Numerical Solution of Advection-Diffusion Equation by a Modified Cubic B-Spline Collocation Method

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Abstract: - In this work, advection-diffusion equation (ADE) has been studied by using a modified cubic B-spline numerical method. This method produces good results with high accuracy. We have discretized the advection-diffusion equation using the finite difference method. To demonstrate the fourth-order convergence of the proposed method, error analysis is performed. A stability analysis is done by Von-Neumann method and the method is shown to be unconditionally stable. Seven different advection-diffusion equations with Periodic, Neumann, and Dirichlet boundary conditions have been solved to test the effectiveness of the proposed method. The results from these examples highlight the benefits of the method, and comparisons with other methods reveal that the proposed method performs better.

Keywords: *Advection-Diffusion equation, Cubic B-spline method, Finite difference, Error analysis, Stability analysis*

1. Introduction

The advection-diffusion equation is a mathematical model that describes the phenomena where energy is transformed due to diffusion and advection processes. This equation is useful in different areas, such as engineering, environmental science, and fluid dynamics. The flow of fluid through the medium [1], the transfer of energy [2], the diffusion of pollutants in rivers and streams [3], and the transportation of pollutants in the atmosphere [4] may all be described by the advection-diffusion equation.

We consider the advection-diffusion equation as follows:

$$\frac{\partial X}{\partial t} + \eta \frac{\partial X}{\partial u} = \rho \frac{\partial^2 X}{\partial u^2}, \qquad m < u < n \text{ and } 0 < t \leq T, \quad (1.1)$$

here, η is advection coefficient and ρ is diffusion coefficient.
The boundary conditions are defined as follows:
(1) The Periodic boundary conditions are defined by

$$X(m,t) = X(n,t). \qquad (1.2)$$

The total heat is conserved when the boundary conditions are periodic

$$M(t) = \int_m^n X(u,t)dt, \qquad (1.3)$$
$$= \text{h(constant)}.$$

(2) The Neumann boundary conditions are given by

$$\frac{\partial X}{\partial u}(m,t) = f_1(t), \qquad \frac{\partial X}{\partial u}(n,t) = f_2(t). \qquad (1.4)$$

(3) The Dirichlet boundary conditions are defined as follows

$$X(m,t) = g_1(t), \qquad X(n,t) = g_2(t). \qquad (1.5)$$

The initial condition is given by

$$X(u,0) = \phi(x). \qquad (1.6)$$

Many researchers have studied numerical solution of advection-diffusion equation (ADE) using different schemes. Gorgulu and Irk [5] developed Galerkin finite element method. Octic B-spline technique was used to find numerical solution of advection-diffusion equation by Jena and Gebremedhin [6]. Anskari and Adibi [7] presented meshless method to study ADE. Dag *et al.* [8] developed least square scheme for ADE. Quartic and Quintic DQM (differential quadrature method) are used by Korkmaz and Dag [9]. Mittal and Jain [10] studied the ADE using cubic B-spline collocation scheme. Finite element scheme was developed by Dhawan *et al.* [11] to find numerical solution of advection-diffusion equation. Malik *et al.* [12] presented collocation scheme to examine the advection-diffusion equation. Palav and Pradhan [13] used the uniform hyperbolic polynomial (UHP) B-spline scheme to find numerical solutions of ADE. Finite difference technique was used by Andallah and Khatun [14] to examine advection-diffusion equation. Nazir *et al.* [15] developed trigonometric B-splines scheme to study the advection-diffusion equation. Operator splitting technique was used by Bahar and Gurarslan [16]. Dursun *et al.* [17] presented extended cubic B-spline approach to find numerical results of ADE. Finite element approach was used by Mojtabi and Deville [18]. Gurarslan *et al.* [19] developed sixth-order compact finite difference technique to solve ADE.

The purpose of this study is to present a practical and effective numerical method for solving equation (1.1) with variable diffusion and advection parameters, by employing various space and time grid point choices. First, we discretize the equation (1.1) using finite difference scheme to get a system of linear equations. Von-Neumann approach has been used for the stability analysis. It is observed that the cubic B-spline scheme is unconditionally stable.

The outline of this research work is as follows: In Section 2, we have discussed the proposed method. The discretization of the advection-diffusion equation (1.1) using finite difference scheme has been explained in Section 3. The convergence has been checked in Section 4. Stability analysis has been done in Section 5. In Section 6, the outcomes of numerical problem are shown in tables and illustrated graphically. A brief summary is given in Section 7.

2. Cubic B-Spline Functions

Consider an interval $[m,n\ ]$ where $m = u_0 \le u_1 \le \cdots u_M = n$, which is divided into equal parts. The mesh points are denoted by $u_k = u_0 + kl$, where $k = 0, 1, 2, \cdots, M$ and the step size is given by $l = n - m/M$. The cubic B-spline basis function [20] is expressed as follows:

$$\psi_k(u) = \frac{1}{6l^3}\begin{cases}(u-u_{k-2})^3, & \mathrm{u} \in [u_{k-2}, u_{k-1}) \\ l^3 + 3l^2(u-u_{k-1}) + 3l(u-u_{k-1})^2 - 3(u-u_{k-1})^3, & \mathrm{u} \in [u_{k-1}, u_k) \\ l^3 + 3l^2(u_{k+1}-u) + 3l(u_{k+1}-u)^2 - 3(u_{k+1}-u)^3, & \mathrm{u} \in [u_k, u_{k+1}) \\ (u_{k+2}-u)^3, & \mathrm{u} \in [u_{k+1}, u_{k+2}) \\ 0 & \text{otherwise.}\end{cases} \quad (2.1)$$

The cubic spline functions are defined by the basis $\{\psi_{-1}(u), \psi_0(u), \psi_1(u), \cdots, \psi_M(u), \psi_{M+1}(u)\}$ over the interval $[m,n]$. Suppose $H(u)$ is an approximate solution to the exact solution $X(u)$. This approximation can be represented as a linear combination of the basis functions ψ_k, where the unknown parameters S_k are time-dependent and determined by boundary conditions. As a result, the values of the function $H(u)$ at the grid points (u_k, t) are obtained in the following manner:

$$H(u_k, t) = \sum_{k=-1}^{M+1} S_k \psi_k(u_k). \quad (2.2)$$

We assume that the approximate solution $H(u)$ satisfies the interpolating condition

$$H(u_k) = X(u_k), \quad \forall k = 0, 1, 2, \cdots M, \quad (2.3)$$

here, the function $X(u)$ is smooth enough, and the cubic spline interpolant $H(u)$ satisfies the specified boundary conditions:

$$H'(m) = X'(m), \quad H'(n) = X'(n), \quad (2.4)$$
$$H''(m) = X''(m), \quad H''(n) = X''(n). \quad (2.5)$$

The cubic B-spline approximation is applied to function X and its first two derivatives at the $i - th$ node. For simplicity, we will use c_i and C_i in place of H_i' and H_i'', respectively.

