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ON SYMMETRICAL EVOLUTION OF DOUGLOUS SPACE OF 2^{nd} KIND WITH AN UNIQUE (α, β)- METRIC

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Abstract

In the present paper, we prove that the Douglous space of $\mathbf{2^{nd}}$ kind with a unique type of $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-metric can be transformed symmetrically on a Douglous space of next kind again. For, we have used the theories of Douglous space and conformal transformation. Consequently, some special cases are discussed to show that the $\mathbf{2^{nd}}$ kind Douglous space with different $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-metrics including Randers metric and some other types, is also conformally invariant.

Keywords: Douglous space, Berwald space,Randers metric, conformal transformation.

1 Introduction

The conception of Douglous space of 2^{nd} kind with (α, β) metric was first introduced by Lee [3]. A Finsler space with (α, β)-metric is defined as Douglous space of second kind if the Douglous tensor D^h_{ijk} vanishes identically. M. Matsumoto and S. Bacso further investigated this space in the light of geodesic equation and found it as generalization

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of Berwald spaces. In Finsler geometry [12], the theory of Berwald spaces containing (α, β)-metrics ([1], [11], [15], and [14]) plays an essential part. An another extension of Berwald spaces can be seen as Landsberg space. The idea of the existence of conforming transformations on Finsler spaces was first i ntroduced b y M . S . K neblman i n [1], and M. Hashiguchi [6] expanded it in various manners. Some tensorial quantities were found conformally invariant by Lee and Prasad ([10], [17]) in a Finsler space with (α, β)-metric with respect to a conforming β-transformation.

Here, we have investigated that the Douglous space of 2^{nd} kind with unique (α, β)-metric, $F = \alpha + \upsilon\beta + \mu\frac{\beta^3}{\alpha^2}$ remains invariant under a conformal transformation. Subsequently, we have derived certain findings that demonstrate the symmetry of the Douglous space of second kind under a conforming alterations, even when the (α, β)-metrics are of various kinds, such as the Randers metric and some others.

2 Preliminaries

If $F(\alpha, \beta)$ is a positively homogeneous function of linear degree in α and β then Finsler space $F^n = (M, F(\alpha, \beta))$ is said to have a (α, β) metric. In this case α denotes is the Riemannian metric which is denoted as $\alpha^2 = a_{ij}(x)y^iy^j$ and β indicates one form is $\beta = b_i(x)y^i$. Keep in mind that the Riemannian space connected to F^n denoted by the space $R^n = (M, \alpha)$. The following illustrates the notations that were used:

$$b^i = a^{ir}b_r,\ b^2 = a^{rs}b_rb_s$$

$$2r_{ij} = b_{i|j} + b_{j|i},\ 2s_{ij} = b_{i|j} - b_{j|i}$$

$$s^i_j = a^{ir}s_{rj},\ s_j = b_rs^r_j$$

The Berwald interrelation

$$B\Gamma = \{G^i_{jk}(x,y), G^i_j\}$$

of F^n has an unique contribution in the present work. B^i_{jk} represents the difference tensor of G^i_{jk} and γ^i_{jk} given as following

$$G^i_{jk}(x,y) = \gamma^i_{jk}(x) + B^i_{jk}(x,y). \tag{1}$$

By transvecting by y^i with the subscript 0, we generate

$$G^i_j = \gamma^i_{0j} + B^i_j \ \ and \ \ 2G^i = \gamma^i_{00} + 2B^i, \tag{2}$$

and then $B^i_j = \dot{\partial}_jB^i$ and $B^i_{jk} = \dot{\partial}_kB^i_j$

A Douglous space [13] is defined as a Finsler space F^n of dimension n if

$$D^{ij} = G^i(x,y)Y^i - G^j(x,y)y^i, \tag{3}$$

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are 3-degree homogeneous polynomial of (y^i).

Next, on differentiating e quation (3) w . r . t . y^m, w e c an g enerate t he following definitions as;

De inition 1 . *([15]) If* $D^i_{im} = (n+1)G^i - G^{im}_m y^i$ *is a h omogeneous b inomial in* (y^i)*, then* F^n *a Finsler space is terminated in a douglous space of* 2^{nd} *kind with unique* (α, β)*-metric.*

Secondly, a Finsler space with (α, β)-metric is termed as second kind Douglous space if and only if

$$B^{im}_m = (n+1)B^i - B^m_m y^i, \tag{4}$$

are 2-degree homogeneous equations in (y^i), where B^m_m is the same as provided by [15].

