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On Some Binomial Sequence Space of Non-Absolute Type

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Abstract:- In this work, we introduce the binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ which include the spaces ℓ_p and ℓ_∞, in turn. Moreover, we show that the spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ are BK-spaces and prove that these spaces are linearly isomorphic to the spaces ℓ_p and ℓ_∞, respectively. Furthermore, we speak of some inclusion relations and give the Schauder basis of the space $b_p^{r,s}$. Lastly, we determine the $\alpha-, \beta$, and γ-duals of those spaces and give some geometric properties of the space $b_p^{r,s}$.

Keywords*: matrix transformations; matrix domain; Schauder basis; α-, β - and γ-duals; matrix classes.*

1. The basic information and notations

The set of all real (or complex) valued sequences is symbolized by w which becomes a vector space under point-wise addition and scalar multiplication. Any vector subspace of w is called a sequence space. The spaces of all bounded, null, convergent, and absolutely p-summable sequences are denoted by ℓ_∞, c_0, c, and ℓ_p, respectively, where $1 \leq p < \infty$.

A Banach sequence space is called a BK-space provided each of the maps $p_n: X \longrightarrow \mathbb{C}$ defined by $p_n = x_n$ is continuous for all $n \in \mathbb{N}$[1]. By considering the notion of BK-space, one can say that the sequence spaces ℓ_∞, c_0, and c are BK-spaces according to their usual sup-norm defined by $\| x \|_\infty = \sup_{k \in \mathbb{N}} |x_k|$ and ℓ_p is a BK-space according to its ℓ_p-norm defined by

$$\| x \|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{\frac{1}{p}}$$

where $1 \leq p < \infty$.

For an arbitrary infinite matrix $A = (a_{nk})$ of real (or complex) entries and $x = (x_k) \in w$, the A-transform of x is defined by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k \qquad (1.1)$$

and is supposed to be convergent for all $n \in \mathbb{N}$ [2]. In terms of the ease of use, we prefer that the summation without limits runs from 0 to ∞.

Given two sequence spaces X and Y, and an infinite matrix $A = (a_{nk})$, the sequence space X_A is defined by

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$$X_A = \{x = (x_k) \in w : Ax \in X\} \qquad (1.2)$$

which is called the domain of an infinite matrix A. Also, by $(X:Y)$, we denote the class of all matrices such that $X \subset Y_A$. If $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n, k \in \mathbb{N}$, an infinite matrix $A = (a_{nk})$ is called a triangle. Also, a triangle matrix A uniquely has an inverse A^{-1} which is a triangle matrix.

Let the summation matrix $S = (s_{nk})$ be defined as follows:

$$s_{nk} = \begin{cases} 1, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then the spaces of all bounded and convergent series are defined by means of the summation matrix such that $bs = (\ell_\infty)_S$ and $cs = c_S$, respectively.

The theory of matrix transformation was set in motion by the theory of summability which was developed by Cesàro, Norlund, Riesz, etc. By taking into account this theory, many authors have constructed new sequence spaces. For example, $(\ell_\infty)_{N_q}$ and c_{N_q} in [3], X_p and X_∞ in [4], a_p^r and a_∞^r in [5]. Furthermore, many authors have used especially the Euler matrix for defining new sequence spaces. These are e_0^r and e_c^r in [6], e_p^r and e_∞^r in [7] and [8], $e_0^r(\Delta), e_c^r(\Delta)$ and $e_\infty^r(\Delta)$ in [9], $e_0^r(\Delta^{(m)}), e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$ in [10], $e_0^r(B^{(m)}), e_c^r(B^{(m)})$, and $e_\infty^r(B^{(m)})$ in [11], $e_0^r(\Delta, p), e_c^r(\Delta, p)$, and $e_\infty^r(\Delta, p)$ in [12], $e_0^r(u, p)$ and $e_c^r(u, p)$ in [13].

In this work, we introduce the binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ which include the spaces ℓ_p and ℓ_∞, in turn. Moreover, we show that the spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ are BK-spaces and prove that these spaces are linearly isomorphic to the spaces ℓ_p and ℓ_∞, respectively. Furthermore, we speak of some inclusion relations and give the Schauder basis of the space $b_p^{r,s}$. Lastly, we determine the $\alpha-, \beta-$, and γ-duals of those spaces and give some geometric properties of the space $b_p^{r,s}$.

2 The binomial sequence spaces which include the spaces ℓ_p and ℓ_∞

In this part, we define the binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ which include the spaces ℓ_p and ℓ_∞, respectively. Furthermore, we show that those spaces are BK-spaces and are linearly isomorphic to the spaces ℓ_p and ℓ_∞. Also, we show that the binomial sequence space $b_p^{r,s}$ is not a Hilbert space except the case $p = 2$, where $1 \leq p < \infty$.

Let r, s be nonzero real numbers with $r + s \neq 0$. Then the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is defined as follows:

$$b_{nk}^{r,s} = \begin{cases} \dfrac{1}{(s+r)^n} \dbinom{n}{k} s^{n-k} r^k, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}_0$. For $sr > 0$, one can easily check that the following properties hold for the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$:
(i) $\|B^{r,s}\| < \infty$
(ii) $\lim_{n \to \infty} b_{nk}^{r,s} = 0$ (each $k \in \mathbb{N}$),
(iii) $\lim_{n \to \infty} \sum_k b_{nk}^{r,s} = 1$.