$$H_i = \sum_{k=i-1}^{i+1} S_k\psi_k(u) = \frac{1}{6}(S_{i-1} + 4S_i + S_{i+1}), \quad (2.6)$$

$$c_i = \sum_{k=i-1}^{i+1} S_k\psi'_k(u) = \frac{1}{2l}(-S_{i-1} + S_{i+1}), \quad (2.7)$$

$$C_i = \sum_{k=i-1}^{i+1} S_k\psi''_k(u) = \frac{1}{l^2}(S_{i-1} - 2S_i + S_{i+1}). \quad (2.8)$$

Using (2.6) - (2.8), we get

$$c_i = X'(u_i) - \frac{1}{180}l^4X^{(5)}(u_i) + \cdots, \quad (2.9)$$

$$C_i = X''(u_i) - \frac{1}{12}l^2X^{(4)}(u_i) + \frac{1}{360}l^4X^{(6)}(u_i) + \cdots, \quad (2.10)$$

2.1. New approximation for $\mathbf{X''(u)}$: For the new approximation of $X''(u)$, we establish a representation for C_{i-1} at the u_i node using equation (2.10) in the following manner:

$$C_{i-1} = X''(u_{i-1}) - \frac{1}{12}l^2X^{(4)}(u_{i-1}) + \frac{1}{360}l^4X^{(6)}(u_{i-1}) + \cdots,$$
$$= X''(u_i) - lX^{(3)}(u_i) + \frac{5}{12}l^2X^{(4)}(u_i) - \frac{1}{12}l^3X^{(5)}(u_i) + \cdots,$$

similarly,

$$C_{i+1} = X''(u_i) + lX^{(3)}(u_i) + \frac{5}{12}l^2X^{(4)}(u_i) + \frac{1}{12}l^3X^{(5)}(u_i) + \cdots. (2.11)$$

Let $\tilde{C}_i$ be the new approximation for $X''(u)$,

$$\tilde{C}_i = N_1C_i + N_2C_{i-1} + N_3C_{i+1}. \quad (2.12)$$

To increase the order of error in $\tilde{C}_i$, the values N_1, N_2, and N_3 are chosen. The system of linear equations are derived from the (2.12)

$$N_1 + N_2 + N_3 = 1,$$
$$-N_2 + N_3 = 0,$$
$$-N_1 + 5N_2 + 5N_3 = 0.$$

After solving these equations, we get $N_1 = \frac{5}{6}$ and $N_2 = N_3 = \frac{1}{12}$. Substituting the values of N_1, N_2, and N_3 in (2.12), we obtain

$$\tilde{C}_i = \frac{1}{12l^2}(S_{i-2} + 8S_{i-1} - 18S_i + 8S_{i+1} + S_{i+2}). (2.13)$$

Four neighboring values at the node u_0 are used for new approximation of $X''(u)$ as follow:

$$\tilde{C}_0 = N_0C_0 + N_1C_1 + N_2C_2 + N_3C_3, \quad (2.14)$$

here,

$$C_1 = X''(u_0) + lX'''(u_0) + \frac{5}{12}l^2X^{(4)}(u_0) + \frac{1}{12}l^3X^{(5)}(u_0) + \cdots,$$
$$C_2 = X''(u_0) + 2lX'''(u_0) + \frac{23}{12}l^2X^{(4)}(u_0) + \frac{7}{6}l^3X^{(5)}(u_0) + \cdots,$$
$$C_3 = X''(u_0) + 3lX'''(u_0) + \frac{53}{12}l^2X^{(4)}(u_0) + \frac{17}{4}l^3X^{(5)}(u_0) + \cdots.$$

Now using (2.14) as follows:

$$N_0 + N_1 + N_2 + N_3 = 1,$$
$$N_1 + 2N_2 + 3N_3 = 0,$$
$$-N_0 + 5N_1 + 23N_2 + 53N_3 = 0,$$
$$N_1 + 14N_2 + 51N_3 = 0.$$

We get, $N_0 = \frac{7}{6}, N_1 = -\frac{5}{12}, N_2 = \frac{1}{3}$, and $N_3 = -\frac{1}{12}$ after solving above equations. Substituting the values in (2.14), we obtain

$$\tilde{C}_0 = \frac{1}{12l^2}(14S_{-1} - 33S_0 + 28S_1 - 14S_2 + 6S_3 - S_4). \quad (2.15)$$

Similarly, at the node u_M approximation is given by

$$\tilde{C}_M = \frac{1}{12l^2}(-S_{M-4} + 6S_{M-3} - 14S_{M-2} + 28S_{M-1} - 33S_M + 14S_{M+1}). \quad (2.16)$$

These relations will be used in approximating the partial differential equation.

3. Implementation

The finite difference method is used to discretize the equation (1.1) as follows:

$$\frac{X^{n+1} - X^n}{\Delta t} + \eta \frac{X_u^{n+1} + X_u^n}{2} = \rho \frac{X_{uu}^{n+1} + X_{uu}^n}{2}. \quad (3.1)$$

After rearranging the terms, we obtain

$$X^{n+1} + \frac{\eta \Delta t X_u^{n+1}}{2} - \frac{\rho \Delta t X_{uu}^{n+1}}{2} = X^n - \frac{\eta \Delta t X_u^n}{2} + \frac{\rho \Delta t X_{uu}^n}{2}, \quad (3.2)$$

for $i = 0$

$$(S_{-1}^{n+1} + 4S_0^{n+1} + S_1^{n+1}) + \frac{\eta \Delta t}{2}\frac{3}{l}(S_1^{n+1} - S_{-1}^{n+1}) - \frac{\rho \Delta t}{4l^2}(14S_{-1}^{n+1} - 33S_0^{n+1} + 28S_1^{n+1} - 14S_2^{n+1} + 6S_3^{n+1} - S_4^{n+1})$$
$$= (S_{-1}^n + 4S_0^n + S_1^n) - \frac{\eta \Delta t}{2}\frac{3}{l}(S_1^n - S_{-1}^n) + \frac{\rho \Delta t}{4l^2}(14S_{-1}^n - 33S_0^n + 28S_1^n - 14S_2^n + 6S_3^n - S_4^n),$$

we may rewrite the equation as follows:
$a_1 S_{-1}^{n+1} + a_2 S_0^{n+1} + a_3 S_1^{n+1} + a_4 S_2^{n+1} + a_5 S_3^{n+1} + a_6 S_4^{n+1} = b_1 S_{-1}^n + b_2 S_0^n + b_3 S_1^n + b_4 S_2^n + b_5 S_3^n + b_6 S_4^n$,
(3.3)

here,
$a_1 = 1 - \frac{3\eta \Delta t}{2l} - \frac{7\rho \Delta t}{2l^2}, a_2 = 4 + \frac{33\rho \Delta t}{4l^2}, a_3 = 1 + \frac{3\eta \Delta t}{2l} - \frac{7\rho \Delta t}{l^2}, a_4 = \frac{7\rho \Delta t}{2l^2}, a_5 = -\frac{3\rho \Delta t}{2l^2}, a_6 = \frac{\rho \Delta t}{4l^2}$,
$b_1 = 1 + \frac{3\eta \Delta t}{2l} + \frac{7\rho \Delta t}{2l^2}, b_2 = 4 - \frac{33\rho \Delta t}{4l^2}, b_3 = 1 - \frac{3\eta \Delta t}{2l} + \frac{7\rho \Delta t}{l^2}, b_4 = -\frac{7\rho \Delta t}{2l^2}, b_5 = \frac{3\rho \Delta t}{2l^2}, b_6 = -\frac{\rho \Delta t}{4l^2}$,
for $1 \le i \le M-1$