Now, differentiating the equation (4) with respect to y^h, y^j *and* y^k, we can have

$$B^{im}_{hjkm} = B^i_{hjk} = 0. \tag{5}$$

Definition 2. *A Finsler space* F^n *with* (α, β)*-metric is termed as second kind Douglous space if* $B^{im}_m = (n+1)B^i - B^m_m y^i$ *is a 2-degree homogeneous polynomial in* (y^i)*.*

3 DOUGLOUS SPACE OF 2^{nd} KIND WITH A UNIQUE (α, β)- METRIC

In this part, we examine the fundamentals of a Finsler space that is a second class Douglous space [2] with a (α, β)-metric.

Assuming that $G^i(x, y)$ represents the spraying coefficients of F^n, as given by [4].

$$2G^i = \gamma^i_{00} + 2B^i \tag{6}$$

$$B^i = \frac{\alpha F_\beta}{F_\alpha} s^i_0 + C^* \left[\frac{\beta F_\beta}{\alpha F} y^i - \frac{\alpha F_{\alpha\alpha}}{F_\alpha} \left(\frac{y^i}{\alpha} - \frac{\alpha b^i}{\beta} \right) \right], \tag{7}$$

where

$$C^* = \frac{\alpha\beta \left(r_{00} F_\alpha - 2\alpha s_0 F_\beta \right)}{2\left(\beta^2 F_\alpha + \alpha \gamma^2 F_{\alpha\alpha} \right)},$$

$$\gamma^2 = b^2\alpha^2 - \beta^2. \tag{8}$$

Since $\gamma^i_{00} = \gamma^i_{jk}(x) y^j y^k$ is hp(2), equation (7) yields

$$B^{ij} = \frac{\alpha F_\beta}{F_\alpha} \left(s^i_0 y^j - s^j_0 y^i \right) + \frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha} C^* \left(b^i y^j - b^j y^i \right). \tag{9}$$

Using equations (3) and (9), we can have the lemma given as [14];

Lemma 1. *A Douglas space is defined as a Finsler space* F^n *with an* (α, β)*-metric, if it exists and only if* $B^{ij} = B^i y^j - B^j y^i$ *are 3-degree homogeneous polynomial.*

Differentiating equation (9) w. r. t. y^h, y^k, y^p and y^q, we obtain $D^{ij}_{hkpq} = 0$ i. e. correspond to $D^{im}_{hkpm} = (n+1)D^i_{hkp} = 0$. Therefore, a Finsler space F^n which agrees with $D^{ij}_{hkpq} = 0$ is termed as Douglas space. Now in the resulting equation, differentiation of equation (9) w. r. t. y^m and then contraction of m and j , provides

$$\begin{aligned} B^{im} =& \frac{(n+1)\,\alpha F_\beta s^i_0}{F_\alpha} + \frac{\alpha\left\{(n+1)\,\alpha^2\Omega F_{\alpha\alpha} b^i + \beta\gamma^2 A y^i\right\} r_{00}}{2\Omega^2} \\ &- \frac{\alpha^2\left\{(n+1)\,\alpha^2\Omega F_\beta F_{\alpha\alpha} b^i + B y^i\right\} s_0}{F_\alpha \Omega^2} - \frac{\alpha^3 F_{\alpha\alpha} y^i r_0}{\Omega} \end{aligned} \tag{10}$$

where $\Omega = \left(\beta^2 F_\alpha + \alpha\gamma^2 F_{\alpha\alpha}\right)$, with $\Omega \neq 0$,, $A = \alpha F_\alpha F_{\alpha\alpha\alpha} + 3F_\alpha F_{\alpha\alpha} - 3\alpha(F_{\alpha\alpha})^2$ and

$$B = \alpha\beta\gamma^2 F_\alpha F_\beta F_{\alpha\alpha\alpha} + \beta\left\{\left(3\gamma^2 - \beta^2\right) F_\alpha - 4\alpha\gamma^2 F_{\alpha\alpha}\right\} F_\beta F_{\alpha\alpha} + \Omega F F_{\alpha\alpha} \tag{11}$$

