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Self-Similar Shock Wave Behavior for the Inviscid Burgers Equation in Various Geometries

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Abstract:- In this research, we will investigate closed-form self-similar shock wave solutions for the inviscid Burgers equation in planar, cylindrical, and spherical geometries. We will derive the self-similar forms of the equations, examine the corresponding ordinary differential equations, and discuss the physical implications of these solutions in different geometric contexts. The approach employed is based on Lee's method for deriving self-similar solutions to the Euler equations in compressible fluid dynamics. This includes two types of self-similarity: one constrained by integral relations and the other by the need for solution regularity along limiting characteristics. The findings highlight the theoretical basis for Taylor-Sedov blast waves (first kind) and Guderley implosion problems (second kind). This thorough analysis aims to enhance our understanding of shock wave behavior and offer a unified framework for exploring-similar phenomena across various geometries.

Keywords*: Inviscid Burgers' Eauation, Similarity solutions, Shock Waves.*

1. Introduction

. Self-similar shock wave dynamics is a key area in the study of nonlinear partial differential equations, especially in the context of fluid dynamics. Also the study of shock waves in various geometries provides crucial insights into the behavior of dynamic systems described by nonlinear partial differential equations. Among the most significant equations in this field is the inviscid Burgers' equation, a fundamental model for shock wave phenomena in one-dimensional compressible flow. One prominent equation in this domain is the inviscid Burgers' equation, which is given by:

$$\frac{\partial c}{\partial t} + c\frac{\partial c}{\partial x} = 0 \tag{1}$$

where c is a scalar representing the wave's amplitude. This equation serves as a onedimensional model for studying compressible flow and fluid dynamics. The inviscid Burgers' equation incorporates the convective non-linearity inherent in the Euler equations for inviscid flow. Specifically, the convection speed c is equal to the transported variable within the partial derivatives of both time and space. This characteristic leads to wave steepening in compressive regions, ultimately resulting in shock formation.

The Burgers equation is often utilized to model various dynamical systems where the Euler equations simplify. In gas dynamics, weak shocks are frequently represented by the inviscid Burgers equation [2, 3]. Moreover, in the realm of reactive compressible flows, reactive forms of the inviscid Burgers equation are commonly employed [4-6]. Additionally, several ad hoc extensions of the inviscid Burgers equation have been proposed as simplified models to explore certain nonlinear dynamics in compressible reactive flows [7-10].

In this study, we seek to derive similarity solutions for shock waves that satisfy the inviscid Burgers' equation. These solutions will draw analogies with well-established results such as the Taylor-Sedov solutions for point blast explosions in compressible flows [11, 12] and the Guderley solution for self-similar shock implosions [13]. To achieve this, we will use the systematic approach developed by Lee to unify self-similar shock propagation problems in perfect gases [14]. Notably, Mi et al. have previously proposed explosion solutions for planar problems modeled by the inviscid Burgers equation using the method of characteristics [10].

2. THE BURGERS-FICKETT EQUATION

The continuity equation, fundamental in the study of fluid dynamics and other fields, is expressed as:

$$\frac{\partial \rho}{\partial t}+\frac{\partial q}{\partial x}=-q\frac{j}{x} \tag{2}$$

where:

- ρ represents the density of the conserved quantity.
- q denotes the flux, which is the rate of transfer of the quantity per unit time per unit area.
- j is a parameter that takes the values 0,1 , or 2 for planar, cylindrical, and spherical geometries, respectively.

The equation reflects the principle of conservation of mass (or another conserved quantity) and describes how changes in density and flux are related in different geometric contexts.

Following Fickett [15,16], who explored the application of the continuity equation in the context of gas dynamics, we consider a simplified approach for modeling. In this approach, the flux q is modeled as a function of ρ alone. One of the simplest and most common forms of flux is the Burgers flux, given by:

$$q=\frac{\rho^2}{2}$$

Substituting this into the continuity equation, we obtain a Burgers-like equation:

$$\frac{\partial \rho}{\partial t}+\rho\frac{\partial \rho}{\partial x}=-\frac{\rho^2}{2}\frac{j}{x} \tag{3}$$

This equation is a non-linear partial differential equation that describes the dynamics of the conserved quantity under the influence of the specified flux function.

The Burgers-like equation provides a foundation for understanding more complex models. Similarity solutions, which reduce the partial differential equation to an ordinary differential equation by assuming specific forms for the solution, can be derived for this model equation. These solutions illustrate how the methodology applies to various extensions of the model.

