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Non-Isolation of weighted composition operators on $L^p(\mu)(1 \leq p \leq \infty)$ spaces

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Abstract: Assuming X is a compact Hausdorff space and $\mathcal{C}(X)$ denotes the Banach algebra of continuous complex-valued functions on X with the supremum norm, comp $\mathcal{C}(X)$ refers to the collection of all composition operators on $\mathcal{C}(X)$. We show that each composition operator is isolated in comp $\mathcal{C}(X)$ under the norm topology as well as the strong operator topology. In addition, we demonstrate that every weighted composition operator on $L^p(\mu)(1 \leq p \leq \infty)$ spaces is non-isolated under the norm topology.

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1. Introduction

The study of composition operators leads to an interplay between function theory and operator theory. Composition operator first arose in 1931 B. O. Koopman's formulation of classical mechanics [9]. These operators also found expression, at least implicitly, in connection with Markov processes. Composition operators also appear in the celebrated work of Hardy and Littlewood, called "Littlewood Subordinate Principle". It also finds its presence in the work of J. V. Ryff 1966 [16]. However, the word "Composition Operator" was used for the first time in 1968 by E. A. Nordgren [13]. In his paper titled "Composition Operator". The investigation of composition operators has been primarily conducted within two main contexts. The first is one the L^p space of measure space and the other is in on the Banach space of analytic function such that as H^p spaces of the unit disk ball in $\mathbb{C}^n(n \geq 1)$. It is interesting to note that in certain situation this of operators possesses a distinct identity different from several known classes of operators, like the multiplication operators and the integral operator.

The investigation of weighted composition operators on L^2 spaces began with Parrot's work in [14], where he focused on studying weighted translation operators. Peterson [15] delved into the spectrum and commutant of these operators, while Bastian examined their decomposition. Embry and Lambert [5], as well as Hoover, Lambert, and Quinn [6], continued to explore the properties and practical uses of these operators. Dharmadhikari and Kumar [10] conducted a detailed study of these operators in their respective Ph.D. dissertations. Carlson obtained the spectra and commutant of certain weighted composition operators on $\ell^2(\mathbb{N})$. Takagi [19] studied the compactness of weighted composition operators on L^p spaces, and Kamowitz and Wortman focused on their behavior on Sobolev type spaces.

The research to be discussed in this paper builds upon Earl Berkson's 1981 result which demonstrated that certain non-compact composition operators on the Hardy space $H^2(\mathbb{D})$ are isolated under the norm topology. **Berkson Isolation Theorem** [1] Let φ be an analytic map of $\mathbb{D}$ into $\mathbb{D}$ such that $\sigma(A) > 0$, where $A = \tilde{\varphi}^{-1}(K)$. Let Ψ: $\mathbb{D} \to \mathbb{D}$ be an analytic map and Cφ, Cψ and be the corresponding composition operators on $H^2(\mathbb{D})$ $(1 \leq p < \infty)$. If $\| C\psi - C\varphi \| < \sqrt[p]{\frac{\sigma(A)}{2}}$, then φ = Ψ. In the above theorem K denotes the unit circle in the complex plane, $\tilde{\varphi}$ denotes the radial limit of φ and σ denotes the normalized Lebesgue measure on K.

Berkson's theorem, which is a topological statement about the compH^p the space of all composition operators on $H^p(\mathbb{D})$ endowed with norm topology, says that whenever the radial limit function $\tilde{\varphi}$ of φ has modulus one on a subset of unit circle having positive Lebesgue measure, then Cφ is isolated. Later, there was further sharpening and elaborations of the work of Berkson by MacCluer, Jabbarzadeh-Pourreza [7, 8], Shapiro and Sundberg [17], Chandra [3,12], Bourdon [2], Hammond-MacClure [11] and most recently, by Cheng and Dai [4].

Intuitively, an isolated point is a point that is separated from all other points in a set by a positive distance. In the context of Banach algebras, an isolated point is a point that is not approximal by any other element in the algebra. This concept is important in the study of the spectra of elements in Banach algebras, as isolated points have a special role in the structure of the spectrum.

Definition 1.1: Let Y be a Banach space and $\mathcal{B}$ denote the Banach algebra of all bounded linear operators on Y. Let T ∈ A ⊂ $\mathcal{B}$(Y). Then T is said to be isolated in A under the norm topology if there exists ∈ > 0 such that $B_{\in}(T)$ ∩ A = {T}, where. $B_{\in}(T)$ = {S ∈ $\mathcal{B}$(Y): || S – T|| < ∈}.

