ and $\{y_i\}_{i=1}^n$ are the zeros of the Laguerre polynomials $L_n^{(k)}(x)$ and $L_n^{(k)'}(x)$, respectively. Then, for the prescribed numbers $\{z_i\}_{i=1}^n$ and $\{z_i'\}_{i=1}^{n-1}$ there exists a unique polynomial $B_m(x)$ of degree $\le 2n+k-1$ satisfying the conditions (2), (3), and (4). The polynomial $B_m(x)$ is explicitly given by:

$$B_m(x) = \sum_{j=1}^{n-1} z_j U_j(x) + \sum_{j=1}^{n} z_j' V_j(x) + \sum_{j=0}^{k} z_0^{(j)} W_j(x) \tag{15}$$

Where $\{U_j(x)\}_{j=1}^{n-1}$, $\{V_j(x)\}_{j=1}^{n}$, and $\{W_j(x)\}_{j=0}^{k}$ are the polynomials having the degree $\le 2n+k-1$. These polynomials are unique and satisfy the following conditions:
for $j = 1, 2, \ldots, n-1$

$$\begin{aligned} U_j(y_i) &= \delta_{ij}, \quad (i = 1,2,\ldots,n-1) \\ [w(x)U_j(x)]'_{x=x_i} &= 0, \quad (i = 1,2,\ldots,n) \\ U_j^{(l)}(0) &= 0, \quad (l = 0,1,\ldots,k) \end{aligned} \tag{16}$$

for $j = 1, 2, \ldots, n$

$$\begin{aligned} V_j(y_i) &= 0, \quad (i = 1,2,\ldots,n-1) \\ [w(x)V_j(x)]'_{x=x_i} &= \delta_{ij}, \quad (i = 1,2,\ldots,n) \\ V_j^{(l)}(0) &= 0, \quad (l = 0,1,\ldots,k) \end{aligned} \tag{17}$$

for $l = 0, 1, \ldots, k$

$$\begin{aligned} W_k(y_i) &= 0, \quad (i = 1,2,\ldots,n-1) \\ [w(x)W_k(x)]'_{x=x_i} &= 0, \quad (i = 1,2,\ldots,n) \\ W_j^{(l)}(0) &= \delta_{lk}, \quad (l = 0,1,\ldots,k) \end{aligned} \tag{18}$$

Here δ_{ij} is a Kronecker delta,

$$\begin{aligned} \delta_{ij} &= 1, \ i = j \\ \delta_{ij} &= 0, \ i \ne j \end{aligned} \tag{19}$$

The explicit forms of the $U_j(x), V_j(x),$ and $W_j(x)$ are given in the following lemma.

Lemma 1: For *k* and *n* positive integers, on the nodal points (1) the fundamental polynomial $\{U_j(x)\}_{j=1}^{n-1}$ of the interpolational problem in (1)-(4) satisfying the interpolatory condition (16) is given by: for $j=1,2,\ldots,n-1$

$$U_j(x)=\frac{1}{y_j^k L_n^{(k)}(y_j)}\left[x^k l_j^*(x)L_n^{(k)}(x)-\frac{x^k L_n^{(k)'}(x)}{L_n^{(k)''}(y_j)}\int_0^x \frac{tL_n^{(k)'}(t)-x_jL_n^{(k)}(t)}{t-y_j}dt\right] \tag{20}$$

Where $l_j(x)$ is given by (10).

Proof: For $j=1,2,\ldots,n-1$, let

$$U_j^*(x)=u_1\left[x^k l_j^*(x)L_n^{(k)}(x)+u_2x^kL_n^{(k)'}(x)\int_0^x\frac{tL_n^{(k)'}(t)+u_3L_n^{(k)}(t)}{t-y_j}dt\right] \tag{21}$$

be a polynomial of degree $\leq 2n+k-1$. We can easily check that $U_j^*(x)$ satisfies the equation (16) provided

$$u_1=\frac{1}{y_j^kL_n^{(k)}(y_j)} \tag{22}$$

and

$$u_2=\frac{-u_1}{L_n^{(k)''}(y_j)} \tag{23}$$

Note that the $U_j^*(x)$ is a polynomial of minimal degree $2n+k-1$, so the integrand in (20) must be a polynomial which implies $tL_n^{(k)'}(t)+u_3L_n^{(k)}(t)=0$. Thus, by using the equations (14) and (11), we get $u_3=-x_j$. Hence,

$$U_j^*(x)\equiv U_j(x) \tag{24}$$

which completes the proof of the lemma.