In the succeeding section, we shall use the theorem given below :
Theorem 1. *When B^{im}_m are second degree homogeneous polynomials in (y^m) , where B^{im}_m is provided by equation (10) and equation (11), with $\Omega \neq 0$, then a Finsler space F^n is termed as a 2^{nd} kind Douglous space.*

4 SYMMETRICAL TRANSFORMATION OF DOUGLOUS SPACE OF 2^{nd} KIND WITH A UNIQUE (α, β)-METRIC

The essentials for a second class Douglous space to being symmetrically invariant have been investigated in this section.

On taking Finsler spaces; $F^n = (M, F)$ and $\overline{F}^n = (M, \overline{F})$ and if a function $\sigma(x)$ exists in each coordinate neighbourhood of the underlying manifold M^n then $\overline{F}(x, y) = e^\sigma F(x, y)$, and hence the space F^n is said to be symmetrical to $\overline{F}^n$. Convolutionally is the name given to this transformation $F \to \overline{F}$.

A symmetrical transformation of the metric (α, β) is provided by $(\alpha, \beta) \to \left(\overline{\alpha}, \overline{\beta}\right)$, where $\overline{\alpha} = e^\sigma \alpha$, $\overline{\beta} = e^\sigma \beta$

that is,

$$\overline{a}_{ij} = e^{2\sigma} a_{ij},\ \overline{b}_i = e^\sigma b_i \tag{12}$$

$$\overline{a}^{ij} = e^{-2\sigma} a^{ij},\ \overline{b}^i = e^{-\sigma} b^i \tag{13}$$

and

$b^2 = a^{ij} b_i b_j = \overline{a}^{ij} \overline{b}_i \overline{b}_j$

Using equation (13), we can have the Christoffel symbols given as:

$$\overline{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma^i a_{jk}, \tag{14}$$

where $\sigma_j = \partial_j \sigma$ and $\sigma^i = a^{ij} \sigma_j$.

From equations (13) and (14), we get the identities as below:

$$\overline{\nabla}_j \overline{b}_i = e^\sigma \left(\nabla_j b_i + \rho a_{ij} - \sigma_i b_j\right),$$

$$\overline{r}_{ij} = e^{\sigma}\left[r_{ij} + \rho a_{ij} - \frac{1}{2}\left(b_i\sigma_j + b_j\sigma_i\right)\right],$$
$$\overline{s}_{ij} = e^{\sigma}\left[s_{ij} + \frac{1}{2}\left(b_i\sigma_j - b_j\sigma_i\right)\right],$$
$$\overline{s}^i_j = e^{-\sigma}\left[s^i_j + \frac{1}{2}\left(b^i\sigma_j - b_j\sigma^i\right)\right],$$
$$\overline{s}_j = s_j + \frac{1}{2}\left(b^2\sigma_j - \rho b_j\right), \tag{15}$$

where $\rho = \sigma_r b^r$.
From equations (14) and (15), we obtain the followings:

$$\overline{\gamma}^i_{00} = \gamma^i_{00} + 2\sigma_0 y^i - \alpha^2\sigma_j, \tag{16}$$

$$\overline{r}_{00} = e^{\sigma}\left(r_{00} + \rho\alpha^2 - \sigma_0\beta\right), \tag{17}$$

$$\overline{s}^i_0 = e^{-\sigma}\left[s^i_0 + \frac{1}{2}\left(\sigma s_0 b^i - \beta\sigma^i\right)\right], \tag{18}$$

$$\overline{s}_0 = s_0 + \frac{1}{2}\left(\sigma_0 b^i - \rho\beta\right). \tag{19}$$

Now, we will investigate the symmetrical variation of B^{ij} obtained by equation (9).
Suppose that $\overline{F}(\alpha,\beta) = e^{\sigma}F(\alpha,\beta)$ then we get

$$\overline{F}_{\overline{\alpha}} = F_\alpha,\ \overline{F}_{\overline{\alpha\alpha}} = e^{-\sigma}F_{\alpha\alpha},\ \overline{F}_{\overline{\beta}} = F_\beta,\ \overline{\gamma}^2 = e^{2\alpha}\gamma^2 \tag{20}$$