In particular, the approach pioneered by Lighthill and Whitham [16,17] has been instrumental in extending these concepts to other types of kinematic waves. Their work encompasses a range of applications, including traffic flow, pedestrian flow, and flood waves, demonstrating the versatility of the continuity equation in modeling diverse flow phenomena.

By analyzing similarity solutions and exploring different flux functions, researchers can gain insights into the behavior of material flows in various contexts and develop more accurate models for practical applications.

3. BEHAVIOR OF BURGERS' SHOCKS USING SIMILARITY VARIABLES

Our focus is on shock dynamics. We will analyze a scenario involving strong shocks where the density ρ is zero in front of the shock. Let $F(t)$ denote the shock's path and $\dot{F}(t)$ its velocity. We introduce similarity variables.

$$\xi = \frac{x}{F(t)} \tag{4}$$

$$\phi = \frac{\rho}{\dot{F}(t)} \tag{5}$$

$$\left(\frac{\partial}{\partial t}\right)_x = \left(\frac{\partial}{\partial \xi}\right)_{t'} \left(\frac{\partial \xi}{\partial t}\right)_x + \left(\frac{\partial}{\partial t'}\right)_\xi \left(\frac{\partial t'}{\partial t}\right)_x \tag{6}$$

$$= -\xi \frac{\dot{F}}{F} \left(\frac{\partial}{\partial \xi}\right)_{t'} + \left(\frac{\partial}{\partial t'}\right)_\xi \tag{7}$$

and

$$\left(\frac{\partial}{\partial x}\right)_t = \left(\frac{\partial}{\partial \xi}\right)_{t'} \left(\frac{\partial \xi}{\partial x}\right)_t = \frac{1}{F} \left(\frac{\partial}{\partial \xi}\right)_{t'} \tag{8}$$

By applying these transformations and looking for solutions where $\phi = \phi(\xi)$ only, 1 can be reformulated as an ordinary differential equation:

$$\dot{\phi} = -\frac{\frac{j}{2}\frac{\phi^2}{\psi} + \theta\phi}{\phi - \xi} \tag{9}$$

provided

$$\theta \equiv \frac{\ddot{F}(t)F(t)}{\dot{F}(t)^2} \tag{10}$$

It does not rely on t. Because it is fundamentally not a function of ξ, it must be a constant. Thus, our objective is to find θ and the distribution $\phi(\xi)$ as described by (9).

Since we can re-write (10) as:

$$\theta = \frac{\ddot{F}F}{\dot{F}^2} = \frac{\ddot{F}/\dot{F}}{\dot{F}/F} = \frac{\ln\ \dot{\mathrm{F}}}{\ln\ \mathrm{F}} = \text{ constant} \tag{11}$$

we obtain immediately that the shock speed depends on distance as a power law:

$$\dot{F} = AF^{\theta} \tag{12}$$

Where A is an integration constant, the shock decay coefficient θ varies depending on the specific problem. Below, we present the method for determining θ for two types of selfsimilar problems related to the Burgers equation: similarity solutions of the first kind and the second kind. Our approach is based on blast wave theory in gas dynamics [14].

4. SHOCK MOTION FROM POINT SOURCE MATERIAL RELEASE: SIMILARITY SOLUTIONS OF THE FIRST KIND

Our goal is to address a problem similar to the Taylor-Sedov blast wave scenario, where energy is added to a gas at a specific location, along a line, or across a plane. This creates a diminishing strong shock wave and a self-similar pattern in variables such as speed, density, and pressure. For the Burgers equation, a parallel situation involves abruptly introducing a fixed quantity of material m_j into a medium initially with $\rho = 0$. Here, m_1 denotes the material per unit area in a planar setup, m_2 refers to the material per unit length in a cylindrical setup, and m_3 indicates the material in a spherical context. We seek to determine the shock decay law and the self-similar density distribution behind the shock $\rho(x, t)$, which must satisfy (1) for each geometry.

When material is added at a point, line, or plane, the total amount of material within the decaying shock must remain constant since $\rho = 0$ ahead of the shock. This is expressed by:

$$\int_0^{F(t)} \rho k_j x^j \, \mathrm{d}x = m_j$$

where m_j is a constant. The geometrical factor k_j is $4\pi, 2\pi$, and 1 for spherical, cylindrical, and planar waves, respectively. By applying similarity variables and (12), we derive:

$$m_j = AX(t)^{\theta+j+1} I_j$$

where:

$$I_j = \int_0^1 \phi k_j \xi^j \, \mathrm{d}\psi$$

Since m_j remains constant over time, for a self-similar solution, the exponent of $F(t)$ must be zero, which leads to:

$$\theta = -(1 + j)$$

As a result, the blast decay coefficient θ equals -1 for planar problems, -2 for cylindrical problems, and -3 for spherical problems. The relationship between mass deposition and the integral I_j is expressed by:

$$m_j = BI_j$$

The mass integral helps determine the shock decay profile for the given kinematic wave. In the Taylor-Sedov solution, the pertinent integral for the blast solution is the energy integral of the total energy captured by the shock, corresponding to energy deposition.

