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On Projective Geometry Over Simple Matrix Rings

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Abstract: This paper investigating the relationship between projective geometry over a ring R and those over a simple matrix ring R_n, demonstrating these geometries are essentially equivalent. The study extends the fundamental theorem of projective geometry ϕ_n modules, where ϕ is a division ring, traditionally applicable to vector spaces over a division rings, to the border context of modules over rings. Specifically, it focuses on the lattice structure formed by all R-sub-modules of an R-module., which defines the projective geometry in this setting. The equivalence between projective geometries over R and R_n is established, highlighting the structural similarities and providing a framework for understanding projective geometry in the more general context of modules. This work contributes to the border understanding of projective geometry, particularly in non-classical settings, and aligns with the 2020 AMS subject Classification 16S10.

Keywords: Matrix ring, division ring, simple matrix ring, R- module, lattice of all R-sub-modules, R_n -modules and isomorphism.

1. Introduction

Projective geometry, a fundamental area of mathematics, explores the properties of geometric structures that remain invariant under projective transformation. Traditionally, projective geometry is studied over fields or division rings, where it has been well developed and understood. However, the extension of projective geometry to more general algebraic structures, such as rings and modules, opens new avenues for exploration and broadens the scope of its applications.

This paper delves into the projective geometry defined over simple matrix rings R_n, where R is a ring. The primary focus is to establish the equivalence between projective geometry over a rind R and those over the corresponding matrix ring R_n. This equivalence offers a deeper understanding of the geometric structures that can be defined over rings, particularly those that are not division rings.

The study also extends the fundamental theorem of projective geometry, which asserts that every isomorphism of the projective space is induced by a semi-linear transformation, to the context of modules over rings. By considering the lattice of all R-submodules of an R- module as a projective geometry, the paper provides a framework for exploring geometric properties in non-classical settings, thereby enriching the theory of projective geometry and its algebraic foundations.

In the year 1952, R. Baer. at [13] defined the extensions of the fundamental theorem of projective geometry have been made, for the case of R-modules, where R is a "Prime Ring" in his sense and the ring of rational integers [3]. Also in the year 1954, at[12] analyses and extend the fundamental theorem of projective geometry over R-module. R_n - denotes simple matrix ring.

2. Lattice Isomorphism Of A Simple Matrix Rings

Definition 2.1: If $n \geq 3$, any lattice isomorphism of the lattice of all ideals of ϕ_n and that of ψ_m is induced by an isomorphism of ϕ_n and ψ_m, where ϕ and ψ are division rings,

Example 2.2: by a ring always mean an associative ring with unity. Let R be a ring with unit element 1 and A be an additive group of an R- module, if ax is defined for all a in R and all x in A such that

$$a(x+y) = ax + ay,$$

$$(a+b)x = ax + bx,$$

$$a(bx) = (ab)x,$$

$1x = x$,for any a, b in R and any x, y in A.

Let a subgroup B of A is called an R- sub-module, if ax in B for all a in R and all x in B. the set of all R- sub-modules of A forms a modular lattice $L(R, A), w.r.t$ group theoretical union and intersection. We shall call the lattice $L(R, A)$ a projective geometry over the ring R. Now we shall denote R_n is a simple matrix ring of degree n over R.

Theorem 2.3: Let R be a ring and n be a positive integer then for any R_n - module M there exist an R-module A such that

$$L(R_n, M) \cong L(R, A).$$

Proof: Let 'a' be an element in R, we will denote by [a] the diagonal matrix of which all the diagonal elements are equal to a. the set of all [a] forms a sub-ring of R_n isomorphic to R. Let e_{ii} be the $n \times n$ matrix with 1 in the i^{th} row and j^{th} column and 0 otherwise. It is easily seen that e_{ii} commutes with any diagonal matrix.