$$(S_{i-1}^{n+1} + 4S_i^{n+1} + S_{i+1}^{n+1}) + \frac{\eta \Delta t}{2}\frac{3}{l}(S_{i+1}^{n+1} - S_{i-1}^{n+1}) - \frac{\rho \Delta t}{4l^2}(S_{i-2}^{n+1} + 8S_{i-1}^{n+1} - 18S_i^{n+1} + 8S_{i+1}^{n+1} + S_{i+2}^{n+1})$$
$$= (S_{i-1}^n + 4S_i^n + S_{i+1}^n) - \frac{\eta \Delta t}{2}\frac{3}{l}(S_{i+1}^n - S_{i-1}^n) + \frac{\rho \Delta t}{4l^2}(S_{i-2}^n + 8S_{i-1}^n - 18S_i^n + 8S_{i+1}^n + S_{i+2}^n),$$

the above equation is rewritten as

$$-f_1 S_{i-2}^{n+1} + f_2 S_{i-1}^{n+1} + f_3 S_i^{n+1} + f_4 S_{i+1}^{n+1} - f_1 S_{i+2}^{n+1} = f_1 S_{i-2}^n + f_5 S_{i-1}^n + f_6 S_i^n + f_7 S_{i+1}^n + f_1 S_{i+2}^n, \quad (3.4)$$

here,
$f_1 = \frac{\rho \Delta t}{4l^2}, f_2 = 1 - \frac{3\eta \Delta t}{2l} - \frac{2\rho \Delta t}{l^2}, f_3 = 4 + \frac{9\rho \Delta t}{2l^2}, f_4 = 1 + \frac{3\eta \Delta t}{2l} - \frac{2\rho \Delta t}{l^2}, f_5 = 1 + \frac{3\eta \Delta t}{2l} + \frac{2\rho \Delta t}{l^2}, f_6 = 4 - \frac{9\rho \Delta t}{2l^2}$,
$f_7 = 1 + \frac{3\eta \Delta t}{2l} + \frac{2\rho \Delta t}{l^2}$,
for $i = M$

$$(S_{M-1}^{n+1} + 4S_M^{n+1} + S_{M+1}^{n+1}) + \frac{\eta \Delta t}{2}\frac{3}{l}(S_{M+1}^{n+1} - S_{M-1}^{n+1}) - \frac{\rho \Delta t}{4l^2}(14S_{M+1}^{n+1} - 33S_M^{n+1} + 28S_{M-1}^{n+1} - 14S_{M-2}^{n+1} + 6S_{M-3}^{n+1} - S_{M-4}^{n+1})$$
$$= (S_{M-1}^n + 4S_M^n + S_{M+1}^n) - \frac{\eta \Delta t}{2}\frac{3}{l}(S_{M+1}^n - S_{M-1}^n) + \frac{\rho \Delta t}{4l^2}(14S_{M+1}^n - 33S_M^n + 28S_{M-1}^n - 14S_{M-2}^n + 6S_{M-3}^n - S_{M-4}^n),$$

we may rewrite the equation as follows:

$$r_1S_{M-5}^{n+1}+r_2S_{M-4}^{n+1}+r_3S_{M-3}^{n+1}+r_4S_{M-2}^{n+1}+r_5S_{M-1}^{n+1}+r_6S_{M}^{n+1} = t_1S_{M-5}^{n}+t_2S_{M-4}^{n}+t_3S_{M-3}^{n}+t_4S_{M-2}^{n}+t_5S_{M-1}^{n}+t_6S_{M}^{n}, \quad (3.5)$$

here,

$$r_1=\frac{\rho\Delta t}{4l^2}, r_2=-\frac{3\rho\Delta t}{2l^2}, r_3=\frac{7\rho\Delta t}{2l^2}, r_4=1-\frac{3\eta\Delta t}{2l}-\frac{7\rho\Delta t}{l^2}, r_5=4+\frac{33\rho\Delta t}{4l^2}, r_6=1+\frac{3\eta\Delta t}{2l}-\frac{7\rho\Delta t}{2l^2}, t_1=-\frac{\rho\Delta t}{4l^2}$$
$$t_2=\frac{3\rho\Delta t}{2l^2}, t_3=-\frac{7\rho\Delta t}{2l^2}, t_4=1+\frac{3\eta\Delta t}{2l}+\frac{7\rho\Delta t}{l^2}, t_5=4-\frac{33\rho\Delta t}{4l^2}, t_6=1-\frac{3\eta\Delta t}{2l}+\frac{7\rho\Delta t}{2l^2}.$$

Equations (3.3) - (3.5) may be written in the following system $XS^{n+1}=YS^n$. Here, $S=[S_{-1},S_0,S_1,\cdots,S_M,S_{M+1}]^T$

$$X=\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & & & & \\ -f_1 & f_2 & f_3 & f_4 & -f_1 & & & & & \\ & -f_1 & f_2 & f_3 & f_4 & -f_1 & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & & & -f_1 & f_2 & f_3 & f_4 & -f_1 \\ & & & r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \end{pmatrix}$$

$$\text{and } Y=\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & & & \\ f_1 & f_5 & f_6 & f_7 & f_1 & & & & \\ & f_1 & f_5 & f_6 & f_7 & f_1 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & & f_1 & f_5 & f_6 & f_7 & f_1 \\ & & & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \end{pmatrix}$$

In this system, the number of unknowns is $M+3$, while the number of equations is $M+1$ equations. By applying Neumann or Dirichlet boundary conditions, we may eliminate S_{-1} and S_{M+1} to reduce the equations to $M+1$ with $M+1$ unknowns. We only solve $S_0,\cdots,S_{M-1}$ for periodic boundary conditions. Once S_{-1} and S_{M+1} are eliminated, the resulting system may be solved at any time level. By using B-spline approximation initial vector can be calculated.