Lemma 2: For *k* and *n* positive integers, on the nodal points (1) the fundamental polynomial $\{V_j(x)\}_{j=1}^{n}$ of the interpolational problem in (1)-(4) satisfying the interpolatory condition (17) is given by: for $j=1,2,\ldots,n$,

$$V_j(x)=\frac{e^{x_j}x^kL_n^{(k)'}(x)}{y_j^kL_n^{(k)'}(y_j)}\int_0^x l_j(t)dt \tag{25}$$

Proof: For $j=1,2,\ldots,n$, let

$$V_j^*(x)=v_1x^kL_n^{(k)'}(x)\int_0^x l_j(t)dt \tag{26}$$

be a polynomial of degree $\leq 2n+k-1$. We can easily check that $V_j^*(x)$ satisfies the equation (17) provided

$$v_1=\frac{e^{x_j}}{y_j^kL_n^{(k)'}(y_j)} \tag{27}$$

Thus,

$$V_j^*(x)\equiv V_j(x) \tag{28}$$

which completes the proof of the lemma.

Lemma 3: For *k* and *n* positive integers, on the nodal points (1) the fundamental polynomial $\{W_j(x)\}_{j=0}^{k}$ of the interpolational problem in (1)-(4) satisfying the interpolatory condition (18) is given by: for $j=0,1,\ldots,k-1$

$$W_j(x)=a_j(x)x^jL_n^{(k)}(x)L_n^{(k)'}(x)+x^{k-1}L_n^{(k)'}(x)\left[w_j-\int_0^x\frac{L_n^{(k)'}(t)a_j(t)+b_j(t)L_n^{(k)}(t)}{t^{k-j}}dt\right] \tag{29}$$

$$W_k(x) = \frac{1}{k!L_n^{(k)'}(0)} x^k L_n^{(k)'}(x) \tag{30}$$

where $a_j(x)$ and $b_j(x)$ are the polynomials of degree at most $k-j-1$.

Proof: For fixed $j \in \{0,1,\dots,k-1\}$ we will find the polynomial $W_j(x)$ in the form

$$W_j(x) = a_j(x)x^j L_n^{(k)}(x)L_n^{(k)'}(x) + x^{k-1}L_n^{(k)'}(x)c_n(x), \tag{31}$$

where the degree of the polynomial $a_j(x)$ is $k-j-1$ and the degree of the polynomial $c_n(x)$ is *n*. Also it is clear that for $l = 0,1,\dots, j-1, W_j^{(l)}(0) = 0$. We know that $L_n^{(k)}(x_i) = 0$ and $L_n^{(k)'}(y_i) = 0$ so we get $W_j(x_i) = 0$ and $W_j(y_i) = 0$ for $i = 1, 2, \dots, n$. The coefficients of the polynomial $a_j(x)$ are determined by the system

$$W_j^{(l)}(0) = \frac{d^l}{dx^l}\left[a_j(x)x^j L_n^{(k)}(x)L_n^{(k)'}(x)\right]_{x=0} = \delta_{jl} \tag{32}$$

where $l = j,\dots,k-1$. Now for the constants w_j, use the equation $W_j^{(k)}(0) = 0$ and get

$$w_j : c_n(0) = \frac{-1}{k!L_n^{(k)'}(0)} \frac{d^k}{dx^k}\left[a_j(x)x^j L_n^{(k)}(x)L_n^{(k)'}(x)\right]_{x=0} \tag{33}$$