Let M be an R_n -module. if we set $A = Me_{11}$ then we can consider A as an R-module, since every [a] is commutative with e_{11}. If M_1 is an R_n -sub-module of M then $A_1 = M_1 e_{11}$ is an R- sub-module of A. we will show that the mapping $\phi : M_1 \rightarrow A_1$ gives the desired isomorphism of $L(R_n, M)$ and $L(R, A)$.

Let M_1, M_2 be two R_n -sub-module of M such that

$$M_1 e_{11} = M_2 e_{11}.$$

We will show that $M_1 \subseteq M_2$.

.Let foe any $x \in M_1$ then,

$$xe_{i1} \in M_1 \text{ and } xe_{i1} = (xe_{i1})e_{11} \in M_1 e_{11} \subseteq M_2 e_{11} \subseteq M_2$$

. Thus $xe_{i1} \in M_2$ and $xe_{ii} = (xe_{i1})e_{1i} \in M_2 e_{1i} \subseteq M_2$.

Hence $xe_{i1} \in M_2$, for $i = 1, 2, 3, \ldots, n$ and since $e_{11}, e_{22}, e_{33}, \ldots, e_{nn}$ is the unit element of R_n, $x \in M_2$. Thus $M_1 \subseteq M_2$ is proved.

If $M_1 e_{11} = M_2 e_{11}$ then clearly $M_1 = M_2$.

Hence we have proved that ϕ is univalent and preserves inclusion.

Let A_1 be any R_n -sub-module of A and Let M_1 be the set of all elements of x in M such that $xe_{i1} \in A_1$ for $i = 1, 2, 3, \ldots, n$ then M_1 be any R_n -sub-module of M, for if $x \in M_1$ then $xe_{ij} \in M_1$ for $i, j = 1, 2, 3, \ldots, n$ and $x[a] \in M_1$ for all 'a' in R, since

$$(xe_{ij})e_{k1} = xe_{j1}\delta_{ik} \in A_1 \text{ , where } \delta_{ik} \text{ is the Kronecker delta.}$$

Since [a] and e_{i1} permute and A_1 be any R_n -sub-module of A, $x[a] \in M_1$, so that M_1 is proved to be an R_n -sub-module of M. we will show that

$$A_1 = M_1 e_{11}.$$

Since $xe_{11} \in A_1$ for all element of x in M_1 we have $M_1 e_{11} \subseteq A_1$

Let $x \in A_1$ then since $A_1 \subseteq A = M_1 e_{11}$, $x = x' e_{11}$ for some x' in M and $xe_{11} = x \in A_1$, $xe_{i1} = (xe_{11})e_{i1} = 0$ for $i > 1$. Therefore $x \in M_1$ and $x = x' e_{11} \in M_1' e_{11}$. Hence $A_1 = M_1 e_{11}$ is proved.

Corollary 2.4: Let for any R-module A there exist an R_n - module M such that $L(R_n, M) \cong L(R, A)$.

Proof: Let A be a given R-module. Let M be a totality of $x_1, x_2, x_3, \ldots, x_n$ elements of A. if we define addition in M by adding component wise. M becomes an element of R_n and defines

$$(a_{ij})(x_1, x_2, x_3, \ldots, x_n) = (y_1, y_2, y_3, \ldots, y_n) \text{ by } y_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n, \text{ for } i = 1, 2, \ldots, n.$$

Then M becomes an R_n - module of A. that $L(R_n, M) \cong L(R, A)$ follows the first half of the theorem 2.3.

Definition 2.5: Let $J = \{R, S, \ldots\}$ be a family of rings and π be a lattice theoretical condition that the fundamental theorem of projective geometry $(f.t.p.g)$ holds in J under π, if for any ring R in J and R-module A such that $L(R, A)$ satisfies the condition of π.

Definition 2.6: Let S be a ring in J and of B is an s-module such hat $L(R, A) \cong L(S, B)$ then there exist an isomorphism σ of A to B and an isomorphism σ' of R to S such that the lattice isomorphism

$$A_1 \to A_1' = B_1,$$

where A_1 denotes an arbitrary element in $L(R,A)$ is induced by σ that is

$$A_1^{'} = \{y^{\sigma} / y \in A_1\}$$

and such that

$$(xa)^{\sigma} = x^{\sigma} a^{\sigma^{'}}$$, for all a in R and all x in A.