4. Error analysis

In this part, an error bound has been found for the proposed scheme. The following equations may be obtained from
(2.1)

$$H(u_k)=S_{k-1}+4S_k+S_{k+1}, (4.1)$$
$$H'(u_k)=\frac{3}{l}(S_{k+1}-S_{k-1}), (4.2)$$
$$H''(u_k)=\frac{6}{l^2}(S_{k-1}-2S_k+S_{k+1}), (4.3)$$
$$[H'(u_{k-1})+4H'(u_k)+H'(u_{k+1})]=\frac{3}{l}[X(u_{k+1})-X(u_{k-1})], (4.4)$$
$$l^2H''(u_k)=6[X(u_{k+1})-X(u_k)]-2l[2X'(u_k)+X'(u_{k+1})]. (4.5)$$

As stated in [21], we can utilize the operator notation $E\big(H(u_k)\big)=H(u_{k+1})$ and $E=e^{lD}$ where $D\equiv d/dx$. We obtain the following equations

$$e^{lD}+e^{-lD}=2\left(1+\frac{l^2D^2}{2!}+\frac{l^4D^4}{4!}+\frac{l^6D^6}{6!}+\cdots\right), (4.6)$$
$$e^{lD}-e^{-lD}=2\left(lD+\frac{l^3D^3}{3!}+\frac{l^5D^5}{5!}+\frac{l^7D^7}{7!}+\cdots\right). (4.7)$$

Using above equations in (4.4) and (4.5), we get

$$l[E^{-1}+4+E]H'(u_k)=\frac{3}{l}(E-E^{-1})X(u_k), (4.8)$$
$$l^2H''(u_k)=6[E-1]X(u_k)-2l(2+E)X'(u_k). (4.9)$$

After applying equations (4.6) and (4.7) to equation (4.8), we get

$$\left[4+2\left(1+\frac{l^2D^2}{2!}+\frac{l^4D^4}{4!}+\frac{l^6D^6}{6!}+\cdots\right)\right]H'(u_k)=\frac{6}{l}\left[lD+\frac{l^3D^3}{3!}+\frac{l^5D^5}{5!}+\frac{l^7D^7}{7!}+\cdots\right]X(u_k).$$

The given expression can be simplified as follows:

$$H'(u_k)=\frac{6}{l}\left[lD+\frac{l^3D^3}{3!}+\frac{l^5D^5}{5!}+\frac{l^7D^7}{7!}+\cdots\right]\left[4+2\left(1+\frac{l^2D^2}{2!}+\frac{l^4D^4}{4!}+\frac{l^6D^6}{6!}+\cdots\right)\right]^{-1}X(u_k).$$

The above expression can be further simplified as follows:

$$H'(u_k)=\left(D-\frac{l^4D^5}{180}+\cdots\right)X(u_k), (4.10)$$

$$H'(u_k)=\left(X'(u_k)-\frac{l^4X^{(5)}(u_k)}{180}+\cdots\right)+O(l^6). (4.11)$$

Using equations (4.9) and (4.11), we get

$$l^2H''(u_k)=6[e^{lD}-1]X(u_k)-2l[2+e^{lD}]\left[X'(u_k)-\frac{1}{180}l^4X^{(5)}(u_k)+\cdots\right]. (4.12)$$

After reorganizing the terms and expanding e^{lD} in the above equation, we get

$$H''(u_k)=\left(X''(u_k)-\frac{l^2X^{(4)}(u_k)}{12}+\frac{l^4X^{(6)}(u_k)}{360}\dots\right)+O(l^6). \quad (4.13)$$

4.1. Error analysis of advection-diffusion equation: The following expressions have been used for error analysis

$$H'(u_k)=X'(u_k)-\frac{1}{180}l^4X^{(5)}(u_k)+O(l^6), \quad (4.14)$$

$$H''(u_k)=X''(u_k)-\frac{l^2X^{(4)}(u_k)}{12}+\frac{1}{360}l^4X^{(6)}(u_k)+O(l^6). (4.15)$$

The typical finite difference formula can be used to determine the truncation error related to the time discretization of the advection-diffusion equation. Consider,

$$X_t=g(X,X_u,X_{uu}). (4.16)$$

We use finite difference to discretize the time derivative and obtain the following:

$$\frac{X_k^{n+1}-X_k^n}{\Delta t}=\frac{g^n+g^{n+1}}{2}. (4.17)$$

Using the Taylor series expansion, we get

$$(X_t)_k=g_k+\frac{1}{12}\Delta t^2g_{tt}+\cdots. (4.18)$$

The advection-diffusion equation has a truncation error

$$e_k=(X_t)_k+\eta(X_u)_k-\rho(X_{uu})_k. (4.19)$$

The truncation error of the advection-diffusion equation may be expressed as follows, by putting the values of errors resulting from space and time approximations

$$e_k=\left(\frac{1}{12}\right)\Delta t^2g_{tt}-\eta\frac{1}{180}l^4X_k^{(5)}-\rho\frac{1}{360}l^4X_k^{(6)}+O(\Delta t^3,l^6). (4.20)$$

Therefore, the final truncation error is as follows:

$$e_k=O(\Delta t^2)+O(l^4). (4.21)$$

The proposed method uses spline approximation with uniformly distributed interior points, making it accurate to $O(\Delta t^2+l^4)$.

5. Stability analysis

We have used the Von-Neumann method to assess the stability of the proposed method. We get the following equation by using finite difference scheme on (1.1)

$$-f_1S_{i-2}^{n+1}+f_2S_{i-1}^{n+1}+f_3S_i^{n+1}+f_4S_{i+1}^{n+1}-f_1S_{i+2}^{n+1}=f_1S_{i-2}^{n}+f_5S_{i-1}^{n}+f_6S_i^{n}+f_7S_{i+1}^{n}+f_1S_{i+2}^{n}. \quad (5.1)$$

The values for f_j 's are mentioned in the section 3. Putting $S_i^n=A\phi^n\exp(ki\xi l)$, here ξ is mode number, l is step length, $k=\sqrt{-1}$ and A is amplitude, we obtain

$$\phi=\frac{f_1e^{-2k\xi l}+f_5e^{-k\xi l}+f_6+f_7e^{k\xi l}+f_1e^{2k\xi l}}{-f_1e^{-2k\xi l}+f_2e^{-k\xi l}+f_3+f_4e^{k\xi l}-f_1e^{2k\xi l}}. \quad (5.2)$$

By putting values of f_j 's in (5.2), we obtain the following equation

$$\phi=\frac{4l^2(\cos(\xi l)+2)-\rho\Delta t(9-8\cos(\xi l)-\cos(2\xi l))-6k\eta l\Delta t\sin(\xi l)}{4l^2(\cos(\xi l)+2)+\rho\Delta t(9-8\cos(\xi l)-\cos(2\xi l))+6k\eta l\Delta t\sin(\xi l)}, \quad (5.3)$$

above equation may be written as

$$\phi=\frac{C_1-iD}{C_2+iD}, \quad (5.4)$$

here, $C_1=4l^2(\cos(\xi l)+2)-\rho\Delta t(9-8\cos(\xi l)-\cos(2\xi l)), C_2=4l^2(\cos(\xi l)+2)+\rho\Delta t(9-8\cos(\xi l)-\cos(2\xi l))$ and $D=6\eta l\Delta\, t\sin(\xi l)$, which gives $|\phi|^2=C_1^2+D^2/C_2^2+D^2$. $|\phi|\leq 1$ is necessary for stability. That is, $\frac{C_1^2+D^2}{C_2^2+D^2}\leq 1$. For stability, we must prove that $C_1^2\leq C_2^2$. It can be seen that, $C_1^2-C_2^2\leq 0$. Therefore, the proposed approach is stable for any values of Δu and Δt $i.e.$, the proposed approach is unconditionally stable.