For decaying shock waves starting from $F(t=0)=0$, integrating (12) yields:

$$F = Ct^N$$

where $C=(A(1-\theta))^N$ and $N=\frac{1}{1-\theta}$. The growth rates for blast wave radii are $t^{1/4}$ for spherical waves, $t^{1/3}$ for cylindrical waves, and $t^{1/2}$ for planar waves.

The shock speed adheres to the shock jump condition:

$$\dot{F} = \frac{\left[\frac{1}{2}\rho^2\right]_0^s}{[\rho]_0^s} = \frac{1}{2}\rho$$

where $[\alpha]_0^s = \alpha_s - \alpha_0$, and:

$$\phi(\xi = 1) = 2$$

With this boundary condition and the known blast decay coefficient θ, the self-similar profile $\phi(\xi)$ that satisfies (9) is:

$$\phi = 2\xi$$

This profile applies to planar, cylindrical, and spherical geometries.

Given $\phi(\xi)$, the constant I_j is calculated as:

$$I_j = \frac{2k_j}{j+2}$$

with values of $1, 4/3\pi$, and 2π for planar, cylindrical, and spherical waves respectively. Consequently, the constant A in the self-similar solution is $m_1, \frac{3m_2}{4\pi}$, and $\frac{m_3}{2\pi}$ for planar, cylindrical, and spherical waves respectively. Thus, the self-similar shock motion described by (12) is:

$$\text{planar: } \dot{F} = m_1 F^{-1}$$
$$\text{cylindrical: } \dot{F} = \left(\frac{3m_2}{4\pi}\right) F^{-2}$$
$$\text{spherical: } \dot{F} = \left(\frac{m_3}{2\pi}\right) F^{-3}$$

Figure 1 shows the solution for a spherical explosion with $m=1$. The blue line represents the shock trajectory, while the black solid lines depict contours of ρ and the black dashed lines show contours of the similarity variable ξ. The red lines illustrate the characteristics. Notably, the interior solution, given by $\rho = \frac{x}{2t}$, remains unchanged by the addition of initial mass; only the shock trajectory is modified. The characteristics' trajectory, described by simple power laws derived from $\mathrm{d}x/\mathrm{d}t = \rho$, defines the influence region of the lead shock