Using equations (8), (19) and (20) and the Theorem 1,

$$\overline{C}^* = e^{\sigma}\left(C^* + D^*\right), \tag{21}$$

where

$$D^* = \frac{\alpha\beta\left[\left(\beta\alpha^2 - \sigma_0\beta\right)F_\alpha - \alpha\left(b^2\sigma_0 - \rho\beta\right)F_\beta\right]}{2\left(\beta^2 F_\alpha + \alpha\gamma^2 F_{\alpha\alpha}\right)} \tag{22}$$

Therefore, B^{ij} can be represented as:

$$\overline{B}^{ij} = \frac{\alpha F_\beta}{F_\alpha}\left(s^i_0 y^j - s^j_0 y^i\right) + \frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha}C^*\left(b^i y^j - b^j y^i\right)$$
$$+\left(\frac{\alpha\sigma_0 F_\beta}{F_\alpha} + \frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha}D^*\right)\left(b^i y^j - b^j y^i\right) - \frac{\alpha\beta F_\beta}{2F_\alpha}\left(\sigma^i y^j - \sigma^j y^i\right),$$
$$= B^{ij} + C^{ij},$$

where

$$C^{ij} = \left(\frac{\alpha\sigma_0 F_\beta}{F_\alpha} + \frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha}D^*\right)\left(b^i y^j - b^j y^i\right) - \frac{\alpha\beta F_\beta}{2F_\alpha}\left(\sigma^i y^j - \sigma^j y^i\right).$$

From equation (11), we obtain

$$\overline{\Omega}=e^{2\alpha}\Omega,\ \ \overline{A}=e^{-\sigma}A,\ \ \overline{B}=e^{2\alpha}B. \tag{23}$$

Next, On applying symmetrical transformation on B_m^{im} to get

$$\overline{B}_m^{im}=B_m^{im}+K_m^{im} \tag{24}$$

where

$$\begin{aligned}2K_m^{im}=&\frac{(n+1)\,\alpha F_\beta}{F_\alpha}\left(\sigma_0 b^i-\beta\sigma^i\right)+\alpha\left\{\frac{(n+1)\,\alpha^2\Omega F_{\alpha\alpha}b^i+\beta\gamma^2Ay^i}{\Omega^2}\right\}\left(\rho\alpha^2-\sigma_0\beta\right)\\&-\left[\frac{\alpha^2\left\{(n+1)\,\alpha^2\Omega\right\}F_\beta F_{\alpha\alpha}b^i+By^i}{F_\alpha\Omega^2}\right]\left(b^2\sigma_0-\rho\beta\right).\end{aligned} \tag{25}$$

Therefore, the following theorem have been developed
Theorem 2. *A Douglous space of second class is invariant under a symmetrical variation iff $K_m^{im}(x)$ are second-degree homogeneous polynomials in (y^i).*

5 SYMMETRICAL TRANSFORMATION OF DOUGLOUS SPACE 2^{nd} OF KIND WITH UNIQUE METRIC (α,β) AND SPACE $F=\alpha+\upsilon\beta+\mu\frac{\beta^3}{\alpha^2}$

Suppose a space of Finsler manifold with unique metric (α,β) defined by

$$F=\alpha+\upsilon\beta+\mu\frac{\beta^3}{\alpha^2},$$

where υ and μ are constants.
Then,

$$\begin{aligned}F_\alpha&=1-2\mu\frac{\beta^3}{\alpha^3},\\F_\beta&=\upsilon+3\mu\frac{\beta^2}{\alpha^2},\\F_{\alpha\alpha}&=6\mu\frac{\beta^3}{\alpha^4}\\F_{\alpha\alpha\alpha}&=\frac{-6\mu\beta^2}{\alpha^4}.\end{aligned} \tag{26}$$