Now use the conditions $[e^{-x}x^{-k}W_k(x)]'_{x=x_i} = 0$, and

$$\frac{d}{dx}\left[x^k L_n^{(k)}(x)\right] = (n+k)x^{k-1}L_n^{(k)'}(x) \tag{34}$$

we have

$$c_n'(x_i) = -(x_i)^{j-k} L_n^{(k)'}(x_i)a_j(x_i) \tag{35}$$

this will imply the value of $c_n'(x)$ as:

$$c_n'(x) = -\frac{L_n^{(k)'}(x)a_j(x) + b_j(x)L_n^{(k)}(x)}{x^{k-j}} \tag{36}$$

where the polynomial $b_j(x)$ is of degree $k-j-1$. The function $c_n'(x)$ will be a polynomial if and only if

$$\frac{d^r}{dx^r}\left[L_n^{(k)'}(x)a_j(x) + b_j(x)L_n^{(k)}(x)\right]_{x=0} = 0 \tag{37}$$

for $r = 0,\dots,k-j-1$. By using these equations, we can uniquely determined the coefficients of $b_j(x)$. Now integrate the equation (35) to get, $c_n(x) = c_n(0) + \int_0^x c_n'(t)dt$, use the value of $c_n(0)$ from (33) we get the desired result as the proof of the theorem.

Theorem 1: For some fixed integers *k* and $n \geq 1$ if $\{z_i\}_{i=1}^{n-1}, \{z_i'\}_{i=1}^{n}$, and $\{z_0^{(j)}\}_{j=0}^{k}$ are arbitrary real numbers, then on the nodal points (1) there exists a unique polynomial $B_m(x)$ having the at most degree $2n+k-1$ satisfying the equations (2), (3), and (4). The existing polynomial can be written as:

$$B_m(x) = \sum_{j=1}^{n-1} z_j U_j(x) + \sum_{j=1}^{n} z_j' V_j(x) + \sum_{j=0}^{k} z_0^{(j)} W_j(x) \tag{38}$$

where the fundamental polynomials $U_j(x), V_j(x)$ and $W_j(x)$ are defined in the previous Lemmas.

Proof: By Lemmas 1, 2, and 3, the polynomial $B_m(x)$, defined in the theorem's statement, holds the equations (2), (3), and (4), it implies that the existence of the polynomial is valid. To prove the uniqueness let us consider

the following problem: Find the polynomial $S_m(x)$ having the least possible degree $2n+k-1$ satisfying the following interpolatory conditions:
for $j=1,2,\ldots,n-1$

$$S_m(y_i)=0, \quad (i=1,2,\ldots,n-1)$$
$$[e^{-x}x^{-k}S_m(x)]'_{x=x_i}=0, \quad (i=1,2,\ldots,n)$$
$$S_m^{(l)}(0)=0, \quad (l=0,1,\ldots,k)$$

After taking into consideration of these equations it can be seen that

$$S_m(x)=x^{k-1}b_n(x)L_n^{(k)'}(x),$$

where $b_n(x)$ is a polynomial of degree at most *n*. Use the equation (34) and get

$$[e^{-x}x^{-k}S_m(x)]'_{x=x_i}=e^{-x_i}L_n^{(k)'}(x_i)b_n'(x_i)=0,$$

from which $b_n'(x_i)=0$ implies $b_n'(x)\equiv 0$, thus $b_n(x)\equiv c$. Therefore, $S_m(x)=cx^kL_n^{(k)'}(x)$, but

$$\frac{d^kS_n}{dx^k}(0)=ck!L_n^{(k)'}(0)=0.$$

Since $L_n^{(k)'}(0)\neq 0$, therefore $c=0$, hence $S_m(x)\equiv 0$. This proves that the polynomial $B_m(x)$ is unique.
Now we state our main theorem.