6. Numerical Results and Discussion

In this section, seven examples with Periodic, Neumann, and Dirichlet boundary conditions have been solved using the proposed method. The maximum absolute error norm formula has been used to evaluate accuracy of proposed method.

$$L_\infty=\|X^{\text{exact}}-X^M\|_\infty=\max|X_k^{\text{exact}}-X_k^M|, \quad (6.1)$$

here, X^M is numerical solution. X_k^{exact} and X_k^M are exact and numerical solution at the node u_k, respectively. The formula to determine the rate of convergence is

$$r=\frac{\ln\left(E(M_2)/E(M_1)\right)}{\ln(M_1/M_2)}, \quad (6.2)$$

here, $E(M_1)$ and $E(M_2)$ are absolute error norms with grid points M_1 and M_2, respectively.

Periodic Problems:

Example 1: We consider advection-diffusion equation (1.1) with the following exact solution [22]

$$X(u,t)=e^{-4\pi^t}\cos\big(2\pi(u-t)\big), \qquad 0<u<1, \quad (6.3)$$

with the boundary and initial conditions given by

$$X(0,t)=X(1,t), \quad (6.4)$$
$$X(u,0)=\cos(2\pi u). \quad (6.5)$$

For numerical experiment, we take $\eta=\rho=1$. In Table 1, calculated error norms at time $t=0.1, 0.2, 0.3$, and 0.4 are depicted. Numerical outcomes at $t=0.1, 0.2, 0.3$, and 0.4 with $\eta=1$ and $\rho=\Delta t=0.001$ have been shown in Figure 1. It is evident that the B-spline method produces very good results.

The value of M(t) has been calculated at time $t=0.1, 0.2, 0.3$, and 0.4

$$M(t)=\int_m^n X(u,t)dt \quad (6.6)$$
$$\mathrm{M(t)}=1.87\mathrm{E}-08$$

We can observe that the value of $M(t)$ is constant at different values of t.

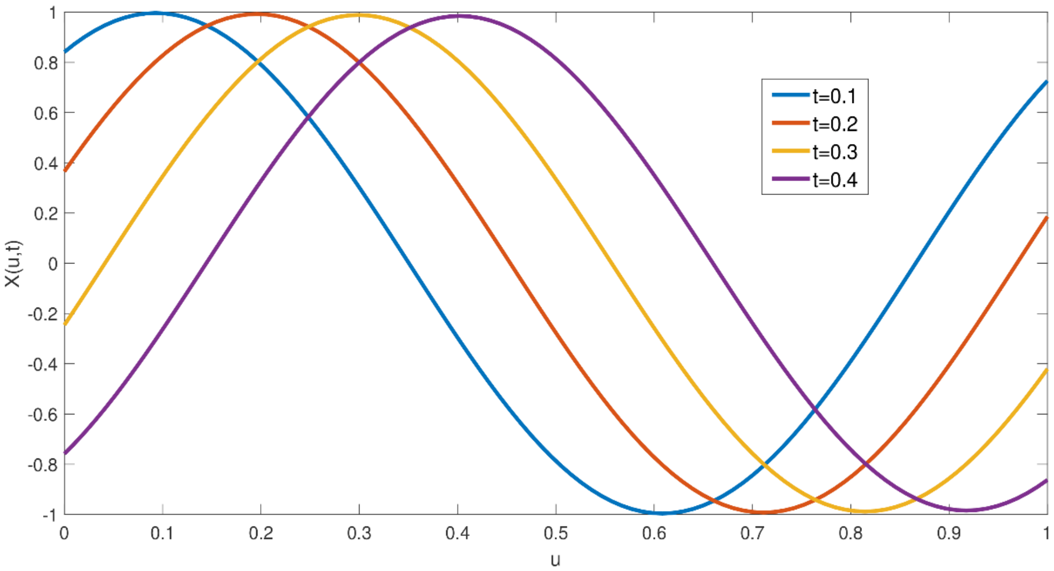

FIGURE 1. Numerical simulation of Example 1 at $n = 100$

TABLE 1. Error norms for Example 1 at $n = 100$ with $\Delta t = 0.0001$

u	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$
0.1	7.37e − 08	1.31e − 06	5.65e − 09	1.88e − 08
0.2	2.98e − 08	1.45e − 06	6.90e − 09	1.85e − 08
0.3	3.26e − 08	1.03e − 06	9.66e − 09	1.83e − 08
0.4	8.98e − 08	1.12e − 07	1.56e − 09	1.82e − 08
0.5	1.20e − 07	6.96e − 07	1.43e − 08	1.84e − 08
0.6	1.11e − 07	1.35e − 06	3.19e − 08	1.86e − 08
0.7	6.66e − 08	1.49e − 06	4.44e − 08	1.90e − 08
0.8	4.08e − 09	1.07e − 06	4.72e − 08	1.92e − 08
0.9	5.27e − 08	2.50e − 07	3.91e − 08	1.93e − 08

Example 2: The exact solution to the advection-diffusion equation (1.1) is

$$X(u,t) = e^{-\rho t}\sin\big(2\pi(u-\eta t)\big), \qquad 0 < u < 1, (6.7)$$

the boundary condition is given by

$$X(0,t) = X(1,t), (6.8)$$

the initial condition is

$$X(u,0) = \sin(2\pi u), (6.9)$$

For numerical simulation, we take $\Delta t = 0.001, n = 50, \eta = 1$, and $\rho = 10^{-5}, 10^{-6}$. In Table 2, evaluated error norms at $t = 0.2$ and 0.4 have been presented. The numerical and exact values at $t = 0.1, 0.2, 0.3$, and 0.4 have been illustrated in Figure 2. It can be observed that the numerical values closely match the exact solution. We computed $M(t)$ and found that the value of $M(t)$ is constant at different times.

$$M(t) = \int_m^n X(u,t)dt = 0.0025. (6.10)$$

Problems having Neumann's boundary conditions:
Example 3: We consider (1.1) with the exact solution [10][23]

$$X(u,t) = pe^{qt-ru}, \quad 0 < u < 1, (6.11)$$

the initial and Neumann's boundary conditions are given by

$$X(u,0) = p\exp(-ru), (6.12)$$

$$\left(\frac{\partial X}{\partial u}\right)_{(0,t)} = -pr\exp(qt), \quad \left(\frac{\partial X}{\partial u}\right)_{(1,t)} = -pr\exp(qt-r), \quad (6.13)$$

here, $r = \frac{-\eta+\sqrt{\eta^2+4\rho b}}{2\rho}$, $p = 1$, and $q = 0.1$. For numerical simulation, we take $\eta = 1$ and $\rho = 0.001$. In Table 3, with $l = 0.1$ and $\Delta t = 0.001$ the calculated outcomes have been reported. We can see that our proposed technique gives better results. In Figure 3, calculated results have been shown and the figure is identical to the one shown by [10], [23].