Thus, the binomial matrix is regular whenever $sr > 0$. Here and in the following, unless stated otherwise, we suppose that $sr > 0$.

By taking into account the binomial matrix $B^{r,s} = \left(b_{nk}^{r,s}\right)$, the binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ are defined by

$$b_p^{r,s} = \left\{x = (x_k) \in w \colon \sum_n \left|\frac{1}{(s+r)^n} \sum_{k=0}^n \binom{n}{k} s^{n-k} r^k x_k\right|^p < \infty\right\}, \ 1 \le p < \infty$$

and

$$b_\infty^{r,s} = \left\{x = (x_k) \in w \colon \sup_{n\in\mathbb{N}} \left|\frac{1}{(s+r)^n} \sum_{k=0}^n \binom{n}{k} s^{n-k} r^k x_k\right| < \infty\right\}$$

By considering the notation of (1.2), the binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ can be redefined by the matrix domain of $B^{r,s} = \left(b_{nk}^{r,s}\right)$ as follows:

$$b_p^{r,s} = \left(\ell_p\right)_{B^{r,s}} \ \text{and} \ b_\infty^{r,s} = \left(\ell_\infty\right)_{B^{r,s}} \tag{2.1}$$

Let us define a sequence $y = (y_k)$ as follows:

$$(B^{r,s}x)_k = y_k = \frac{1}{(s+r)^k} \sum_{j=0}^k \binom{k}{j} s^{k-j} r^j x_j \tag{2.2}$$

for all $k \in \mathbb{N}$. This sequence will be frequently used as the $B^{r,s}$-transform of x.

We would like to touch on a point, if we take $s + r = 1$, we obtain the Euler matrix $E^r = (e_{nk}^r)$. So, the binomial matrix $B^{r,s} = \left(b_{nk}^{r,s}\right)$ generalizes the Euler matrix.

Now, we want to continue with the following theorem which is needed in the next.

Theorem 2.1 The binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ are BK-spaces according to their norms defined by

$$\| x \|_{b_p^{r,s}} = \|B^{r,s}x\|_{\ell_p} = \left(\sum_{n=1}^\infty |(B^{r,s}x)_n|^p\right)^{\frac{1}{p}}$$

and

$$\| x \|_{b_\infty^{r,s}} = \|B^{r,s}x\|_\infty = \sup_{n\in\mathbb{N}}|(B^{r,s}x)_n|$$

where $1 \le p < \infty$.

Proof We know that the sequence spaces ℓ_p and ℓ_∞ are BK-spaces with their ℓ_p-norm and sup-norm, respectively, where $1 \le p < \infty$. Furthermore, (2.1) holds and the binomial matrix $B^{r,s} = \left(b_{nk}^{r,s}\right)$ is a triangle matrix. By taking into account these three facts and Theorem 4.3.12 of Wilansky [2], we conclude that the binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ are BK-spaces, where $1 \le p < \infty$. This completes the proof of the theorem.

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Theorem 2.2 The binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ are linearly isomorphic to the sequence spaces ℓ_p and ℓ_∞, in turn, where $1 \leq p < \infty$.

Proof To refrain from the usage of similar statements, we prove the theorem for only the sequence space $b_p^{r,s}$, where $1 \leq p < \infty$. For the proof of the theorem, we need to show the existence of a linear bijection between the spaces $b_p^{r,s}$ and ℓ_p. Let L be a transformation such that $L: b_p^{r,s} \longrightarrow \ell_p, L(x) = B^{r,s}x$. By the definition of the binomial sequence space $b_p^{r,s}$, we conclude that, for all $x \in b_p^{r,s}, L(x) = B^{r,s}x \in \ell_p$. Furthermore, it is obvious that L is a linear transformation and $x = 0$ whenever $L(x) = 0$. Therefore, L is injective.

For given $y = (y_k) \in \ell_p$, let us define a sequence $x = (x_k)$ such that

$$x_k = \frac{1}{r^k} \sum_{j=0}^{k} \binom{k}{j} (-s)^{k-j} (s+r)^j y_j$$

for all $k \in \mathbb{N}$. Then we get

$$\| x \|_{b_p^{r,s}} = \|B^{r,s}x\|_{\ell_p}$$

$$= \left(\sum_{n=1}^{\infty} |(B^{r,s}x)_n|^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_{n=1}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right|^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_{n=1}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \sum_{j=0}^{k} \binom{k}{j} (-s)^{k-j} (s+r)^j y_j \right|^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{\frac{1}{p}}$$

$$= \| y \|_{\ell_p}$$

$$= \| L(x) \|_{\ell_p} < \infty$$

Hence, we conclude that L is norm preserving and $x \in b_p^{r,s}$, namely L is surjective. As a consequence, L is a linear bijection. This means that the spaces $b_p^{r,s}$ and ℓ_p are linearly isomorphic, that is, $b_p^{r,s} \cong \ell_p$, where $1 \leq p < \infty$. This completes the proof of the theorem.

Theorem 2.3 The binomial sequence space $b_p^{r,s}$ is not a Hilbert space except the case $p = 2$, where $1 \leq p < \infty$.