Theorem 2.7: Let $J = \{R, S, \ldots\}$ be a family of rings, n be a positive integer and π be a lattice theoretical condition. If the fundamental theorem of projective geometry holds in J under π then the fundamental theorem of projective geometry also holds in $J_n = \{R_n, S_n, \ldots\}$ under π. This isomorphism $\sigma^{'}$ of R_n to S_n needed in the fundamental theorem of projective geometry in J_n, can always be chosen , so that $\sigma^{'}$ is of the type

$$(a_{ij})^{\sigma^{'}} = (a_{ij}^{\sigma^{''}}),$$

where $\sigma^{''}$ is a suitable isomorphism of R to S.

Proof: Suppose the fundamental theorem of projective geometry holds in $J = \{R, S, \ldots\}$ under π, that M is an R_n -module and that $L(R_n, M)$ satisfies π.

Let $M_1 \to M_1^{'}$ be a lattice isomorphism of $L(R_n, M)$ and $L(S_n, N)$, where N is an S_n -module, by theorem2.3 there exist an R_n -module a and S-module J such that,

$$L(R_n, M) \cong L(R, A),\ L(S_n, N) \cong L(S, B).$$

then
$$L(R, A) \cong L(S, B)$$

and clearly both of these lattices satisfy π. In the view of the proof of theorem2.3, we may assume that $A = Me_{11}$ and $B = Ne_{11}$. Now by our assumption there is an isomorphism σ of Me_{11} to Ne_{11} and an isomorphism $\sigma^{'}$ of R to S such that $M_1^{'} e_{11} = \{y^{'} / y \in M_1 e_{11}\}$ for any R_n -sub-module M_1 of M and such that

$$((xe_{11})[a])^{\sigma} = (xe_{11})^{\sigma} [a\sigma^{''}]$$ for all 'a' in R and all x in N.

now we define a map $\tau : x \to x^{\tau}$ from M to N by

$$x^{\tau} = (xe_{11})^{\sigma} e_{11} + (xe_{21})^{\sigma} e_{12} + (xe_{31})^{\varpi} e_{13} + \ldots + + (xe_{n1})^{\sigma} e_{1n}$$

Note that $(xe_{i1})^{\sigma}$ is a meaningful for $x \in M$ and for $i = 1, 2, 3, \ldots, n$, since $xe_{i1} = (xe_{i1}) e_{11} \in Me_{11}$. We will show that τ is an isomorphism of M and N. Since σ is an isomorphism of Me_{11} to Ne_{11},

we have easily verify that

$$(x + y)^{\tau} = x^{\tau} + y^{\tau}.$$

If $x^{\tau}=0$ then $x^{\tau}e_{i1}=0$ and therefore $(xe_{i1})^{\sigma}e_{1i}e_{i1}=(xe_{i1})^{\sigma}e_{11}=0$. Since $(xe_{i1})^{\sigma}\in Ne_{11}$, we have $(xe_{i1})^{\sigma}=(xe_{i1})^{\sigma}e_{11}=0$. Hence $xe_{i1}=0$, since σ is an isomorphism then $xe_{ii}=(xe_{i1})e_{1i}=0$ for $i=1,2,3,\ldots,n$, therefore $x=0$,Now we have to show that for any $z\in N$ there is an $x\in M$ such that $z=x^{\tau}$, since $ze_{i1}=(ze_{i1})e_{11}\in Ne_{11}$ there exist an element $x_i\in Me_{11}$ such that $ze_{i1}=x_i^{\tau}$.

Put $$x=x_1e_{11}+x_2e_{21}+x_3e_{31}+\ldots+x_ne_{n1}$$

then

$$x_ie_{i1}=x_ie_{11}=x_i.$$

Hence $$ze_{i1}=(xe_{i1})^{\sigma} \text{ and } ze_{i1}=(xe_{i1})^{\sigma}e_{1i}.$$

Therefore $z=x^{\tau}$. Thus we have proved that τ is an isomorphism of M to N.