Definition 1.2: An operator T is said to be isolated in A under strong operator topology, if there exists exists ∈ > 0 and y ∈ Y such that $B(T, y, \in)$ ∩ A = {T}, where. $B(T, y, \in)$ = {S ∈ $\mathcal{B}$(Y): || (S – T) y || < ∈}.

Weighted composition operator: Let X be a non-empty set and V(X) be a linear space of complex valued functions on X under point wise addition and scalar multiplication. If φ: X → X and u: X → ℂ be function such that u f ∘ φ ∈ V(X) for all f ∈ V(X), then φ and u induce a linear transformation uCφ on V(X), defined as

$$(uC\varphi)f = u\,(f \circ \varphi) \text{ for all } f \in V(X).$$

If V(X) is a Banach, Hilbert space and uCφ is a bounded linear operator on V(X), then uCφ is called weighted composition operator. When u is identically equal to one, we get the composition Cφ. Let (X, $\mathfrak{B}, \mu$) be a σ -finite measure space. Let $u \in L^{\infty}(\mu)$ and φ: X → X be a non-singular measurable transformation define

$$\upsilon(E) = \int |u|^p \, d\mu$$

Let $\frac{d\upsilon}{du}$ denote Radon-Nikodym derivative of υ with respect μ and $\Psi = \sqrt[p]{\frac{d\upsilon}{du}}$. Now we have the following theorem.

Theorem 1.3: The necessary and sufficient condition of uCφ: $L^p(\mu) \to L^p(\mu)(1 \le p < \infty)$ is a weighted composition operator is that $\psi \in L^{\infty}(\mu)$. In this case || uCφ || = || ψ ||$_{\infty}$.

Remark 1.4: When X = ℕ, $\mathfrak{B}$ = P(X), and μ = counting measure then we have the following theorem.

Theorem 1.5: A weighted composition operator uCφ: $\ell^p \longrightarrow \ell^p$ (1 ≤ p < ∞) is a bounded linear operator if $\sup_{n \ge 1}\{|u(n)| \sqrt[p]{|\varphi^{-1}(n)|}\} < \infty$. In this case $\|uC\varphi\| = \sup_{n \ge 1}\{|u(n)| \sqrt[p]{|\varphi^{-1}(n)|}\}$.

Definition 1.6: Let φ: ℕ ⟶ ℕ, then φ induces a linear transformation Cφ on ℓ^p defined by

$$C\varphi\left(\sum_{n=1}^{\infty} x_n \chi_n\right) = \sum_{n=1}^{\infty} x_n \chi_{\varphi-1(n)}$$

where χ_n stands for the characteristic function of the set {n}. The following results can be found in [18].

Theorem 1.7: A necessary and sufficient condition that a function φ on ℕ into itself induces a composition operator on ℓ^p ($1 \le p < \infty$) is that

i. **(i)**. $|\varphi^{-1}(n)|$ is finite for each n in ℕ.
ii. **(ii)**. The set { $\varphi^{-1}(n)$: n ∈ ℕ } is a bounded set.

Now we state a proposition about the norm of the Composition operator Cφ on ℓ^p ($1 \le p < \infty$).

Theorem 1.8: Let Cφ be a Composition operator on ℓ^p ($1 \le p < \infty$). Then || Cφ || = Max $\{|\varphi^{-1}(n)|\}^{1/p}$, n ∈ ℕ.

Theorem 1.9: Let Cφ be a Composition operator on ℓ^p ($1 \le p < \infty$). Then null space N(Cφ) is given by

$$N(C\varphi) = \{f \in \ell^p : f \mid \varphi(\mathbb{N}) = 0\}.$$

Theorem 1.10: Let Cφ be a Composition operator on ℓ^p induced by a function φ in ℕ into itself. Then range space R(Cφ) is given by

$$R(C\varphi) = \{f \in \ell^p : f \mid \varphi^{-1}(n) = \text{constant}, \forall\, n \in \mathbb{N}\}.$$

Theorem 1.11: Range space a Composition operator on ℓ^p is always closed. We now construct the example of the composition operator in the following manner.