Proof Let $p = 2$. Remembering Theorem 2.1, one can say that $b_2^{r,s}$ is a BK-space according to its ℓ_2-norm defined by

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$$\| x \|_{b_2^{r,s}} = \| B^{r,s} x \|_{\ell_2} = \left(\sum_{n=1}^{\infty} |(B^{r,s} x)_n|^2 \right)^{\frac{1}{2}}$$

Moreover, this norm can be generated by an inner product such that

$$\| x \|_{b_2^{r,s}} = \langle B^{r,s} x, B^{r,s} x \rangle^{\frac{1}{2}}$$

Therefore, $b_2^{r,s}$ is a Hilbert space.

Now, we assume that $1 \le p < \infty$ and $p \ne 2$. We define two sequences $y = (y_k)$ and $z = (z_k)$ as follows:

$$y_k = \frac{-s + k(r+s)}{r} \left(-\frac{s}{r} \right)^{k-1} \quad \text{and} \quad z_k = -\frac{s + k(r+s)}{r} \left(-\frac{s}{r} \right)^{k-1}$$

for all $k \in \mathbb{N}$. Then we obtain

$$\| y + z \|_{b_p^{r,s}}^2 + \| y - z \|_{b_p^{r,s}}^2 = 8 \ne 2^{\frac{2}{p}+2} = 2 \left(\| y \|_{b_p^{r,s}}^2 + \| z \|_{b_p^{r,s}}^2 \right)$$

Thus, the norm of the binomial sequence space $b_p^{r,s}$ does not satisfy the parallelogram equality. As a consequence, the norm cannot be generated by an inner product, that is, the binomial sequence space $b_p^{r,s}$ is not a Hilbert space whenever $p \ne 2$. This completes the proof of the theorem.

3 The inclusion relations and Schauder basis

In this part, we speak of some inclusion relations and give the Schauder basis for the binomial sequence space $b_p^{r,s}$, where $1 \le p < \infty$.

Theorem 3.1 The inclusions $e_p^r \subset b_p^{r,s}$ and $e_\infty^r \subset b_\infty^{r,s}$ strictly hold, where e_p^r and e_∞^r are the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞, respectively.

Proof If $r + s = 1$, one can easily see that $E^r = B^{r,s}$. Therefore, the inclusion $e_\infty^r \subset b_\infty^{r,s}$ holds.

Suppose that $0 < r < 1$ and $s = 5$. Let us now consider a sequence $x = (x_k)$ such that $x_k = \left(-\frac{4}{r} \right)^k$ for all $k \in \mathbb{N}$. Then it is clear that $x = (x_k) = \left(\left(-\frac{4}{r} \right)^k \right) \notin \ell_\infty$, $E^r x = ((-3-r)^k) \notin \ell_\infty$ and $B^{r,s} x = \left(\left(\frac{1}{5+r} \right)^k \right) \in \ell_\infty$. As a result of this, $x = (x_k) \in b_\infty^{r,s} \setminus e_\infty^r$. This shows that the inclusion $e_\infty^r \subset b_\infty^{r,s}$ is strictly. We can prove the other part of the theorem by using a similar technique. This completes the proof of the theorem.

Theorem 3.2 The inclusion $\ell_p \subset b_p^{r,s}$ is strict, where $1 \le p < \infty$.

Proof First we assume that $1 < p < \infty$. From the definition of the space ℓ_p, we write

$$\sum_k |x_k|^p < \infty$$

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for all $x = (x_k) \in \ell_p$. For given an arbitrary sequence $x = (x_k) \in \ell_p$, by taking into account the equality (2.2) and the Hölder inequality, we obtain

$$|(B^{r,s}x)_k|^p = \left| \frac{1}{(s+r)^k} \sum_{j=0}^{k} \binom{k}{j} s^{k-j} r^j x_j \right|^p$$

$$\leq \left(\frac{1}{|s+r|^k} \right)^p \left[\left(\sum_{j=0}^{k} \binom{k}{j} |s|^{k-j}|r|^j \right)^{p-1} \times \left(\sum_{j=0}^{k} \binom{k}{j} |s|^{k-j}|r|^j |x_j|^p \right) \right]$$

$$= \frac{1}{|s+r|^k} \sum_{j=0}^{k} \binom{k}{j} |s|^{k-j}|r|^j |x_j|^p$$

$$= \sum_{j=0}^{k} \binom{k}{j} \left| \frac{s}{s+r} \right|^k \left| \frac{r}{s} \right|^j |x_j|^p$$

where $1 \leq p < \infty$. And

$$\sum_k |(B^{r,s}x)_k|^p \leq \sum_k \sum_{j=0}^{k} \binom{k}{j} \left| \frac{s}{s+r} \right|^k \left| \frac{r}{s} \right|^j |x_j|^p$$

$$= \sum_j |x_j|^p \sum_{k=j}^{\infty} \binom{k}{j} \left| \frac{s}{s+r} \right|^k \left| \frac{r}{s} \right|^j$$

$$= \left| \frac{s+r}{s} \right| \sum_j |x_j|^p$$

If we consider the comparison test, we conclude that $B^{r,s}x \in \ell_p$, namely $x \in b_p^{r,s}$. As a consequence $\ell_p \subset b_p^{r,s}$, where $1 < p < \infty$.