Now, from equation (11), we get

$$\Omega=\frac{-8\mu\beta^5+\left[\alpha^2\beta+6b^2\alpha\beta^2\right]\alpha\beta}{\alpha^3}$$

$$A = 6\mu\frac{\beta^3}{\alpha^4}\left[-1 - 16\mu\frac{\beta^3}{\alpha^3}\right] \tag{27}$$

$$B = \bigcup_1 + \bigcup_2 + \bigcup_3$$

where,

$$\bigcup_1 = -24\mu\frac{\beta^4}{\alpha^4}\left[\upsilon + 3\mu\frac{\beta^2}{\alpha^2} - 2\upsilon\mu\frac{\beta^3}{\alpha^3} - 6\mu^2\frac{\beta^5}{\alpha^5}\right]\left(b^2\alpha^2 - \beta^2\right),$$

$$\bigcup_2 = 6\mu\frac{\beta^4}{\alpha^4}\left(\upsilon + 3\mu\frac{\beta^2}{\alpha^2}\right)\left[\left(3 - 30\frac{\beta^3}{\alpha^3}\right)b^2\alpha^2 + \left(8\mu\frac{\beta^3}{\alpha^3} - 1\right)4\beta^2\right],$$

$$\bigcup_3 = 6\mu\frac{\beta^4}{\alpha^4}[(\alpha\beta + \upsilon\beta^2) + 6\frac{\beta^2}{\alpha^2}\{(b^2\alpha^2 + \upsilon b^2\alpha\beta - \mu\beta^2 - \upsilon\mu\beta^3\alpha^{-1}) + \frac{\beta^3}{\alpha^3}(b^2\alpha^2 - \mu^2\beta^2)\}],$$

Therefore, from equations (25) and (26), K_m^{im} have been derived as

$$2K_m^{im} = (n+1)\alpha\left[\upsilon + 3\mu\frac{\beta^2}{\alpha^2}\right](\sigma_0 b^i - \beta\sigma^i) + (\alpha\Pi_1 + \alpha\Pi_2)(\rho\alpha^2 - \sigma_0\beta) - \left[\Lambda_0 + (\Lambda_1 + \Lambda_2 + \Lambda_3)y^i - \Gamma\right](b^2\sigma_0 - \rho\beta). \tag{28}$$

where,

$$\alpha\Pi_1 = \frac{6\mu(n+1)\alpha^2\beta^3}{\{\alpha^3\beta^2 + 6b^2\alpha^2\beta^3\} - 6\mu\beta^5}b^i,$$

$$\alpha\Pi_2 = \frac{6\mu\left[-\alpha^3 - 16\mu\beta^4\right]\beta^2\gamma^2}{\left[\{\alpha^3\beta + 6b^2\alpha^2\beta^2\} - 8\mu\beta^4\right]^2}y^i$$

$$\Lambda_0 = \frac{6(n+1)\mu\alpha^4\beta^2(\varepsilon\alpha^2 + 3\mu\beta^2)}{(\alpha^3 - 2\mu\beta^3)\left[\alpha^3\beta + 6b^2\alpha^2\beta^2 - 8\mu\beta^4\right]}b^i$$

$$\Lambda_1 = \frac{-24\mu\alpha^2\beta^4\left(\upsilon\alpha^5 + 3\mu\alpha^3\beta^2 - 2\upsilon\mu\alpha^2\beta^3 - 6\mu^2\beta^5\right)}{(\alpha^3 - 2\mu\beta^3)[\alpha^3\beta^2 + 6b^2\alpha^2\beta^3 - 6\mu\beta^5]^2}\gamma^2,$$

$$\Lambda_2 = \frac{24\mu\alpha^2\beta^4(\varepsilon\alpha^2 + 3\mu\beta^2)}{(\alpha^3 - 2\mu\beta^3)[\alpha^3\beta^2 + 6b^2\alpha^2\beta^3 - 8\mu\beta^5]^2}[3b^2\alpha^5 - 30\mu b^2\alpha^2\beta^3 - 4\alpha^3\beta^2 + 32\mu\beta^5].$$

$$\Lambda_3 = \frac{6\mu(\alpha\beta)^4}{(\alpha^3 - 2\mu\beta^3)[\alpha^3\beta^2 + 6b^2\alpha^2\beta^3 - 8\mu\beta^5]^2}[\alpha^4\beta + 6\upsilon\alpha^3\beta^2(b^2\alpha^3\beta^2 - \mu\alpha\beta^4) + (6 + \upsilon b^2)\alpha^2\beta^4 - (6\mu\upsilon + \mu^2)\beta^5]$$

$$\Gamma = \frac{-6\mu\alpha^2\beta^3}{[\alpha^3\beta^2 + 6b^2\alpha^2\beta^3 - 8\mu\beta^5]} y^i.$$