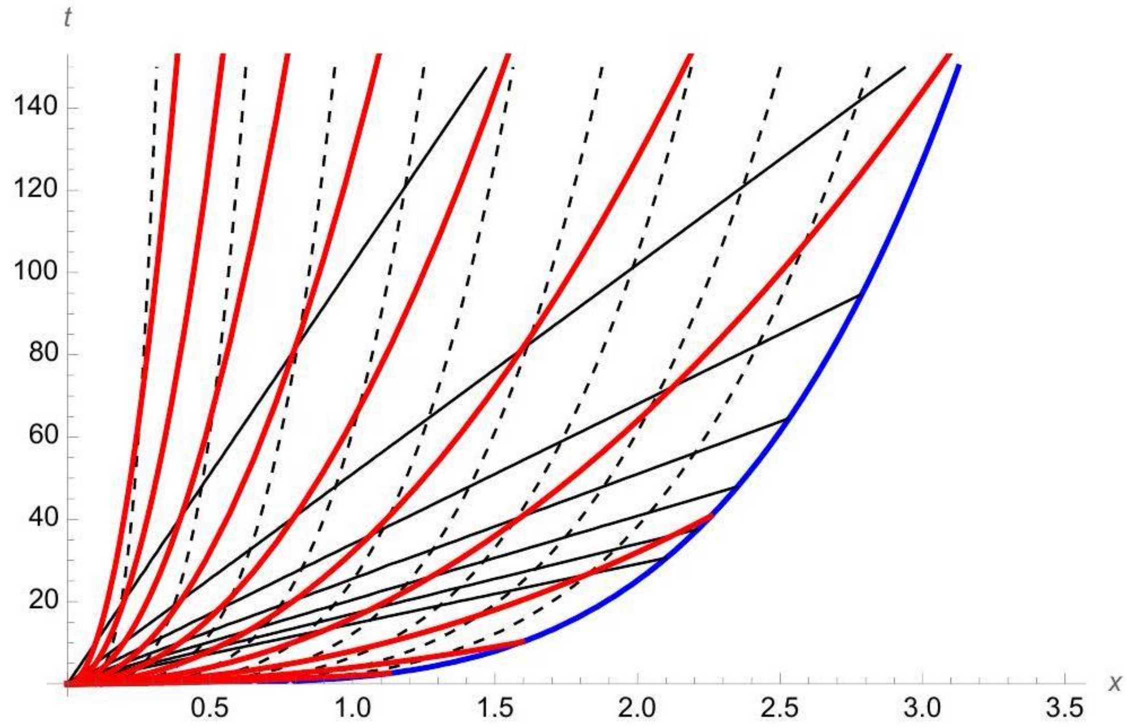

FIG. 1. Space-time diagram for the spherical explosion problem with $m = 1$; the blue line represents the shock trajectory, the black solid lines indicate the contours of ρ, the black dashed lines show the contours of the similarity variable ξ, and the red lines depict the characteristics.

motion, with the characteristics eventually aligning with it. Similar trends are seen in the planar and cylindrical problems, with the planar problem's solution matching that provided by Mi et al. [10].

5. SECOND-KIND SIMILARITY SOLUTIONS: SELF-SIMILAR SHOCK IMPLOSION

The second issue we need to address involves a situation where the shock decay coefficient θ cannot be derived from an integral conservation law of mass, momentum, or energy. In blast wave theory [14], these scenarios are referred to as similarity solutions of the second kind. We will now tackle the problem of shock implosion in cylindrical and spherical cases, focusing specifically on the Guderley problem in shock dynamics [13].

Self-similarity still requires power-law solutions for shock dynamics, as described by (12), with the distribution behind the shock represented by (9). Analyzing (9) reveals that the profiles become singular when the denominator approaches zero, i.e., when $\xi = \phi$. To ensure a regular solution that avoids singularity, the numerator of (9) must also be zero at this point. Denoting this internal point as *, we have $\xi^* = \phi^*$. Setting the numerator to zero immediately provides:

$$\theta = -j/2 \quad (13)$$

The profiles can now be derived by directly integrating the ODEs (9). With the boundary condition at the shock $\phi(1) = 2$, the wave structure solutions are:

$$\phi = \frac{2}{\xi} \text{ for } j = 2 \text{ (spherical)}$$
$$\phi = \frac{2}{\sqrt{\xi}} \text{ for } j = 1 \text{ (cylindrical)} \tag{15}$$

The saddle points in the solutions are at $\xi^* = 2^{1/2}$ for spherically imploding waves and $\xi^* = 2^{3/2}$ for cylindrically imploding waves.

Although the solution is complete, it is useful to further explain the regularization criterion used to obtain the solution and its connection to the underlying dynamics of Burgers' equation. Burgers' equation (1) is a hyperbolic equation with a single set of characteristic curves given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \rho \tag{16}$$

which can be rewritten as:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \phi\dot{F} \tag{17}$$

A line of $\xi =$ constant corresponds to:

$$\mathrm{d}\xi(x,t) = \left(\frac{\partial\xi}{\partial x}\right)_t \mathrm{d}x + \left(\frac{\partial\xi}{\partial t}\right)_x \mathrm{d}t = 0 \tag{18}$$

which simplifies to:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\left(\frac{\partial\xi}{\partial t}\right)_x}{\left(\frac{\partial\xi}{\partial x}\right)_t} \tag{19}$$

By evaluating the partial derivatives from (5), this equation can be rewritten as:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \xi\dot{F} \tag{20}$$

Comparing (17) and (20) shows that the curve where $\xi = \phi$ is where a characteristic surface intersects a constant ψ line. This curve functions as a limiting characteristic, acting as an event horizon for the dynamics of the imploding shock wave. On this limiting characteristic, ϕ remains constant since the right-hand side of (9) is zero, indicating that the numerator