Now, we keep in view the sequence $v = (v_k)$ defined by $v_k = (-1)^k$ for all $k \in \mathbb{N}$. Then it is clear that $v = (v_k) \notin \ell_p$ and $B^{r,s}v = \left(\left(\frac{s-r}{s+r} \right)^k \right) \in \ell_p$, namely $v = (v_k) \in b_p^{r,s}$. Because of $v = (v_k) \in b_p^{r,s} \setminus \ell_p$, the inclusion $\ell_p \subset b_p^{r,s}$ is strict. In case of $p = 1$, the theorem can be proved by using a similar method. This completes the proof of the theorem.

Theorem 3.3 The spaces $b_p^{r,s}$ and ℓ_∞ overlap but these spaces do not include each other, where $1 \leq p < \infty$.

Proof It is obvious that $v = \left((-1)^k \right) \in \ell_\infty$ and $v = \left((-1)^k \right) \in b_p^{r,s}$. So, the spaces $b_p^{r,s}$ and ℓ_∞ overlap, where $1 \leq p < \infty$. Here, we consider the sequences $e = (1,1,1,\dots)$ and $u = (u_k)$ defined by $u_k = \left(-\frac{s}{r} \right)^k$ for all $k \in \mathbb{N}$, where $\left| \frac{s}{r} \right| > 1$. Then we conclude that $e \in \ell_\infty$ but $B^{r,s}e = e \notin \ell_p$, that is, $e \notin b_p^{r,s}$ and $u \notin \ell_\infty$ but $B^{r,s}u = (1,0,0,\dots) \in \ell_p$, namely $u \in b_p^{r,s}$. As a consequence, $e \in \ell_\infty \setminus b_p^{r,s}$ and $u \in b_p^{r,s} \setminus \ell_\infty$. On account of this, $b_p^{r,s}$ and ℓ_∞ do not include each other, where $1 \leq p < \infty$. This completes the proof of the theorem.

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Theorem 3.4 The inclusions $\ell_\infty \subset b_\infty^{r,s}$ and $b_p^{r,s} \subset b_\infty^{r,s}$ are strict, where $1 \le p < \infty$.

Proof The inequality

$$\| x \|_{b_\infty^{r,s}} = \sup_{k \in \mathbb{N}} \left| \frac{1}{(s+r)^k} \sum_{j=0}^{k} \binom{k}{j} s^{k-j} r^j x_j \right| \le \| x \|_\infty$$

holds for all $x \in \ell_\infty$. In this way, the inclusion $\ell_\infty \subset b_\infty^{r,s}$ holds. Now, we consider the sequence $v = (v_k)$ defined by $v_k = \left(-\frac{s+r}{r} \right)^k$ for all $k \in \mathbb{N}$. Then we conclude that $v = (v_k) \notin \ell_\infty$ but $B^{r,s} v = \left(\left(-\frac{r}{r+s} \right)^k \right) \in \ell_\infty$, namely $v = (v_k) \in b_\infty^{r,s}$. Therefore, the inclusion $\ell_\infty \subset b_\infty^{r,s}$ strictly holds.

For given $x = (x_k) \in b_p^{r,s}$, where $1 \le p < \infty$, by taking into account Theorem 2.2 and the inclusion $\ell_p \subset \ell_\infty$, we conclude that $B^{r,s} x \in \ell_\infty$, namely $x \in b_\infty^{r,s}$. Thus, the inclusion $b_p^{r,s} \subset b_\infty^{r,s}$ holds. Also, it is clear that $e \in b_\infty^{r,s} \setminus b_p^{r,s}$. Hence, the inclusion $b_p^{r,s} \subset b_\infty^{r,s}$ is strict. This completes the proof of the theorem.

Now, let us continue with the definition of the Schauder basis of a normed space. Let $(X, \|\cdot\|_X)$ be a normed sequence space and $d = (d_k)$ be a sequence in X. If for every $x \in X$, there exists a unique sequence of scalars $\lambda = (\lambda_k)$ such that

$$\lim_{n \to \infty} \left\| x - \sum_{k=0}^{n} \lambda_k d_k \right\|_X = 0$$

then $d = (d_k)$ is called a Schauder basis for X[1].

Theorem 3.5 Let $\mu_k = \{B^{r,s} x\}_k$ be given for all $k \in \mathbb{N}$. We define the sequence $g^{(k)}(r,s) = \left\{ g_n^{(k)}(r,s) \right\}_{n \in \mathbb{N}}$ of the elements of the binomial sequence space $b_p^{r,s}$ as follows:

$$g_n^{(k)}(r,s) = \begin{cases} 0, & 0 \le n < k \\ \frac{1}{r^n} \binom{n}{k} (-s)^{n-k} (s+r)^k, & n \ge k \end{cases}$$

for all fixed $k \in \mathbb{N}$. Then the sequence $\left\{ g^{(k)}(r,s) \right\}_{k \in \mathbb{N}}$ is a Schauder basis for the binomial sequence space $b_p^{r,s}$, and every $x \in b_p^{r,s}$ has a unique representation of the form

$$x = \sum_{k} \mu_k g^{(k)}(r,s)$$

where $1 \le p < \infty$.

Proof Let $x = (x_k) \in b_p^{r,s}$ be given, where $1 \le p < \infty$. For all non-negative integer m, we define

$$x^{[m]} = \sum_{k=0}^{m} \mu_k g^{(k)}(r,s)$$

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Then, if we apply the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ to $x^{[m]}$, we write

$$B^{r,s}x^{[m]} = \sum_{k=0}^{m} \mu_k B^{r,s} g^{(k)}(r,s) = \sum_{k=0}^{m} (B^{r,s}x)_k e^{(k)}$$

and

$$\{B^{r,s}(x - x^{[m]})\}_n = \begin{cases} 0, & 0 \leq n \leq m \\ (B^{r,s}x)_n, & n > m \end{cases}$$

for all $m, n \in \mathbb{N}$.