TABLE 2. Error norms for Example 2

	$\rho = 10^{-5}$		$\rho = 10^{-6}$	
u	$t = 0.2$	$t = 0.4$	$t = 0.2$	$t = 0.4$
0.1	4.99e − 05	1.42e − 04	9.27e − 06	1.10e − 05
0.2	5.88e − 06	1.50e − 04	5.87e − 06	1.83e − 05
0.3	4.04e − 05	9.99e − 04	2.32e − 07	1.85e − 05
0.4	7.13e − 05	1.17e − 05	5.50e − 06	1.17e − 05
0.5	7.51e − 05	8.10e − 05	9.13e − 06	4.64e − 07
0.6	4.96e − 05	1.42e − 04	9.27e − 06	1.10e − 05
0.7	5.87e − 06	1.50e − 04	5.87e − 06	1.83e − 05
0.8	4.04e − 05	9.99e − 04	2.32e − 07	1.85e − 05
0.9	7.14e − 05	1.17e − 05	5.50e − 06	1.17e − 05

Example 4: We consider advection-diffusion equation (1.1) with the exact solution [10]

$$X(u,t) = e^{\epsilon u+\gamma t}, \quad 0 < u < 1, (6.14)$$

with the initial and boundary conditions as follows:

$$X(u,0) = e^{\epsilon u}, \quad (6.15)$$

$$\left(\frac{\partial X}{\partial u}\right)_{(0,t)} = \epsilon e^{\gamma t}, \quad \left(\frac{\partial X}{\partial u}\right)_{(1,t)} = \epsilon e^{\epsilon+\gamma t}. (6.16)$$

For numerical experiment, we choose $\Delta t = 0.001, n = 40, \epsilon = 0.02854797991928, \gamma = -0.0999, \eta = 3.5$, and $\rho = 0.022$. In Table 4, calculated outcomes have been reported and compared with those calculated by [10]. We can see that our proposed scheme gives better results than the method given in [10]. The numerical results have been illustrated in Figure 4.

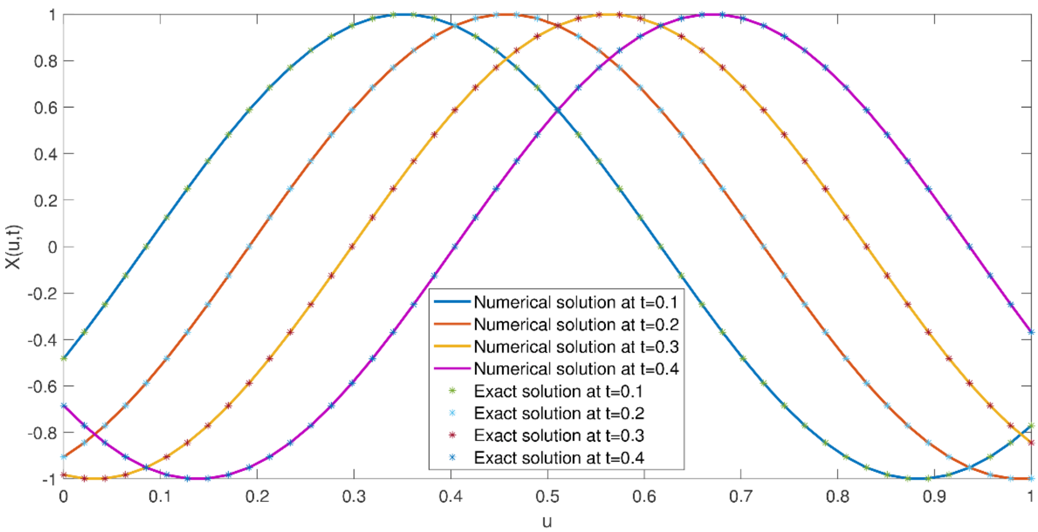

FIGURE 2. Numerical simulation of Example 2 with $\rho = 10^{-6}$

TABLE 3. Error norms for Example 3 at $n = 10$ with $\eta = 1$ and $\rho = 0.001$

u	$t = 1$	$t = 5$	$t = 20$
0.1	8.50e − 11	5.87e − 10	6.32e − 09
0.2	8.54e − 11	5.69e − 10	6.20e − 09
0.3	8.38e − 11	5.81e − 10	6.29e − 09
0.4	8.40e − 11	5.62e − 10	6.16e − 09
0.5	8.27e − 11	5.76e − 10	6.26e − 09
0.6	8.26e − 11	5.56e − 10	6.12e − 09
0.7	8.12e − 11	5.70e − 10	6.22e − 09
0.8	8.04e − 11	5.48e − 10	6.08e − 09
0.9	7.90e − 11	5.64e − 10	6.19e − 09

Problems having Dirichlet boundary conditions:

Example 5: We consider advection-diffusion equation (1.1) with the exact solution [2]

$$X(u,t) = \frac{1}{r}\exp\left(-50\frac{(u-t)^2}{r}\right), \qquad 0 < u < 1, \qquad (6.17)$$

the initial and boundary conditions are given by

$$X(u,0) = \frac{1}{r}\exp\left(\frac{-50u^2}{r}\right),$$
$$X(0,t) = \frac{1}{r}\exp\left(-50\frac{t^2}{r}\right),$$
$$X(1,t) = \frac{1}{r}\exp\left(-50\frac{(1-t)^2}{r}\right),$$

here, $r = 1 + 200\rho t$. Computed error norms at various time levels have been depicted in Table 5. In Table 6, the outcomes of the current approach are contrasted with those of Nazir *et al.* [15]. We found that the proposed method gives better results than the results given in [15]. In Figure 5, exact and numerical solutions have been depicted. It is evident that the numerical solutions align closely with the exact solutions.

Example 6: The exact solution [24] to the advection-diffusion equation (1.1) is given by

$$X(u,t) = \frac{e^{\frac{\eta u}{\rho}} - 1}{e^{\frac{\eta}{\rho}} - 1}, \quad 0 < u < 1, (6.18)$$

the boundary condition is given by

$$X(0,t) = 0, \qquad X(1,t) = 1 \ (6.19)$$

Initial conditions are taken from the exact solution. Calculated error norms at time $t = 2, 4$, and 10 have been displayed in Table 7. We also calculate the rate of convergence at various values of n and illustrated in Table 8. We can observe that our method gives minimum error. In Figure 6, the obtained numerical results have been presented.

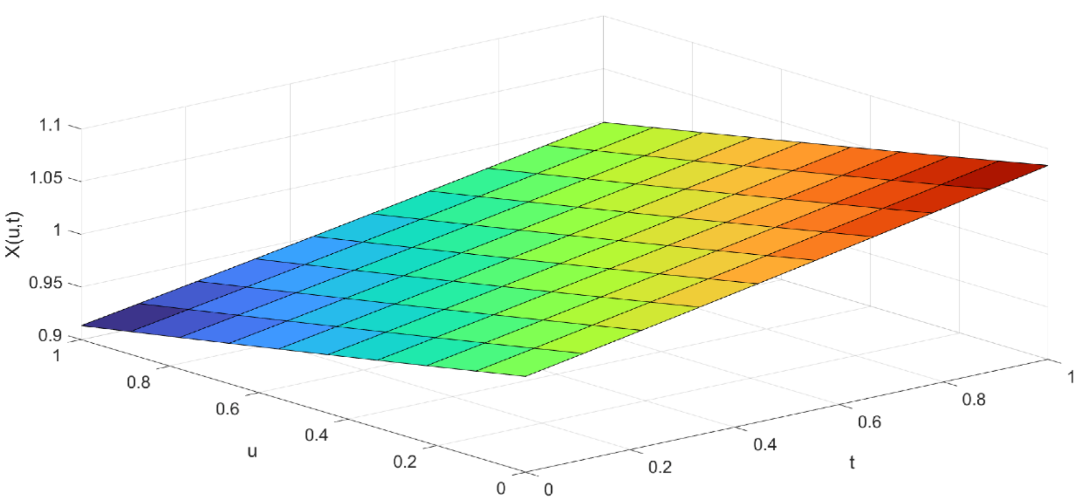

FIGURE 3. Numerical simulation of Example 3 with $n = 10$

TABLE 4. Comparison of error norms for Example 4

t	Present Method	Method I [10]	Method II [10]
0.2	1.66e − 11	1.67e − 09	8.46e − 06
0.4	3.28e − 11	3.29e − 09	1.75e − 05
0.6	4.87e − 11	4.87e − 09	2.70e − 05