Theorem 2: Assuming that the interpolatory function $f:R\to R$ is continuous as well as differentiable such that $C(m)=\{f(x): f(x)=O(x^m) \text{ as } x\to\infty;\}$ where m is a non negative integer, f is continuous function in the interval $[0,\infty)$, then for each $f\in C(m)$ and a non negative *k*,

$$B_m(x)=\sum_{j=1}^{n-1}z_jU_j(x)+\sum_{j=1}^{n}z_j'V_j(x)+\sum_{j=0}^{k}z_0^{(j)}W_j(x) \tag{39}$$

satisfies the relation:

$$|B_m(x)-f(x)|=O(n^{-1})\omega\left(f,\tfrac{\log n}{\sqrt{n}}\right) \quad \text{for } 0\le x\le cn^{-1} \tag{40}$$
$$|B_m(x)-f(x)|=O(n^{-1})\omega\left(f,\tfrac{\log n}{\sqrt{n}}\right) \quad \text{for } cn^{-1}\le x\le \Omega \tag{41}$$

here ω represents the modulus of continuity.
Before proving the theorem 2, first estimate the values of the following fundamental polynomials, which are listed below:

4. Estimation of the Fundamental Polynomials

First estimate the values of the following fundamental polynomials, which are listed below:

Theorem 3: Let us assume the fundamental polynomial $U_j(x)$, for $j=1,2,\ldots,n-1$ is presented by:

$$U_j(x)=\frac{x^kl_j^*(x)L_n^{(k)}(x)}{y_j^kL_n^{(k)}(y_j)}-\frac{x^kL_n^{(k)'}(x)}{y_j^kL_n^{(k)}(y_j)L_n^{(k)''}(y_j)}\int_0^x\frac{tL_n^{(k)'}(t)-x_jL_n^{(k)}(t)}{t-y_j}dt \tag{42}$$

then we have

$$\sum_{j=1}^{n-1}e^{x_j}x_j^k|U_j(x)|=O(n^{-2}) \quad \text{for } 0\le x\le\Omega \tag{43}$$

Proof: From the polynomial $U_j(x)$ we have

$$\sum_{j=1}^{n-1} e^{x_j} x_j^k |U_j(x)| \le \sum_{j=1}^{n-1} \frac{e^{x_j} x_j^k |x^k| |l_j^*(x)| |L_n^{(k)}(x)|}{|y_j^k| |L_n^{(k)}(y_j)|} + \sum_{j=1}^{n-1} \frac{e^{x_j} x_j^k |x^k| |L_n^{(k)'}(x)|}{|y_j^k| |L_n^{(k)}(y_j)| |L_n^{(k)''}(y_j)|} \left| \int_0^x \frac{tL_n^{(k)'}(t) - x_j L_n^{(k)}(t)}{t - y_j} dt \right| \tag{44}$$

Let $I = \left| \int_0^x \frac{tL_n^{(k)'}(t) - x_j L_n^{(k)}(t)}{t - y_j} dt \right|$. To evaluate I, let

$$\frac{L_n^{(k)}(x)}{x - x_j} = d_{j,n-1} x^{n-1} + d_{j,n-2} x^{n-2} + d_{j,n-3} x^{n-3} + \cdots + d_{j,0} \tag{45}$$

$$L_n^{(k)}(x) = (x - x_j)\left(d_{j,n-1} x^{n-1} + d_{j,n-2} x^{n-2} + d_{j,n-3} x^{n-3} + \cdots + d_{j,0}\right) \tag{46}$$

To find the values of the coefficients, use the general form of the Laguerre polynomial

$$L_n^{(k)}(x) = \sum_{\mu=0}^{n} \binom{n+k}{n-\mu} \frac{(-x)^\mu}{\mu!} \tag{47}$$

and get,

$$d_{j,n-1} = \frac{(-1)^n}{n!} \text{ and } d_{j,n-2} = \frac{(-1)^n}{n!}[x_j - n(n+k)].$$

Now let,

$$\int_0^x \frac{tL_n^{(k)'}(t) - x_j L_n^{(k)}(t)}{t - y_j} dt = \sum_{i=0}^{n} A_{j,i} L_i^{(k)'}(x), \tag{48}$$

on comparing the coefficient with the equation (44) we have

$$A_{j,n} = 1 + \frac{x_j}{n}.$$

Thus, by substituting the value of coefficient and using the equations (6), (7), (8) and (9), we get the desired result.