Example 1.12: Define a self-map φ on ℕ by

$$\varphi(n) = \begin{cases} n+1, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even} \end{cases}.$$

Then $|\varphi^{-1}(n)| = 1$ for each n ∈ ℕ. Hence $C\varphi$ be a Composition operator. Moreover $C\varphi$ is an isometry and hence $\| C\varphi \| = 1$.

Example 1.13: Define a self-map φ on N by

$$\varphi(n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even} \end{cases}$$

In this example we easy to see that $|\varphi^{-1}(n)| = 2$ for every n ∈ ℕ. Hence by using above theorem we get $\| C\varphi \| = \sqrt{2}$.

Definition 1.7: Notation and Terminology: For $1 \leq p < \infty$, ℓ^p denotes the space of all p-summable real or complex sequences and $|\varphi^{-1}(n)|$ denotes the cardinality of function. Further, let ℕ and ℂ denote the set of all positive integers and the set of all complex numbers, respectively. For given a subset A of a set X, the characteristic function of A is denoted by $\chi_A(x)$ and defined as

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Also, $\mathcal{C}(X)$ is the Banach algebra of all continuous complex-valued functions on a compact Hausdorff space X under the supremum norm and comp $\mathcal{C}(X)$ denote the set of all composition operators on $\mathcal{C}(X)$. Further $\mathcal{B}(X)$ denote the Banach algebra of all bounded complex-valued function on X under the supremum norm and comp $\mathcal{B}(X)$ denote the set of all composition operators on $\mathcal{B}(X)$. Finally, S denote the class of all weighted composition operators on $L^p(\mu)(1 \leq p \leq \infty)$.

2. Isolation of composition operators on $\mathcal{C}(X)$

Within this section, we demonstrate that every composition operator $C\varphi$ is isolated in comp $\mathcal{C}(X)$ under the strong operator topology.

Theorem 2.1: Let X be a compact Hausdorff space and $\varphi: X \longrightarrow X$ be a continuous map. Then $C\varphi$ is isolated in comp$\mathcal{C}(X)$ under the strong operator topology.

Proof: Let C_Ψ be another operator on $\mathcal{C}(X)$ such that $C\varphi \neq C_\Psi$. Then $\varphi \neq \Psi$ then there exists $x_0 \in X$, such that $x_1 = \varphi(x_0) \neq \Psi(x_0) = x_2$. Now, using Urysohn's lemma there exits $f_0: X \longrightarrow [0, 1]$ such that $f_0(x_1) = 1$ and $f_0(x_2) = -1$. Further

$$\| C\varphi(f_0) - C_\Psi(f_0) \| = \| f_0 \circ \varphi - f_0 \circ \Psi \|$$
$$\| C\varphi(f_0) - C_\Psi(f_0) \| \geq | f_0(\varphi(x_0)) - f_0(\Psi(x_0)) |$$
$$\| C\varphi(f_0) - C_\Psi(f_0) \| \geq 2.$$

Thus $\{ C_\Psi : \| C\varphi(f_0) - C_\Psi(f_0) \| < 2\} = \{ C\varphi\}$.

Hence $C\varphi$ is isolated in comp$\mathcal{C}(X)$ under the strong operator topology.

The corollary follows from the above theorem.

Corollary 2.2: Let X be a compact Hausdorff space and $\varphi: X \longrightarrow X$ be a continuous map. Then $C\varphi$ is isolated in comp$\mathcal{C}(X)$ under the norm operator topology. Let $\varphi: X \longrightarrow X$ be a function and $\varphi_1 = \varphi$ in more generally $\varphi_n = \varphi \circ \varphi_{n-1}$ ∀ n ≥1. Therefore the composition operator $C^{\varphi}{}_n = C\varphi_n$ ∀ n ≥1.

Remark 2.3: Every positive integer power $C^{\varphi}{}_{n}$ of a composition operator is isolated in comp$\mathcal{C}$(X) under the norm topology.

3. Isolation of composition operators on $\mathcal{B}$(X)

In this section we show that every composition operator Cφ is isolated in comp$\mathcal{B}$(X) under the strong operator topology.

Theorem 3.1: Let X be a non-empty set and φ: X ⟶ X be a bounded map then Cφ is isolated in Comp$\mathcal{B}$(X) under the strong operator topology.