Now, from equation (28) we can have

$$2K^{im}_m = (n+1)\alpha\left[\upsilon + 3\mu(\alpha^{-1}\beta)\right](\sigma_0 b^i - \beta\sigma^i) + u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7. \quad (29)$$

where,

$$u_1 = \alpha\Pi_1(\rho\alpha^2 - \sigma_0\beta)$$

$$u_2 = \alpha\Pi_2(\rho\alpha^2 - \sigma_0\beta)$$
$$u_3 = -\Lambda_0(b^2\sigma_0 - \rho\beta)$$
$$u_4 = -\Lambda_1 y^i(b^2\sigma_0 - \rho\beta)$$
$$u_5 = -\Lambda_2 y^i(b^2\sigma_0 - \rho\beta)$$
$$u_6 = -\Lambda_3 y^i(b^2\sigma_0 - \rho\beta)$$
$$u_7 = \Gamma(b^2\sigma_0 - \rho\beta)$$

which provides that K^{im}_m is 2-degree homogeneous polynomial of in y^i. Using this, we can have the following result

Theorem 3. *A Douglous space of 2^{nd} kind with an unique (α,β)-metric $F = \alpha + \upsilon\beta + \mu\frac{\beta^3}{\alpha^2}$, where υ and μ are constants, and invariant under a symmetrical transformation.*

Therefore, we find with the help of Theorem 3 that a Douglous space of 2^{nd} kind associated by a space of Finsler with special form of (α,β)-metric can be transformed symmetrically to a Douglous space of 2^{nd} kind again. And hence, we can derive following cases;

Case(i). If $\upsilon = 1$ and $\mu = 0$, we get $F = \alpha + \beta$ which is simply the Randers metric. In the case, $2K^{im}_m$ takes the shape

$$2K^{im}_m = (n+1)\,\alpha\left(\sigma_0 b^i - \beta\sigma^i\right), \quad (30)$$

showing that K^{im}_m is 2-degree homogeneous polynomial in (y^i).

It is remarkable that, $u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = u_7 = 0$. Hence, we can have the following corollary

Corollary 1. *A Douglous space of 2^{nd} kind with Randers metric $F = \alpha + \beta$, is invariant under a conformal change.*

Case(ii). If $\upsilon = 0$ and $\mu = 1$, we get $F = \alpha + \frac{\beta^3}{\alpha^2}$. In the case, $2K^{im}_m$ takes the following shape

$$2K^{im}_m = 3(n+1)(\alpha^{-1}\beta)\alpha(\sigma_0 b^i - \beta\sigma^i) + v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7, \quad (31)$$

where

$$v_1 = \frac{6(n+1)\alpha^2\beta^3}{\alpha^3\beta^2 + 6b^2\alpha^2\beta^3 - 8\mu\beta^5} b^i(\sigma_0 b^i - \beta\sigma^i),$$

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$$v_2 = \frac{6\left[(-\alpha^3 - 16\beta^3\right]\beta^2\gamma^2}{\left[\alpha^3\beta + 6b^2\alpha^2\beta^2 - 8\beta^4\right]^2}(\rho\alpha^2 - \sigma_0\beta),$$

$$v_3 = \frac{18(n+1)\alpha^4\beta^4 b^i}{(\alpha^3 - 2\beta^3)\left[\alpha^3\beta + 6b^2\alpha^2\beta^2 - 8\beta^4\right]}(b^2\sigma_0 - \rho\beta),$$

$$v_4 = \frac{72\alpha^2\beta^6\gamma^2}{\left[\alpha^3\beta^2 + 6b^2\alpha^2\beta^3 - 8\beta^5\right]^2}(b^2\sigma_0 - \rho\beta),$$

$$v_5 = \frac{-18\alpha^2\beta^6\left[3b^2\alpha^5 - 30b^2\alpha^2\beta^3 - 4\alpha^2\beta^3 + 32\beta^5\right]}{(\alpha^3 - 2\beta^3)[\alpha^3\beta^2 + 6b^2\alpha^2\beta^3 - 8\mu\beta^5]^2}y^i(b^2\sigma_0 - \rho\beta),$$

