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Sturm-Liouville Boundary Value Problems

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Abstract:- The paper focuses on Sturm- Lioville boundary value problems and general solution to Liovilles equations, Boundary value problems with discontinuties inside an interval and Sturm-Lioville eigen value.

Keywords*: Sturm -Lioville boundary value problems, Eigen value problems, Discontinuties in an interval, Boundary conditions.*

1. Introduction

Charles-Francois Sturm (1803-1855) and Joseph Liouville (1809-1882) published a series of papers in 1836 and 1837 on second order linear ordinary differential equations including boundary value problems. The influences of their work was such that the subject became known as Sturm- Liouvilles Theory. The impact of these papers went well beyond their subject matter to general linear and nonlinear differential equations.

Sturm and Liouville were among the first to realise the limitation of their approach and to see the need for finding properties of solutions directly from the equations even when no analytic expression for solutions are available.

Althrough the subject of Sturm Liouville Problems is over 170 year old,a surprising number of the results surveyed here are of recent origin.

It is well known that boundary value transmission problems have important applications in heat conduction mass transfer and string vibrations problems with nodes located internally.

Sturm -Liouville boundary value problems :The problem consists of a differential equation of the form

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0$$

On the interval $0 < x < 1$

together with boundary conditions

$$a_1 y(0) + a_2 y'(0) = 0$$

and

$$b_1 y(1) + b_2 y'(1) = 0$$

at the end points.

The solution to a Sturm - problem is set of eigen values

$$\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \dots \dots \dots \dots \dots \dots ..\}$$

and a corresponding set of functions $\phi_0(x), \phi_1(x), \phi_2(x), \dots \dots \dots \dots \dots \dots$

Satisfying

$$\frac{d}{dx}\left[p(x).\frac{d\phi_n}{dx}\right] - q(x)\phi_n + \lambda_n r(x).\phi_n = 0$$

n=0,1,2,3,4,……

and boundary conditions

$$a_1\phi_n(0)+a_1\phi_n'(0)=\ 0=b_1\phi_n(1)\ +b_2\phi_n'(1)$$

$$n\ =0,1,2,3,4\ \ldots\ \ldots\ \ldots.$$

Moreover any continuous function

$$f\colon[0,1]\to R$$

Can be expanded in terms of the Sturm -Lioville eigen function

$$\{\phi_n\ /n\in N\}$$

$$f(x)=\sum_{n=0}^{\infty} c_n\ \phi_n(x)$$

With

$c_n := \int_0^1 f(x)\ \phi_n(x)r(x)dx$

The paper focuses on developing solutions $\phi(x)$

of a related non homogeneous differential equation of the form

$$\frac{d}{dx}\left[p(x).\frac{d\phi}{dx}\right]-q(x)\phi+\mu r(x)\phi=f(x)$$

Satisfying the same boundary conditions

$$a_1\phi(0)+a_1\phi'(0)=\ 0=b_1\phi(1)\ +b_2\phi'(1)$$

We stress that the parameter μ need not be one of the Sturm- Liouville Eigen values λ_n

We suppose that $\phi(x)$

is a continious

$$f(x)=\sum_{n=0}^{\infty} c_n\ \phi_n(x)$$

$$c_n := \int_0^1 f(x)\ \phi_n(x)r(x)dx$$

We will have an expansion

$$\phi(x)=\sum_{n=0}^{\infty} c_n\ \phi_n(x)$$

$$c_n := \int_0^1 \phi(x)\ \phi_n(x)r(x)dx$$

Let $\{\lambda_0,\lambda_1,\lambda_2,\lambda_3,\ldots\ldots\ldots\ldots\ldots\ldots..\}$ be set of eigen values of homogeneous Sturm- Liouville problem

$$\frac{d}{dx}\left[p(x).\frac{d\phi}{dx}\right]-q(x)\phi+\lambda r(x)\phi=0$$

$$a_1\phi(0)+a_1\phi'(0)=\ 0=b_1\phi(1)\ +b_2\phi'(1)$$

and let $\{\phi_1,\phi_2,\phi_3,\ldots\ldots\ldots\ldots\}$

be a corresponding set of Sturm- Liouville eigen function normalized so that

$$\int_0^1 \phi_n(x)\ \phi_n(x)r(x)dx=1$$

The non-homogenious boundary problem

$$\frac{d}{dx}\left[p(x).\frac{d\phi}{dx}\right]-q(x)\phi+\mu r(x)\phi=f(x)$$

$$a_1\phi(0)+a_1\phi'(0)=\ 0=b_1\phi(1)\ +b_2\phi'(1)$$

has unique solution whenever $\mu\notin$

$$\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \dots \dots \dots \dots\}$$

It is given by

$$\phi(x) = \sum_{n=0}^{\infty} c_n \, \phi_n(x)$$

$$c_n = \frac{1}{\lambda_n - \mu} \int_0^1 f(x) \, \phi_n(x) dx$$

If on the other hand $\mu = \lambda_m$

Then the non -homogenious problem has no solution unless

$$\int_0^1 f(x) \phi_m(x) dx = 0$$

If in fact $\mu = \lambda_m$

and

$$\int_0^1 f(x) \phi_m(x) dx = 0$$

is true then there is one parameter solution.

Sturm-Liovilles problem with discontinuities in the case when an Eigen -Parameter appear not only in the differential equation but also in one of the Transmission Conditions.

Boundary value problems with discontinuities in an interval and eigen value contained in the boundary conditions often appear in many branches of natural sciences.

In the Analysis we consider one discontinuous eigen-value of the problem that consists of the Sturm Liouville equation

$$-a(x)u'' + q(x)u = \lambda u$$

$$x \in [1,0) \cup (0,1]$$

With boundary conditions at the end points

$$\cos \alpha \, u(-1) + \sin \alpha \, u'(-1) = 0$$

$$\cos \beta \, u(1) + \sin \beta \, u'(1) = 0$$

And transmissions conditions at the points of discontinuities are

$$u(-0) - u(+0) = 0$$

$$u'(-0) - u'(+0) = 0$$

Where $a(x) = a_1^2 \; for \; x \; \in [-1,0) \; and \; a(x) = a_2^2 \; for \; x \in \; (0,1]$

a_1, a_2 are positive real numbers.

We shall construct a special fundamental system of solution for λ

is not an Eigen value.

Consider the following initial value problem

$$-a_1^2 \frac{d^2u}{dx^2} + q(x)u(x) = \lambda u(x)$$

$$u(-1) = \sin \alpha$$

$$\frac{du - 1)}{dx} = -\cos \alpha$$

The problem has unique solution

$$u \approx \phi_1(x) \approx \phi_1(x, \lambda)$$

Which is an entire function of the parameter

$\lambda \in C$ for each fixed $x \in [-1,0)$

Similarly the problem

$$-a_2^2 \frac{d^2u}{dx^2} + q(x)u(x) = \lambda u(x)$$

$$u(1) = -\sin\beta$$

$$\frac{du(1)}{dx} = \cos\beta$$

has unique solution

$$u \approx \lambda_2(x) \approx \lambda_2(x, \lambda)$$

Which is an entire function of the parameter

$$\lambda \in C$$

for each fixed $x \in [0,1)$

2. Conclusion

In this paper we have presented Sturm-Liovilles Boundary value problems with their general solutions and Sturm -Lioville problem with discontinuities.

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