Proof. Let |X| = 1 then there exists only one function φ: X ⟶ X. Hence comp $\mathcal{B}$(X) is singleton. If |X| ≥ 2 and Cφf ∈ $\mathcal{B}$(X) ∀ f ∈ $\mathcal{B}$(X). Then || Cφ || = 1(Since || Cφf || ≤ ||f|| ∀ f ∈ $\mathcal{B}$(X)). Now suppose Ψ: X ⟶ X is a function with $\varphi \neq \Psi$ then there exists $x_0 \in X$, such that $x_1 = \varphi(x_0) \neq \Psi(x_0) = x_2$. Now, we define a function f_0: X ⟶ ℂ such that $f_0(x_1)$ = -1 and $f_0(x_2)$ = 1, and $f_0(x)$ = 0 for all x ∈ $X \setminus \{x_1, x_2\}$ and || f_0 || = 1

Now

$$\| (C\varphi - C_{\Psi}) f_0 \| = \| f_0 \circ \varphi - f_0 \circ \Psi \|$$

This implies that

$$\| C\varphi(f_0) - C_{\Psi}(f_0) \| \geq 2.$$

Thus { C_{Ψ} : || C$\varphi(f_0)$ - $C_{\Psi}(f_0)$ || < 2} = { Cφ}.

Hence Cφ is isolated in comp$\mathcal{B}$(X) under the strong operator topology.

The corollary follows from the above theorem.

Corollary 3.2: Every composition operator is isolated in comp$\mathcal{B}$(X) under the norm operator topology.

Remark 3.3: Every positive integer power $C^{\varphi}{}_{n}$ of a composition operator is isolated in comp$\mathcal{B}$(X) under the norm topology.

Definition 3.4: Let X be a compact Hausdorff space and φ: X ⟶ X is continuous function, let $f_0 \in \mathcal{C}$(X) define $T^{\varphi}_{f_0}$: $\mathcal{C}$(X) ⟶ $\mathcal{C}$(X) by $T^{\varphi}_{f_0}$(f) = f_0. f∘ φ for all f ∈ $\mathcal{C}$(X) then $T^{\varphi}_{f_0} \in \mathcal{C}$(X) and $\|T^{\varphi}_{f_0}\| = \| f_0\|$. The operator $T^{\varphi}_{f_0}$ is called a Weighted composition operator with weight f_0.

Theorem 3.5: Let X be a compact Hausdorff space and φ: X ⟶ X is continuous function and $f_0 \in \mathcal{C}$(X). Then weighted composition operator is non- isolated in S under the norm topology, where S= { $T^{\varphi}_{f_0}$: $f_0 \in \mathcal{C}$(X), φ: X ⟶ X is continuous}.

Proof: Case (i) - Let $f_0 \neq 0$ and $\{\alpha_n\}$ be a sequence of complex numbers such that α_n ⟶ 1. For each n ≥ 1 let $g_n = \alpha_n f_0$, $f_0 \neq 0$, $g_n \in \mathcal{C}$(X) for all n ≥ 1.

Now

$$\| T^{\varphi}_{f_0} - T^{\varphi}_{g_n} \| = \| f_0 - g_n \|$$
$$\| T^{\varphi}_{f_0} - T^{\varphi}_{g_n} \| = \| f_0 - \alpha_n f_0 \|$$
$$= |1 - \alpha_n| \, \| f_0 \| \text{ as } n \longrightarrow \infty$$

Hence $T^{\varphi}_{f_0}$ is not isolated under the norm topology.

Case (ii) - Suppose $f_0 = 0$, f_0(x) ≡ 0 for all x ∈ X then $T^{\varphi}_{f_0} \equiv 0$. Let $g_n(x) = \frac{1}{n}$, x ∈ X,

For all n ≥ 1. Hence $T^{\varphi}_{g_n} \longrightarrow T^{\varphi}_{f_0} = 0$ and $\| T^{\varphi}_{g_n} \| = \| g_n \| = \frac{1}{n} \longrightarrow 0$ as n ⟶ ∞.

Remark 3.6: From the theorem (3.5) it follows that every composition operator is non-isolated in S under the strong operator as well as the weak operator topology.

4. Non-isolation amongst weighted composition operators on $L^p(\mu)(1 \leq p \leq \infty)$ spaces

In this section we show that every weighted composition operator on $L^p(\mu)(1 \leq p \leq \infty)$ spaces is non-isolated under the norm operator topology.