$$v_6 = \frac{-6(\alpha\beta)^4\left[\alpha^4\beta + 6\{b^2\alpha^3\beta^2 + \alpha^2\beta^3 - \alpha\beta^4\} - \beta^5\right]}{(\alpha^3 - 2\beta^3)[\alpha^3\beta^2 + 6b^2\alpha^2\beta^3 - 8\mu\beta^5]^2}y^i(b^2\sigma_0 - \rho\beta),$$

$$v_7 = \frac{6\alpha^2\beta^3}{\alpha^3\beta^2 + 6b^2\alpha^2\beta^3 - 8\mu\beta^5}y^i(b^2\sigma_0 - \rho\beta),$$

showing that K_m^{im} is 2-degree homogeneous polynomial in (y^i).
Hence, the following corollary have been developed;

Corollary 2. *A Douglous space of 2^{nd} kind with special (α,β)-metric $F = \alpha + \frac{\beta^3}{\alpha^2}$ is invariant under a symmetrical transformation.*

Case(iii). If $\upsilon = 1$ and $\mu = 1$, we get $F = \alpha + \beta + \frac{\beta^3}{\alpha^2}$. In the case, $2K_m^{im}$ takes the shape

$$2K_m^{im} = (n+1)\left[1 + 3(\alpha^{-1}\beta)\right]\alpha(\sigma_0 b^i - \beta\sigma^i) + w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7, \quad (32)$$

where

$$w_1 = \frac{6(n+1)\alpha^2\beta^3}{\alpha^3\beta + 6b^2\alpha^2\beta^2 - 8\mu\beta^4}b^i(\rho\alpha^2 - \sigma_0\beta),$$

$$w_2 = \frac{6\left[(-\alpha^3 - 16\beta^3\right]\beta^2\gamma^2}{\left[\alpha^3\beta + 6b^2\alpha^2\beta^2 - 8\beta^4\right]^2}y^i(\rho\alpha^2 - \sigma_0\beta),$$

$$w_3 = \frac{-6(n+1)\alpha^4\beta^2(\alpha^2 + 3\beta^2)b^i}{(\alpha^3 - 2\beta^3)\left[\alpha^3\beta + 6b^2\alpha^2\beta^2 - 8\beta^4\right]}(b^2\sigma_0 - \rho\beta),$$

$$w_4 = \frac{24\alpha^2\beta^4(\alpha^5 + 3\alpha^3\beta^2 - 2\alpha^2\beta^3 - 6\beta^5)\gamma^2}{(\alpha^3 - 2\beta^3)[\alpha^3\beta^2 + 6b^2\alpha^2\beta^3 - 8\beta^5]^2}y^i(b^2\sigma_0 - \rho\beta),$$

$$w_5 = \frac{-6\alpha^2\beta^4(\alpha^2 + (3)\beta^2)}{(\alpha^3 - 2\beta^3)[\alpha^3\beta^2 + 6b^2\alpha^2\beta^3 - 8\mu\beta^5]^2}[3b^2\alpha^5 - 30b^2\alpha^2\beta^3 - 4\alpha^3\beta^2 + 32\beta^5]y^i(b^2\sigma_0 - \rho\beta),$$

$$w_6 = \frac{6(\alpha\beta)^4}{(\alpha^3 - 2\beta^3)[\alpha^3\beta^2 + 6b^2\alpha^2\beta^3 - 8\mu\beta^5]^2}[\alpha^4\beta + \alpha^3\beta^2 + 6(b^2\alpha^3\beta^2 - \alpha\beta^4) + (5 + b^2)\alpha^2\beta^3 - 7\beta^5]y^i(b^2\sigma_0 - \rho\beta),$$

$$w_7 = \frac{-6\alpha^2\beta^2}{\alpha^3\beta + 6b^2\alpha^2\beta^2 - 8\mu\beta^4} y^i(b^2\sigma_0 - \rho\beta).$$

showing that K_m^{im} is a 2-degree homogeneous polynomial in (y^i).
Hence, the following corollary have been developed;;
Corollary 3. *A Douglous space of 2^{nd} kind with certain form of (α, β)-metric $F = \alpha + \beta + \frac{\beta^3}{\alpha^2}$ is invariant under a symmetrical transformation.*

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