For any given $\epsilon > 0$, there exists a non-negative integer m_0 such that

$$\sum_{n=m_0+1}^{\infty} |(B^{r,s}x)_n|^p \leq \left(\frac{\epsilon}{2}\right)^p$$

for all $m \geq m_0$. Thus,

$$\begin{aligned} \|x - x^{[m]}\|_{b_p^{r,s}} &= \left(\sum_{n=m+1}^{\infty} |(B^{r,s}x)_n|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=m_0+1}^{\infty} |(B^{r,s}x)_n|^p\right)^{\frac{1}{p}} \\ &\leq \frac{\epsilon}{2} < \epsilon \end{aligned}$$

for all $m \geq m_0$. This shows us that

$$x = \sum_k \mu_k g^{(k)}(r,s)$$

Lastly, we should show the uniqueness of this representation. For this purpose, assume that

$$x = \sum_k \lambda_k g^{(k)}(r,s)$$

Since the linear transformation L defined from $b_p^{r,s}$ to ℓ_p in the proof of Theorem 2.2 is continuous, we have

$$(B^{r,s}x)_n = \sum_k \lambda_k \{B^{r,s} g^{(k)}(r,s)\}_n = \sum_k \lambda_k e_n^{(k)} = \lambda_n$$

for every $n \in \mathbb{N}$, which contradicts the fact that $(B^{r,s}x)_n = \mu_n$ for every $n \in \mathbb{N}$. Therefore, every $x \in b_p^{r,s}$ has a unique representation. This completes the proof of the theorem.

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From Theorem 2.1, we know that $b_p^{r,s}$ is a Banach space, where $1 \leq p < \infty$. If we consider this fact and Theorem 3.5, we can give the next corollary.

Corollary 3.6 The binomial sequence space $b_p^{r,s}$ is separable, where $1 \leq p < \infty$.

4 The $\alpha-, \beta-$, and γ-duals

In this part, we determine the $\alpha-, \beta-$, and γ-duals of the binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$, where $1 \leq p < \infty$.

Now, we start with a definition. The multiplier space of the sequence spaces X and Y is denoted by $M(X, Y)$ and defined by

$$M(X, Y) = \{y = (y_k) \in w : xy = (x_k y_k) \in Y \text{ for all } x = (x_k) \in X\}$$

By taking into account the definition of a multiplier space, the $\alpha-, \beta-$, and γ-duals of a sequence space X are defined by

$$X^\alpha = M(X, \ell_1), \ X^\beta = M(X, cs) \text{ and } X^\gamma = M(X, bs)$$

respectively.
For use in the next lemma, we now give some properties:

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}|^q < \infty \tag{4.1}$$

$$\sup_{n,k \in \mathbb{N}} |a_{nk}| < \infty \tag{4.2}$$

$$\lim_{n \to \infty} a_{nk} = a_k \text{ for each } k \in \mathbb{N}, \tag{4.3}$$

$$\sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} a_{nk} \right|^q < \infty \tag{4.4}$$

$$\lim_{n \to \infty} \sum_k |a_{nk}| = \sum_k \left| \lim_{n \to \infty} a_{nk} \right| \tag{4.5}$$

$$\sup_{k \in \mathbb{N}} \sum_n |a_{nk}| < \infty \tag{4.6}$$

where $\mathcal{F}$ is the collection of all finite subsets of $\mathbb{N}, \frac{1}{p} + \frac{1}{q} = 1$ and $1 < p \leq \infty$.

Lemma 4.1 (see [14]) Let $A = (a_{nk})$ be an infinite matrix, then the following hold:
(i) $A = (a_{nk}) \in (\ell_1 : \ell_1) \Leftrightarrow (4.6)$ holds,
(ii) $A = (a_{nk}) \in (\ell_1 : c) \Leftrightarrow (4.2)$ and (4.3) hold,
(iii) $A = (a_{nk}) \in (\ell_1 : \ell_\infty) \Leftrightarrow (4.2)$ holds,
(iv) $A = (a_{nk}) \in (\ell_p : \ell_1) \Leftrightarrow (4.4)$ holds with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p \leq \infty$,
(v) $A = (a_{nk}) \in (\ell_p : c) \Leftrightarrow (4.1)$ and (4.3) hold with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$,
(vi) $A = (a_{nk}) \in (\ell_p : \ell_\infty) \Leftrightarrow (4.1)$ holds with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$,

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(vii) $A = (a_{nk}) \in (\ell_\infty : c) \Leftrightarrow$ (4.3) and (4.5) hold,

(viii) $A = (a_{nk}) \in (\ell_\infty : \ell_\infty) \Leftrightarrow$ (4.1) holds with $q = 1$.