0.8	6.42e − 11	6.42e − 09	3.63e − 05
1.0	7.94e − 11	7.93e − 09	4.54e − 05
5.0	3.78e − 10	3.27e − 08	1.95e − 04
20.0	7.19e − 10	7.18e − 08	4.30e − 04

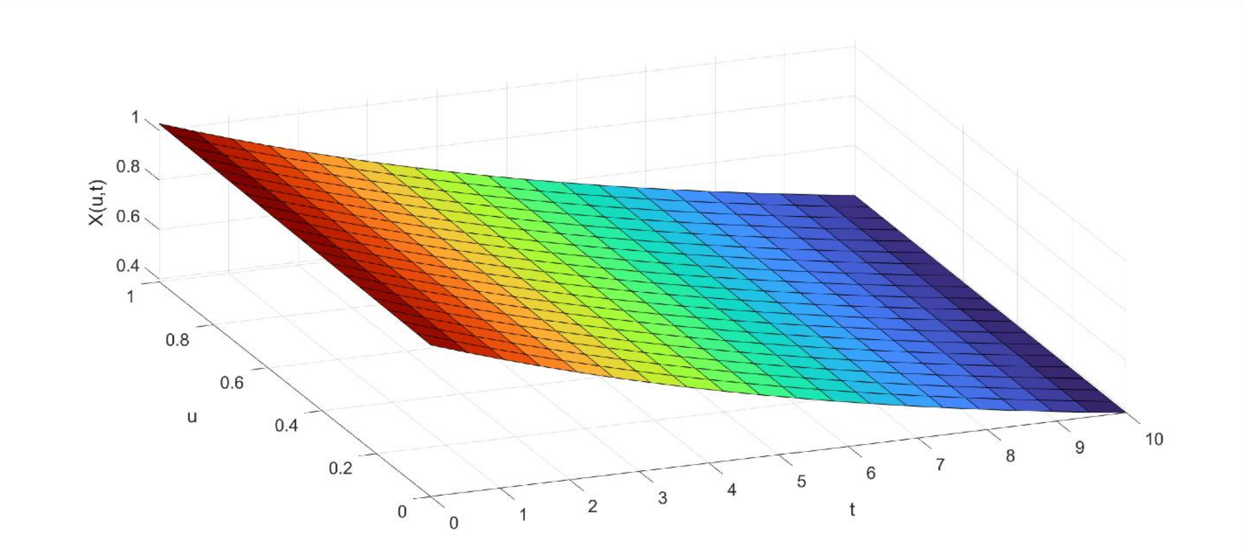

FIGURE 4. Numerical simulation of Example 4 with $\Delta t = 0.001$ and $n = 20$

TABLE 5. Error norms for Example 5 with $\Delta t = 0.001, n = 10$, and $\eta = \rho = 1$

u	$t = 1$	$t = 2$	$t = 5$
0.1	5.44e − 10	1.58e − 05	2.93 − 12
0.2	8.60e − 10	1.65e − 05	3.80e − 12
0.3	4.20e − 12	1.72e − 05	6.38e − 07
0.4	8.28e − 10	1.77e − 05	4.09e − 12
0.5	6.01e − 10	1.82e − 05	3.63e − 12
0.6	3.36e − 10	1.86e − 05	2.96e − 12
0.7	1.25e − 10	1.89e − 05	2.27e − 12
0.8	5.19e − 11	1.91e − 05	1.69e − 12
0.9	3.10e − 10	1.92e − 05	1.69e − 12

Example 7: The exact solution [24] for equation (1.1) is given by

$$X(u,t) = \exp\left(5\left(u - \frac{t}{2}\right)\right)\exp\left(-\frac{\pi^2 t}{40}\right)\left[\frac{1}{4}\sin\left(\frac{\pi u}{2}\right) + \cos\left(\frac{\pi u}{2}\right)\right], \qquad 0 < u < 1, (6.20)$$

the initial condition is

$$X(u,0) = \exp(5u)\left[\frac{1}{4}\sin\left(\frac{\pi u}{2}\right) + \cos\left(\frac{\pi u}{2}\right)\right] \quad (6.21)$$

Boundary conditions are taken from the exact solution. For numerical experiment, we take $\rho = 0.1$ and $\eta = 1$. The calculated error norms have been presented in Table 9 with $t = 2$. It is evident from the table that the present approach gives better outcomes than Mohebbi [25]. In Figure 7, approximate results have been illustrated graphically.

TABLE 6. Comparison of error norms for Example 5 with $\eta = \rho = 1$

	t = 1		t = 2	
	$\Delta t = 0.001$	$\Delta t = 0.001$	$\Delta t = 0.01$	$\Delta t = 0.001$
n	Present Method	Nazir et al. [15]	Present Method	Nazir et al. [15]
4	6.34e − 07	3.38e − 05	9.49e − 09	6.66E − 06
8	3.07e − 08	7.93e − 06	1.47e − 08	1.63e − 06
16	3.90e − 09	1.92e − 06	1.64e − 08	4.08e − 07
32	1.95e − 09	4.66e − 07	1.65e − 08	1.02e − 07
64	1.83e − 09	1.03e − 07	1.65e − 08	2.56e − 08
128	1.83e − 09	1.14e − 08	1.66e − 08	6.54e − 09

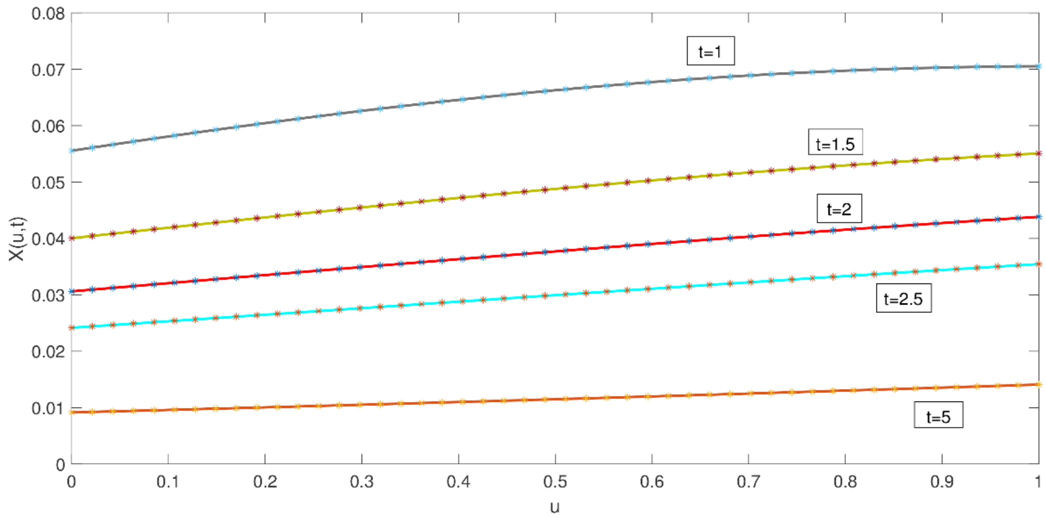

FIGURE 5. Numerical simulation of Example 5 with $\eta = \rho = 1, \Delta t = 0.001$, and $n = 50$