Theorem 4: Let us assume the fundamental polynomial $V_j(x)$, for $j = 1, 2, \ldots, n$ is presented by:

$$V_j(x) = \frac{e^{x_j} x^k L_n^{(k)'}(x)}{y_j^k L_n^{(k)'}(y_j)} \int_0^x l_j(t) dt \tag{49}$$

then we have

$$\sum_{j=1}^{n} |V_j(x)| = O(n^{-1}), \qquad \text{for } 0 \le x \le \Omega \tag{50}$$

Proof: From the polynomial $V_j(x)$ we have:

$$|V_j(x)| \le \frac{e^{x_j} |x^k| |L_n^{(k)'}(x)|}{|y_j^k| |L_n^{(k)'}(y_j)|} \left| \int_0^x l_j(t) dt \right|$$

Then

$$\sum_{j=1}^{n} |V_j(x)| \le \sum_{j=1}^{n} \frac{e^{x_j} |x^k| |L_n^{(k)'}(x)|}{|y_j^k| |L_n^{(k)'}(y_j)|} \left| \int_0^x l_j(t) dt \right| \tag{51}$$

by using the equations (6), (8), and (13) we get the desired result, $\sum_{j=1}^{n} |V_j(x)| = O(n^{-1})$, for $0 \le x \le \Omega$.

Remark: Let $C(m) = \{ f(x) : f$ is continuous in $[0, \infty)$, $f(x) = O(x^m)$ as $x \to \infty; \}$ where $m \geq 0$ is an integer. Then, by Szegö [12] Theorem 14.7,

$$\lim_{n\to\infty} \left| f(x) - H_n^{(\alpha)}(f, x) \right|_I = 0 \tag{52}$$

where $I \subset (0, \infty)$ for $\alpha \geq 0$, or $I \subset (0, \infty)$ for $-1 < \alpha < 0$. Also note that there is a function in $C(m)$ such that $\{H_n^{(\alpha)}(f, x)\}$ diverges for $\alpha \geq 0$ at $x = 0$. And for the convergence rate, we have:

$$\left| f(x) - H_n^{(\alpha)}(f, x) \right|_I = \begin{Bmatrix} O\left(\omega\left(f, n^{-1-\alpha}\right)\right); & -1 < \alpha < 0 \\ O\left(\omega\left(f, \frac{\log n}{\sqrt{n}}\right)\right); & \alpha \geq -\frac{1}{2} \end{Bmatrix} \tag{53}$$

Proof of the main theorem 2:

Let us suppose that $A_n(x)$ be a polynomial of degree $\leq 2n + k - 1$ and $B_m(x)$ be given by (15). Note that $B_m(x)$ is exact for every fundamental polynomial of degree $\leq 2n + k - 1$; therefore,

$$A_n(x) = \sum_{j=1}^{n-1} A_n(y_j) U_j(x) + \sum_{j=1}^{n} A_n^{'}(x_j) V_j(x) + \sum_{j=0}^{k} A_n(x_0) W_j(x) \tag{54}$$

from equations (15) and (54) we get

$$\begin{aligned} \left| f(x) - B_m(x) \right| &\leq \left| f(x) - A_n(x) \right| + \left| A_n(x) - B_m(x) \right| \\ &\leq \left| f(x) - A_n(x) \right| + \sum_{j=1}^{n-1} \left| f(y_j) - A_n(y_j) \right| \left| U_j(x) \right| \\ &\quad + \sum_{j=1}^{n} \left| f^{'}(x_j) - A_j^{'}(x_j) \right| \left| V_j(x) \right| \\ &\quad + \sum_{j=0}^{k} \left| f^{l}(x_0) - A_n^{l}(x_0) \right| \left| W_j(x) \right| \end{aligned} \tag{55}$$

Thus, equation (55) and the conclusions of theorem 3, and 4 complete the proof of the theorem 2.

5. Conclusions

The findings in this work demonstrate the existence, uniqueness, explicit representation, and order of convergence of the given interpolatory problem when the roots $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n-1}$, are specified on the Laguerre polynomials $L_n^{(k)}(x)$ and its derivative $L_n^{(k)'}(x)$, respectively. If the interpolatory function $f : R \to R$ is continuously differentiable, then the equations (2), (3), and (4) are held by a polynomial $B_m(x)$ of degree $\leq 2n + k - 1$.

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