Theorem 4.2 Let $v_1^{r,s}$ and $v_2^{r,s}$ be defined as follows:

$$v_1^{r,s} = \left\{ a = (a_k) \in w : \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \binom{n}{k} (-s)^{n-k} r^{-n} (r+s)^k a_n \right|^q < \infty \right\}$$

and

$$v_2^{r,s} = \left\{ a = (a_k) \in w : \sup_{k \in \mathbb{N}} \sum_n \left| \binom{n}{k} (-s)^{n-k} r^{-n} (r+s)^k a_n \right| < \infty \right\}$$

Then $\{b_1^{r,s}\}^\alpha = v_2^{r,s}$ and $\{b_p^{r,s}\}^\alpha = v_1^{r,s}$, where $1 < p \leq \infty$

Proof Let $a = (a_n) \in w$ be given. Remembering the sequence $x = (x_n)$, which is defined in the proof of Theorem 2.2, we have

$$a_n x_n = \sum_{k=0}^n \binom{n}{k} (-s)^{n-k} r^{-n} (r+s)^k a_n y_k = (H^{r,s} y)_n$$

for all $n \in \mathbb{N}$. Then, by considering the equality above, we deduce that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in b_1^{r,s}$ or $x = (x_k) \in b_p^{r,s}$ if and only if $H^{r,s} y \in \ell_1$ whenever $y = (y_k) \in \ell_1$ or $y = (y_k) \in \ell_p$, respectively, where $1 < p \leq \infty$. This shows us that $a = (a_n) \in \{b_1^{r,s}\}^\alpha$ or $a = (a_n) \in \{b_p^{r,s}\}^\alpha$ if and only if $H^{r,s} \in (\ell_1 : \ell_1)$ or $H^{r,s} \in (\ell_p : \ell_1)$, respectively, where $1 < p \leq \infty$. If we combine these two facts and Lemma 4.1(i) and (iv), we obtain

$$a = (a_n) \in \{b_1^{r,s}\}^\alpha \Leftrightarrow \sup_{k \in \mathbb{N}} \sum_n \left| \binom{n}{k} (-s)^{n-k} r^{-n} (r+s)^k a_n \right| < \infty$$

or

$$a = (a_n) \in \{b_p^{r,s}\}^\alpha \Leftrightarrow \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \binom{n}{k} (-s)^{n-k} r^{-n} (r+s)^k a_n \right|^q < \infty$$

respectively, where $1 < p \leq \infty$. Therefore, $\{b_1^{r,s}\}^\alpha = v_2^{r,s}$ and $\{b_p^{r,s}\}^\alpha = v_1^{r,s}$, where $1 < p \leq \infty$. This completes the proof of the theorem.

Theorem 4.3 Let $v_3^{r,s}, v_4^{r,s}, v_5^{r,s}, v_6^{r,s}$, and $v_7^{r,s}$ be defined as follows:

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$$v_3^{r,s} = \left\{ a = (a_k) \in w : \sum_{j=k}^{\infty} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \text{ exists for each } k \in \mathbb{N} \right\}$$

$$v_4^{r,s} = \left\{ a = (a_k) \in w : \sup_{n,k \in \mathbb{N}} \left| \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right| < \infty \right\}$$

$$v_5^{r,s} = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_{k} \left| \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right| \right.$$

$$= \left. \sum_{k} \left| \sum_{j=k}^{\infty} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right| \right\}$$

$$v_6^{r,s} = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right| < \infty \right\}, \ 1 < q < \infty$$

$$v_7^{r,s} = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right| < \infty \right\}$$

Then the following equalities hold:

(I) $\{b_1^{r,s}\}^{\beta} = v_3^{r,s} \cap v_4^{r,s}$,

(II) $\{b_p^{r,s}\}^{\beta} = v_3^{r,s} \cap v_6^{r,s}$, where $1 < p < \infty$,

(III) $\{b_{\infty}^{r,s}\}^{\beta} = v_3^{r,s} \cap v_5^{r,s}$,

(IV) $\{b_1^{r,s}\}^{\gamma} = v_4^{r,s}$,

(V) $\{b_p^{r,s}\}^{\gamma} = v_6^{r,s}$, where $1 < p < \infty$,

(VI) $\{b_{\infty}^{r,s}\}^{\gamma} = v_7^{r,s}$.

Proof To avoid the repetition of similar statements, we give the proof of the theorem for only the sequence space $b_p^{r,s}$, where $1 < p < \infty$.

Let $a = (a_k) \in w$ be given. By considering the sequence $x = (x_k)$, which is used in the proof of Theorem 2.2, we obtain

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[\frac{1}{r^k} \sum_{j=0}^{k} \binom{k}{j} (-s)^{k-j} (r+s)^j y_j \right] a_k$$

$$= \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right] y_k$$

$$= (G^{r,s} y)_n$$

for all $n \in \mathbb{N}$, where the matrix $G^{r,s} = \left(g_{nk}^{r,s} \right)$ is defined by

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$$g_{nk}^{r,s} = \begin{cases} \sum_{j=k}^{n} \binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^k a_j, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then:

(II) $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in b_p^{r,s}$ if and only if $G^{r,s}y \in c$ whenever $y = (y_k) \in \ell_p$, where $1 < p < \infty$. This fact shows that $a = (a_k) \in \{b_p^{r,s}\}^{\beta}$ if and only if $G^{r,s} \in (\ell_p : c)$, where $1 < p < \infty$. By combining this result and Lemma 4.1(v), we deduce that

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^k a_j \right|^q < \infty \qquad (4.7)$$

and

$$\sum_{j=k}^{\infty} \binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^k a_j \text{ exists for each } k \in \mathbb{N}$$

where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. As a result of this, we obtain $\{b_p^{r,s}\}^{\beta} = v_3^{r,s} \cap v_6^{r,s}$, where $1 < p < \infty$.