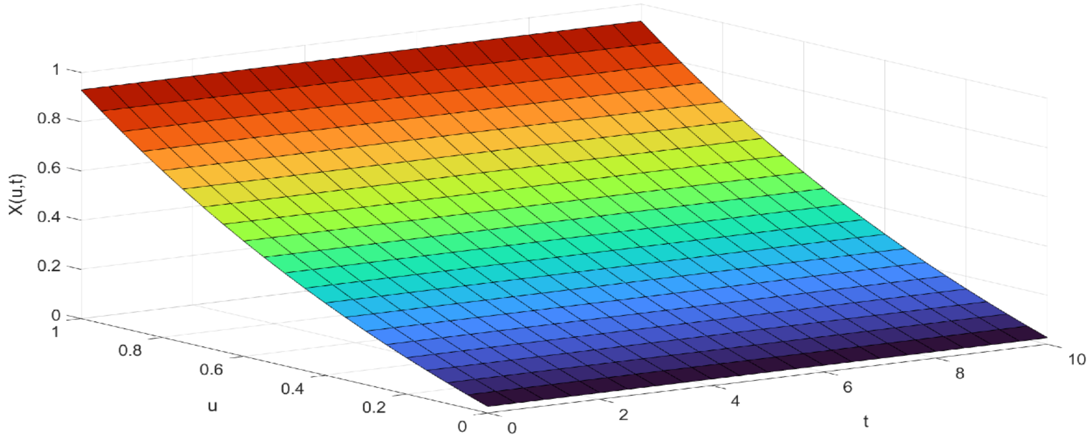

FIGURE 6. Numerical simulation of Example 6 with $\Delta t = 0.1, \eta = \rho = 1$, and $n = 20$

TABLE 7. Error norms of Example 6 with $n = 300$ and $\Delta t = 0.1$

u	$t = 2$	$t = 4$	$t = 10$
0.1	4.58e − 16	1.19e − 15	6.59e − 16
0.2	2.66e − 15	3.58e − 15	5.16e − 15
0.3	8.79e − 15	9.94e − 15	1.68e − 14
0.4	1.48e − 14	2.14e − 14	2.85e − 14
0.5	2.32e − 14	3.30e − 14	4.03e − 14
0.6	2.39e − 14	5.04e − 14	4.57e − 14
0.7	2.68e − 14	5.23e − 14	4.35e − 14
0.8	1.95e − 14	3.33e − 14	1.59e − 14
0.9	1.19e − 14	3.19e − 14	2.33e − 15

TABLE 8. Error norms and rate of convergence for Example 6 at $t = 1$

n	L_∞ errors	order of convergence
10	$9.46e - 08$	-
20	$6.18e - 09$	3.94
40	$3.91e - 10$	3.98
80	$2.45e - 11$	3.99

160	$1.51e-12$	4.02

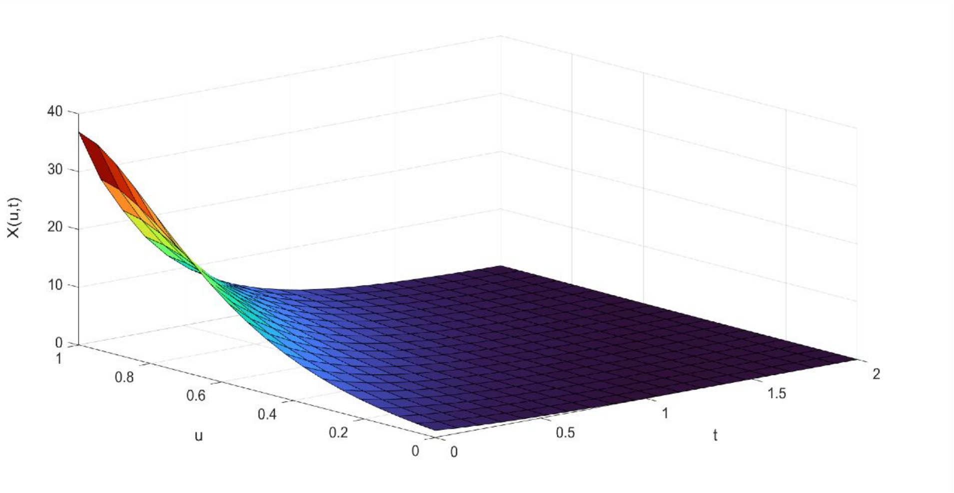

FIGURE 7. Numerical simulation of Example 7 with $\Delta t = 0.01, t \leq 2$, and $n = 20$

TABLE 9. Error norms for Example 7 at $t = 2$

	$l = 0.02$		$l = 0.01$	
u	Mohebbi [25]	Present Method	Mohebbi [25]	Present Method
0.1	$7.17e-06$	$5.03e-07$	$1.80e-06$	$4.99e-07$
0.2	$1.10e-05$	$1.46e-06$	$2.77e-06$	$1.44e-06$
0.3	$1.66e-05$	$3.09e-06$	$4.17e-06$	$3.06e-06$
0.4	$2.46e-05$	$5.69e-06$	$6.17e-06$	$5.62e-06$
0.5	$3.59e-05$	$9.50e-06$	$9.01e-06$	$9.36e-06$
0.6	$5.16e-05$	$1.45e-05$	$1.29e-05$	$1.43e-05$
0.7	$7.32e-05$	$2.01e-05$	$1.84e-05$	$1.98e-05$
0.8	$1.02e-04$	$2.42e-05$	$2.55e-05$	$2.37e-05$
0.9	$1.36e-04$	$2.13e-05$	$3.42e-05$	$2.08e-05$

7. Conclusion

In this work, modified cubic B-spline approach has been applied on advection-diffusion equation to find the numerical solutions. This method is a new approximation to solve the advection-diffusion equation. To discretize the equation, finite difference scheme has been applied. Von Neumann's scheme has been employed to perform the stability analysis. The proposed spline-based fourth-order method has been shown to be unconditionally stable. The convergence study reveals that the suggested technique is $O(\Delta t^2 + l^4)$ accurate. We have studied seven significant advection-diffusion problems. The obtained outcomes have been illustrated in tables and displayed in graphs. It is evident that both diffusion-dominated and advection-dominated problems may be solved using this approach. A wide class of partial differential equations with Periodic, Neumann, or Dirichlet boundary conditions may be solved very effectively by this method.

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