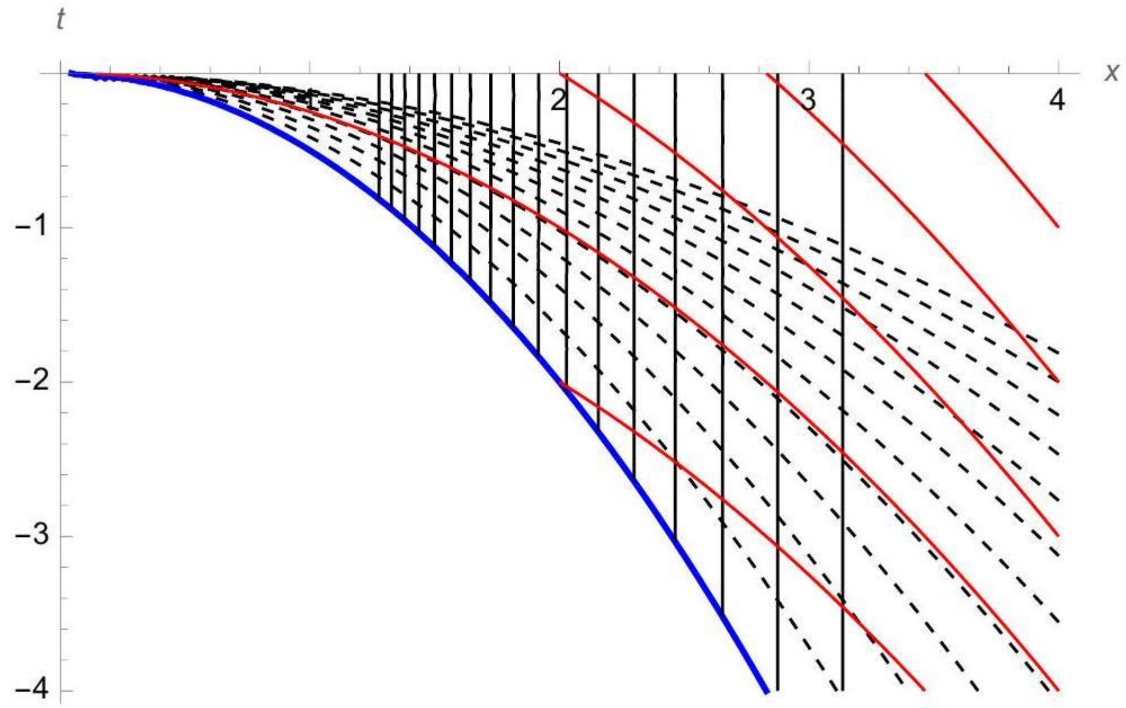

FIG. 2. Space-time diagram illustrating the spherical implosion problem with $A = -1$. The blue line represents the shock trajectory, solid black lines depict contours of ρ, dashed black lines indicate contours of the similarity variable ξ, and red lines show the characteristics.

must also be zero. Within the imploding shock wave structure, non-linearity leads to wave compression as per Burgers' equation, with additional effects from geometric terms involving j in (9). This scenario mirrors the gas dynamics problem described by Guderley.

The solutions for the implosion problem produce time-independent interior solutions, which are influenced by time-dependent shock motion and the time-dependent limiting characteristic. Figure 2 illustrates the spherical implosion solution, where the density field is given by $\rho = 2A/x$. The characteristic trajectories follow simple power laws, with $\mathrm{d}x/\mathrm{d}t = \rho$. The limiting characteristic corresponds to the $\xi = 2^{3/2}$ curve, which differentiates between characteristics affecting the implosion dynamics and those that do not, resembling the Guderley solution in gas dynamics.

The cylindrical implosion problem exhibits qualitatively similar behavior, with the density field described by $\rho = 2A/\sqrt{x}$.

6. CONCLUSIONS

In this comprehensive study, we have successfully derived exact solutions for self-similar shock dynamics that adhere to the inviscid Burgers equation, expanding our understanding of shock wave behavior across different geometries [18-20]. For the first kind of self-similar shock dynamics, our solutions were derived using integral relations, offering a methodical approach to solving these nonlinear problems. For the second kind, the requirement of analyticity on the limiting characteristic provided a robust framework for obtaining solutions.

Our findings reveal that these results are not only consistent with but also directly comparable to those obtained from the inviscid Euler equations. This comparison offers a clear generalization of two classical problems in shock dynamics: the Taylor-Sedov blast wave problem, representing the first kind of self-similar shock dynamics, and the Guderley shock implosion problem, representing the second kind. By establishing these connections, we have demonstrated that our approach provides a unified and powerful method for analyzing self-similar shock waves.

Furthermore, the techniques and methodologies developed in this study are versatile and can be readily applied to other one-way nonlinear wave equations that model kinematic waves. This adaptability underscores the broad applicability of our approach, making it a valuable tool for researchers investigating a wide range of nonlinear wave phenomena.

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