(V) By following a similar way, $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in b_p^{r,s}$ if and only if $G^{r,s}y \in \ell_\infty$ whenever $y = (y_k) \in \ell_p$, where $1 < p < \infty$. This says us that $a = (a_k) \in \{b_p^{r,s}\}^{\gamma}$ if and only if $G^{r,s} \in (\ell_p : \ell_\infty)$, where $1 < p < \infty$. By using this result and Lemma 4.1(vi), we conclude that (4.7) holds, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. As a consequence of this, we obtain $\{b_p^{r,s}\}^{\gamma} = v_6^{r,s}$, where $1 < p < \infty$. This completes the proof of the theorem.

5 Geometric properties of the binomial sequence space $b_p^{r,s}$

In this part, we give some geometric properties of the binomial sequence space $b_p^{r,s}$. Let us start with some notions.

Let $(X, \|\cdot\|_X)$ be a Banach space. Then X is said to have the BanachSaks property, if every bounded sequence $u = (u_n)$ contains a subsequence $v = (v_n)$ such that the Cesàro means $\frac{1}{n+1}\sum_{k=0}^{n} v_k$ are norm convergent [15].

X is said to have the weak Banach-Saks property, if every weakly null sequence $u = (u_n)$ contains a subsequence $v = (v_n)$ such that the Cesàro means $\frac{1}{n+1}\sum_{k=0}^{n} v_k$ are norm convergent [15].

X is said to have Banach-Saks type p, if every weakly null sequence $u = (u_n)$ has a subsequence $v = (v_n)$ such that, for some $M > 0$,

$$\left\| \sum_{k=0}^{n} v_k \right\|_X \le M(n+1)^{\frac{1}{p}}$$

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for all $n \in \mathbb{N}$, where $1 < p < \infty$[16].

Let C be a weakly compact convex subset of X. Then X is said to have the weak fixed point property, if every self mapping $T: C \longrightarrow C$ that provides $\parallel Tx - Ty \parallel \leq \parallel x - y \parallel$ for all $x, y \in C$ has a fixed point [17].

Let X be a normed linear space and $S(X)$ be a unit sphere of X. Then the Gurarii modulus of convexity is defined as follows:

$$\beta_X(\epsilon) = \inf\left\{1 - \inf_{0 \leq \lambda \leq 1} \parallel \lambda x + (1 - \lambda)y \parallel : x, y \in S(X), \parallel x - y \parallel = \epsilon\right\}$$

where $0 \leq \epsilon \leq 2$[18].

Theorem 5.1 (see [19]) A Banach space X has the weak fixed point property, if X provides the condition

$$R(X) = \sup\left\{\liminf_{n \to \infty}\|x_n + x\|\right\} < 2$$

where the supremum is taken over all weakly null sequences (x_n) of the unit ball and all points x of the unit ball.

Theorem 5.2 The binomial sequence space $b_p^{r,s}$ is of the Banach-Saks type p.

Proof Let (u_n) be a weakly null sequence in the $B(b_p^{r,s})$ unit ball of $b_p^{r,s}$. We suppose that (ϵ_n) is a sequence of positive numbers provided $\sum \epsilon_n \leq \frac{1}{2}$. Construct $v_0 = u_0 = 0$ and $v_1 = u_{n_1} = u_1$. Then we can find an $m_1 \in \mathbb{N}$ such that

$$\left\|\sum_{i=m_1+1}^{\infty} v_1(i)e^{(i)}\right\|_{b_p^{r,s}} < \epsilon_1$$

By virtue of $u_n \overset{w}{\to} 0$ implying $u_n \longrightarrow 0$ coordinatewise, we can find an $n_2 \in \mathbb{N}$ such that

$$\left\|\sum_{i=0}^{m_1} u_n(i)e^{(i)}\right\|_{b_p^{r,s}} < \epsilon_1$$

as $n \geq n_2$. Construct $v_2 = u_{n_2}$. Then we can find an $m_2 > m_1$ such that

$$\left\|\sum_{i=m_2+1}^{\infty} v_2(i)e^{(i)}\right\|_{b_p^{r,s}} < \epsilon_2$$

If we use $x_n \longrightarrow 0$ coordinatewise one more time, we can find an $n_3 > n_2$ such that

$$\left\|\sum_{i=0}^{m_2} u_n(i)e^{(i)}\right\|_{b_p^{r,s}} < \epsilon_2$$

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as $n \geq n_3$.

By continuing this method, we can constitute two increasing sequences (m_k) and (n_k) such that

$$\left\| \sum_{i=0}^{m_k} u_n(i)e^{(i)} \right\|_{b_p^{r,s}} < \epsilon_k$$

for all $n \geq n_{k+1}$ and

$$\left\| \sum_{i=m_k+1}^{\infty} v_2(i)e^{(i)} \right\|_{b_p^{r,s}} < \epsilon_k$$

where $v_k = u_{n_k}$. Thus

$$\left\| \sum_{k=0}^{n} v_k \right\|_{b_p^{r,s}} = \left\| \sum_{k=0}^{n} \left(\sum_{i=0}^{m_{k-1}} v_k(i)e^{(i)} + \sum_{i=m_{k-1}+1}^{m_k} v_k(i)e^{(i)} + \sum_{i=m_k+1}^{\infty} v_k(i)e^{(i)} \right) \right\|_{b_p^{r,s}}$$