Theorem 4.1: Let $u \in \ell^\infty$ and $uC\varphi$ be a weighted composition operator on $\ell^p(1 \leq p < \infty)$. Then $uC\varphi$ is non-isolated in S under the norm operator topology.

Proof: For each $m \geq 1$ we define $u_m = (1 - \frac{1}{m})u$ then

$$\| uC\varphi - u_m C\varphi \| = \| (u - u_m) C\varphi \|$$

Let $f = \sum_{n=1}^{\infty} f_n \chi_n$ then

$$\| (u - u_m) C\varphi f \|^p = \| (u - u_m) C\varphi \sum_{n=1}^{\infty} f_n \chi_n \|$$
$$\| (u - u_m) C\varphi f \|^p = \sum_{n=1}^{\infty} | u(n) - u_m(n) |^p |f_n|^p | \varphi^{-1}(n)|$$
$$\| (u - u_m) C\varphi f \|^p = \sum_{n=1}^{\infty} \frac{1}{m} |u(n)|^p |f_n|^p | \varphi^{-1}(n)|$$
$$\| (u - u_m) C\varphi f \|^p = \frac{1}{m} \sum_{n=1}^{\infty} | u(n)|^p |f_n|^p | \varphi^{-1}(n)| \text{ for all } f \in \ell^p$$

Now

$$\| (u - u_m) C\varphi f \| = \frac{1}{(m)^{1/p}} \| uC\varphi f \| \text{ for all } f \in \ell^p$$

Therefore

$$\| (u - u_m) C\varphi \| = sup_{\|f\|=1} \| (u - u_m) C\varphi f \|$$

$$\| (u - u_m) C\varphi \| = \frac{1}{(m)^{1/p}} \| uC\varphi \|$$

This implies that

$$\| (u - u_m) C\varphi \| = \frac{1}{(m)^{1/p}} \| uC\varphi \|.$$

Hence

$$\| (u - u_m) C\varphi \| = \frac{1}{(m)^{1/p}} \| uC\varphi \| \longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

Hence $uC\varphi$ is non-isolated in S under the norm operator topology.

Remark 4.2: Every weighted composition operator on $\ell^p(1 \leq p < \infty)$ is non-isolated in S under the strong operator topology as well as weak operator topology.

Further we generalized the above result in the more general setting of σ-finite measure space.

Theorem 4.3: Let $u \in L^\infty(\mu)$ and $uC\varphi$ be a weighted composition operator on $L^p(1 \leq p < \infty)$. Then $uC\varphi$ is non-isolated in S under the norm operator topology.

Proof. For each $m \geq 1$ we define a function as $u_m = (1 - \frac{1}{m})\, u$ then

$$\| uC\varphi - u_m C\varphi \| = \|(u - u_m) C\varphi\|$$

$$\| uC\varphi - u_m C\varphi \| = \| (u - (1 - \frac{1}{m})\, u) C\varphi \|$$

$$\| uC\varphi - u_m C\varphi \| = \| (u - u - \frac{u}{m}) C\varphi \|$$
$$\| uC\varphi - u_m C\varphi \| = \frac{1}{m} \| uC\varphi \|$$

$$\| uC\varphi - u_m C\varphi \| = \frac{1}{m} \| uC\varphi \| \longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

Hence $uC\varphi$ is non-isolation in S under the norm operator topology.

Remark 4.4: Every weighted composition operator on $L^p(1 \leq p < \infty)$ spaces is non- isolation in S under the strong operator topology as well as weak operator topology.

Theorem 4.5: Let u ∈ ℓ^∞ and $uC\varphi$ be a weighted composition operator on ℓ^∞. Then $|| uC\varphi ||_\infty = || u ||_\infty$.

Proof: Let u ∈ ℓ^∞ and φ: ℕ ⟶ ℕ, then $uC\varphi$: $\ell^\infty \longrightarrow \ell^\infty$ defined as $(uC\varphi f)(n) = u(n) f \circ \varphi(n)$ for all f ∈ ℓ^∞ and for all n ∈ ℕ. Now $\| uC\varphi f \| = sup_{n \geq 1}|u(n)|$ (Since $\|f\| \leq 1$) therefore $\| uC\varphi \| \leq || u ||_\infty$. Now we take $f_0 = (1, 1, 1, \ldots, 1, \ldots)$ and $\| f_0 \| = 1$, then $\| uC\varphi f_0 \| = sup_{n \geq 1}|u(n)| = || u ||_\infty$ this implies that $\| uC\varphi \| \geq || u ||_\infty$. Hence $|| uC\varphi ||_\infty = || u ||_\infty$.