Now for all $a\in R$ and all $x\in N$, then we have

$$(x[a])^{\sigma}=\sum(x[a]e_{i1})^{\sigma}e_{i1}=\sum(xe_{i1}[a])^{\sigma}e_{i1}=\sum(xe_{i1})^{\sigma}\left[a^{\sigma^{'}}\right]e_{i1}=\left[a^{\sigma^{'}}\right]\sum(xe_{i1})^{\sigma}e_{i1}=\left[a^{\sigma^{'}}\right]x^{\sigma}$$

and

$$(xe_{ij})^{\tau}=(xe_{i1})^{\tau}e_{1j}=x^{\tau}e_{ij}.$$

Thus for all $(a_{ij})\in R_n$ and $x\in M$ we have $\left(x(a_{ij})\right)^{\tau}=\left(a_{ij}^{\sigma^{'}}\right)x^{\tau}$. Finally we have to show that $M_1^{'}=\left\{x^{\tau}/x\in M_1\right\}$ for any R_n -sub-module M_1 of M, we note that $y^{\tau}=y^{\sigma}$ for all $y\in Me_{11}$.

Let x be in M_1. Since we have $M_1^{'}e_{11}=\left\{y^{\sigma}/y\in M_1e_{11}\right\}$ and since $xe_{i1}\in M_1e_{11}$, we have $x^{\tau}e_{i1}=(xe_{i1})^{\tau}=(xe_{i1})^{\sigma}\in M_1^{'}e_{11}$. Hence $x^{\tau}e_{ii}\in M_1^{'}e_{i1}\subseteq M_1^{'}$. Therefore $x^{\tau}\in M_1^{'}$.

Conversely, let z be in $M_1^{'}$, then we can find x_i in such that $ze_{i1}=x_i^{\sigma}=x_i^{\tau}$. Hence $ze_{ii}=x_i^{\tau}e_{1i}=(x_ie_{1i})^{\tau}$ and $z=\sum(x_ie_{1i})^{\tau}=\left(\sum x_ie_{1i}\right)^{\tau}$.

Clearly $x=x_ie_{1i}$ is in M_1 and $z=x^{\tau}$. Thus we have proved that the lattice isomorphism of $L(R_n,M)$ to $L(S_n,N)$ is induced by the group isomorphism τ, proof is complete.

Now let R be a ring $A=R\oplus R\oplus\ldots\oplus R$ be a direct sum of n terms of the additive group of R. if we define $xa=a_1a,a_2a,\ldots,a_na$, for $x=(a_1,a_2,\ldots,a_n)\in A$ and $a\in R$, then A become an R-module. We assume that R satisfies the following conditions;

(i) Any lattice automorphism of $L(R,A)$ induced by some automorphism σ of A such that $(xa)^{\sigma}=x^{\sigma}a^{\sigma^{'}}$ for all $a\in R$, $x\in A$, where $\sigma^{'}$ is an automorphism of R.

(ii) Let P, Q be in the simple matrix ring R_n. If $PQ=1, QP=1$. Q will be denoted by P^{-1}.

Theorem 2.8: Let a ring R and an integer n be such that the above two conditions (i),(ii) are satisfied then any automorphism α of the simple matrix ring R_n is of the form $\left(a_{ij}\right)^{\alpha} = W^{-1}\left(a_{ij}^{\alpha'}\right)W$, where α' is an automorphism of R and W is an element of R_n for which W^{-1} is exist.

Proof: obvious

CONCLUSION

From Baer theorem [3, p.39]we know that the ring I of all integers satisfies the above condition(i) for all n. since I can be imbedded in a field, (ii) is also satisfied for all n. from theorem 2.8, therefore, it follows that any automorphism of the simple matrix ring I_n is inner.

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