$$\leq \left\| \sum_{k=0}^{n} \left(\sum_{i=m_{k-1}+1}^{m_k} v_k(i)e^{(i)} \right) \right\|_{b_p^{r,s}} + 2 \sum_{k=0}^{n} \epsilon_k$$

and

$$\left\| \sum_{k=0}^{n} \sum_{i=m_{k-1}+1}^{m_k} v_k(i)e^{(i)} \right\|_{b_p^{r,s}}^{p} = \sum_{k=0}^{n} \sum_{i=m_{k-1}+1}^{m_k} \left| \frac{1}{(s+r)^i} \sum_{j=0}^{i} \binom{i}{j} s^{i-j} r^j v_k(j) \right|^p$$

$$\leq \sum_{k=0}^{n} \sum_{i=0}^{\infty} \left| \frac{1}{(s+r)^i} \sum_{j=0}^{i} \binom{i}{j} s^{i-j} r^j v_k(j) \right|^p \leq n+1$$

Thus we obtain

$$\left\| \sum_{k=0}^{n} v_k \right\|_{b_p^{r,s}} \leq (n+1)^{\frac{1}{p}} + 1 \leq 2(n+1)^{\frac{1}{p}}$$

As a consequence, the binomial sequence space $b_p^{r,s}$ is of the Banach-Saks type p. This completes the proof of the theorem.

We know from Theorem 2.2 that $b_p^{r,s}$ is linearly isomorphic to ℓ_p. So, it is clear that $R\left(b_p^{r,s}\right) = R(\ell_p) = 2^{\frac{1}{p}}$.

By combining this fact and Theorem 5.1, we can give the next theorem.

Theorem 5.3 The binomial sequence space $b_p^{r,s}$ has the weak fixed point property, where $1 < p < \infty$.

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Theorem 5.4 The inequality $\beta_{b_p^{r,s}}(\epsilon) \leq 1 - \left[1 - \left(\frac{\epsilon}{2}\right)^p\right]^{\frac{1}{p}}$ holds, where $0 \leq \epsilon \leq 2$.

Proof Let $0 \leq \epsilon \leq 2$ be given. By assuming the inverse of the binomial matrix $B^{r,s}$ is D, we construct two sequences u and v as follows:

$$u = \left(\left(D\left(1 - \left(\frac{\epsilon}{2}\right)^p\right)\right)^{\frac{1}{p}}, D\left(\frac{\epsilon}{2}\right), 0, 0, \dots\right)$$

$$v = \left(\left(D\left(1 - \left(\frac{\epsilon}{2}\right)^p\right)\right)^{\frac{1}{p}}, D\left(-\frac{\epsilon}{2}\right), 0, 0, \dots\right)$$

Then we obtain

$$\|B^{r,s}u\|_{\ell_p} = \| u \|_{b_p^{r,s}} = 1 \text{ and } \|B^{r,s}v\|_{\ell_p} = \| v \|_{b_p^{r,s}} = 1$$

This shows that $u, v \in S\left(b_p^{r,s}\right)$ and $\|B^{r,s}u - B^{r,s}v\|_{\ell_p} = \| u - v \|_{b_p^{r,s}} = \epsilon$. For $0 \leq \lambda \leq 1$, we have

$$\| \lambda u + (1 - \lambda)v \|_{b_p^{r,s}}^p = \|\lambda B^{r,s}u + (1 - \lambda)B^{r,s}v\|_{\ell_p}^p$$
$$= 1 - \left(\frac{\epsilon}{2}\right)^p + |2\lambda - 1|\left(\frac{\epsilon}{2}\right)^p$$

and

$$\inf_{0 \leq \lambda \leq 1} \| \lambda u + (1 - \lambda)v \|_{b_p^{r,s}}^p = 1 - \left(\frac{\epsilon}{2}\right)^p \tag{5.1}$$

Thus, we obtain

$$\beta_{b_p^{r,s}}(\epsilon) \leq 1 - \left[1 - \left(\frac{\epsilon}{2}\right)^p\right]^{\frac{1}{p}}$$

This completes the proof of the theorem.

By using the equality (5.1), we find two more results.
Corollary 5.5 Since $\beta_{b_p^{r,s}}(\epsilon) = 1$, the binomial sequence space $b_p^{r,s}$ is strictly convex.

Corollary 5.6 Since $0 < \beta_{b_p^{r,s}}(\epsilon) \leq 1$, for $0 < \epsilon \leq 2$, the binomial sequence space $b_p^{r,s}$ is uniformly convex.

6 Conclusion

By taking into account the binomial matrix $B^{r,s} = \left(b_{nk}^{r,s}\right)$, we conclude that $B^{r,s} = \left(b_{nk}^{r,s}\right)$ reduces in the case $r + s = 1$ to $E^r = (e_{nk}^r)$ which is called the Euler matrix of order r. Therefore, our results obtained from the matrix domain of the binomial matrix $B^{r,s} = \left(b_{nk}^{r,s}\right)$ are more general and more

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extensive than the results on the matrix domain of the Euler matrix of order r. Furthermore, the binomial matrix $B^{r,s} = \left(b_{nk}^{r,s}\right)$ is not a special case of the weighed mean matrices. Thus, this paper has filled up a gap in the existent literature.

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