Theorem 5.6: Every weighted composition operator on ℓ^∞ is non-isolated in S under the norm operator topology.

Proof: For each m ≥ 1 we define $u_m = (1 - \frac{1}{m})\, u$ then

$$||(uC\varphi - u_m C\varphi) f ||_\infty = \boldsymbol{sup}_{\boldsymbol{n \geq 1}}|u(n) - u_m(n)|\, |f(\varphi(n))|$$

$$||(uC\varphi - u_m C\varphi) f ||_\infty \leq \boldsymbol{sup}_{\boldsymbol{n \geq 1}}|u(n) - (1 - \frac{1}{m})\, u(n)|\, |f(\varphi(n))|$$

$$||(uC\varphi - u_m C\varphi) f ||_\infty \leq \boldsymbol{sup}_{\boldsymbol{n \geq 1}} \frac{1}{m}\, |u(n)|\, \| f \| \text{ for all } f \in \ell^\infty$$

Therefore

$$||(uC\varphi - u_m C\varphi) ||_\infty \leq \boldsymbol{sup}_{\boldsymbol{n \geq 1}} \frac{1}{m} || u ||_\infty.$$

Taking limit on both side we get

$$\lim_{m \to \infty} || (uC\varphi - u_m C\varphi) ||_\infty \leq \lim_{m \to \infty} \boldsymbol{sup}_{\boldsymbol{n \geq 1}} \frac{1}{m} || u ||_\infty$$

$$\lim_{m \to \infty} || (uC\varphi - u_m C\varphi) ||_\infty \longrightarrow 0.$$

Hence $uC\varphi$ is non-isolated in S under the norm operator topology.

Corollary 4.7: Let u ∈ $L^\infty(\mu)$ and $uC\varphi$ be a weighted composition operator on $L^\infty(\mu)$. Then $uC\varphi$ is non-isolated in S under the norm operator topology as well as weak operator topology.

Remark 4.8: Every weighted composition operator on ℓ^∞ is non-isolated in S under the strong operator topology as well as weak operator topology.

5. Isolation of Composition operators on $\boldsymbol{\ell^p}$ spaces

In this section, we prove that each Composition operator is isolated in comp(ℓ^p)($1 \leq p < \infty$) under the norm topology.

Theorem 5.1: Let $C\varphi$ be a composition operator on ℓ^p($1 \leq p < \infty$) specs. Then $C\varphi$ is isolated in comp(ℓ^p) with norm topology.

Proof: Let C_Ψ be another composition operator on ℓ^p($1 \leq p < \infty$) specs such that $\varphi \neq \Psi$. Then there exists a natural number n ∈ ℕ such that m = $\varphi(n) \neq \Psi(n) = p$.

Let f = $\chi_m - \chi_p$, then || f || = $2^{1/p}$

Further

$$\| (C\varphi - C_\Psi)\, f \| = \| (C\varphi - C_\Psi)\, (\chi_m - \chi_p) \|$$

$\| (C\varphi - C_\Psi)\, f \| = \| (C\varphi - C_\Psi)\, \chi_m - \chi_p\, (C\varphi - C_\Psi) \|$

$\| (C\varphi - C_\Psi)\, f \| = \| \chi_{\varphi^{-1}(m)} - \chi_{\Psi^{-1}(m)} - \chi_{\varphi^{-1}(p)} + \chi_{\Psi^{-1}(p)} \|$.

Since n $\in \varphi^{-1}(m) \cap \Psi^{-1}(p)$ and n $\notin \varphi^{-1}(p)$, n $\notin \Psi^{-1}(p)$.

Therefore

$\| (C\varphi - C_\Psi)\, f \| \geq 2$

$\| (C\varphi - C_\Psi) \| = \frac{\| (C\varphi - C_\Psi)\, f \|}{\| f \|} \geq \frac{2}{\sqrt[p]{2}} \geq 1 \; (1 \leq p < \infty)$

Hence $\| C\varphi - C_\Psi \| \geq 1$.

Hence $C\varphi$ is isolated in comp(ℓ^p) with norm topology.

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