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The Laplacian Minimum Pendant Dominating Energy of a Graph

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Abstract: - Let G be any graph. The pendant dominating set and the associated Laplacian minimum pendant dominating energy (denoted as $LE_{pe}(G)$) provide insights into both the structure of the graph and the energy associated with specific dominating sets. Here's an overview of the key concepts and bounds related to $LE_{pe}(G)$. The study of $LE_{pe}(G)$ involves understanding the spectral properties of the Laplacian matrix relative to pendant dominating sets. By calculating exact values for standard graph families and establishing bounds, one gains deeper insights into both the structural and spectral characteristics of graphs.

Key-words*:* Dominating set, Pendant Dominating set, Laplacian minimum pendant dominating energy.

1. Introduction

The energy of a graph was introduced by I. Gutman [4]. Let $G = (V, E)$ be a graph with n vertices and m edges and let $A = (a_{i,j})$ be an adjacency matrix of a graph G. The eigenvalues of a graph G are the eigenvalues of its adjacency matrix denoted by $A_{pe}(G)$. A graph G is called singular graph if atleast one of its eigenvalues are zero. For singular graphs evidently, $\det A = 0$. A graph G is called non-singular graph if all its eigen values are different from zero. For non-singular graphs evidently, $\det A > 0$. A graph G is called k-regular graph if every vertex in G has degree k. The energy of a graph G is denoted by $E(G)$ and is defined to be sum of absolute values of the eigenvalues of G. i.e.,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

In 2006, I. Gutman and B. Zhou [5] defined the Laplacian energy of a graph. Let G be a graph with n vertices and m edges. The Laplacian matrix of a graph G is denoted by $L = (L_{i,j})$ which is a square matrix of order n. The elements of the Laplacian matrix are defined as follows:

$$L_{i,j} = \begin{cases} -1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ d_i, & \text{if } i = j \end{cases}$$

where d_i is degree of vertex v_i.

The Laplacian energy of a graph G is defined using the eigenvalues of its Laplacian matrix $L_{i,j}$. If $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are eigenvalues of Laplacian matrix and the Laplacian energy of G is defined as

$$LE(G) = \sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|$$

The concept of minimum pendant dominating set and Laplacian energy can be analysed through various bounds and inequalities. The Laplacian energy of a graph, derived from the eigenvalues of its Laplacian matrix, has diverse and significant applications in various fields, including chemical applications, high-resolution satellite image classification and segmentation and finding semantic structures in image hierarchies.

2. The minimum pendant dominating energy of a graph

Given a simple graph $G = (V, E)$ of order n, where n refers to the number of vertices in the graph $V(G) = \{v_1, v_2, \ldots\ldots, v_n\}$ is the set of vertices. A dominating set S of $V(G)$ is called a pendant dominating set if the induced subgraph $\langle S\rangle$ (the subgraph formed by the vertices in S) contains at least one pendant vertex, denoted by $\gamma_{pe}(G)$. In other words, there exists atleast one vertex in S which is a pendant vertex in the subgraph $\langle S\rangle$. For more details on the terms used in this paper refer [9]. The minimum pendant dominating matrix $A_{pe}(G)$ of a graph $G = V, E)$ with a minimum pendant dominating set D is defined as an $n \times n$ matrix, where n is the number of vertices in G. The matrix is constructed as follows:

$$a_{i,j} = \begin{cases} 1, & \text{if } v_i v_j \epsilon E \\ 1, & \text{if } i = j \text{ and } v_i \in D \\ 0, & \text{otherwise} \end{cases}$$

The characteristic polynomial of the matrix $A_{pe}(G)$, denoted by $f_n(G, \lambda)$ is a key concept in spectral graph theory, particularly when studying the properties of the minimum pendant dominating matrix of a graph G and is defined as $f_n(G, \lambda) = \det(\lambda I - A_{pe}(G))$. The concept of least pendant dominating eigenvalues is not a standard term in graph theory, but I can explain how it might relate to known concepts involving the eigenvalues of matrices associated with a graph, particularly the adjacency matrix $A_{pe}(G)$ and related matrices. Since $A_{pe}(G)$ are real and symmetric, the eigenvalues are real numbers. So, the eigenvalues of a matrix are typically arranged in non-increasing order $\lambda_1 \geq \lambda_2 \geq, \ldots\ldots, \geq \lambda_n$ to facilitate their analysis and comparison. The minimum pendant dominating energy of a graph G is a measure derived from the eigenvalues of an adjacency matrix $A_{pe}(G)$ associated with the graph. To define it formally:

$$E_{pe}(G) = \sum_{i=1}^{n} |\lambda_i|$$

3. The Laplacian minimum pendant dominating energy of a graph

Consider a graph $G = (V, E)$ with a minimum pendant dominating set D. The set D is a dominating set such that the induced subgraph $\langle D\rangle$ contains at least one pendant vertex. The Laplacian matrix $LE_{pe}(G)$ of the graph G is defined as $LE_{pe}(G) = D(G) - A_{pe}(G)$ where $D(G)$ is the diagonal matrix of vertex degrees and $A_{pe}(G)$is the adjacency matrix of G. Let the eigenvalues of $LE_{pe}(G)$ be $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ arranged in non-increasing order i.e. $\lambda_1 \geq \lambda_2 \geq, \ldots\ldots, \geq \lambda_n$. These eigenvalues are referred to as the Laplacian minimum pendant dominating eigenvalues of the graph G. The Laplacian minimum pendant dominating energy is defined as:

$$LE_{pe}(G) = \sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|$$

Here, $\left|\lambda_i - \frac{2m}{n}\right|$ represents the absolute value of the difference between each eigenvalue λ_i and the average degree of the graph G and we compute the average degree of the graph G as $\frac{2m}{n}$ where m is the number of edges in G and n is the number of vertices.

The study of the Laplacian minimum pendant dominating energy of a graph touches upon several mathematical aspects and has potential applications in various fields, including chemistry and computer science.

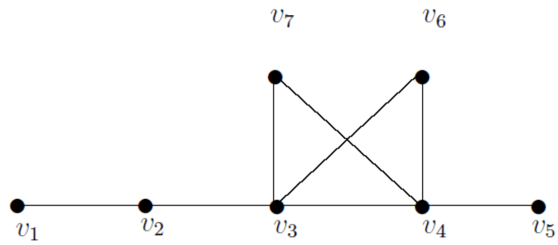

Figure 3.1

Example 3.1. Let G be a graph with 6 vertices as shown in the Figure 3.1. The minimum pendant dominating sets are as follows:

(i) $D_1 = \{v_1, v_3, v_4\}$ (ii) $D_2 = \{v_1, v_2, v_4\}$ (iii) $D_3 = \{v_2, v_3, v_4\}$ (iv) $D_4 = \{v_2, v_3, v_5\}$

(v) $D_5 = \{v_2, v_4, v_5\}$ (vi) $D_6 = \{v_2, v_4, v_6\}$ (vii) $D_7 = \{v_2, v_4, v_7\}$

(i) Let $D_1 = \{v_1, v_3, v_4\}$ then

$$A_{pe,D_1}(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } D(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$L_{pe,D_1}(G) = D(G) - A_{pe,D_1}(G) = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 3 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

The characteristic polynomial of the Laplacian minimum pendant dominating matrix $L_{pe}(G)$ of a graph G:

$$f_n(G, \lambda) = \lambda^7 - 13\,\lambda^6 + 61\,\lambda^5 - 120\,\lambda^4 + 71\,\lambda^3 + 47\,\lambda^2 - 38\lambda - 8 = 0$$

The Laplacian minimum pendant dominating eigenvalues of a graph G as follows:

$\lambda_1 = -0.5597,\ \lambda_2 = -0.1842,\ \lambda_3 = 0.9583, \lambda_4 = 2,\ \lambda_5 = 2.1960,\ \lambda_6 = 4.1923,\ \lambda_7 = 4.3973$

The average degree of the graph G is

$\frac{2m}{n} = \frac{2(8)}{7} = \frac{16}{7} \approx 2.2857$

Based on the calculations with the eigenvalues and the average degree, the Laplacian minimum pendant dominating energy of the graph G is indeed approximately $LE_{pe}G) \approx 10.0263$

(ii) Let $D_2 = \{v_1, v_2, v_4\}$ then

$$A_{pe,D_2}(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } D(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$L_{pe,D_1}(G) = D(G) - A_{pe,D_1}(G) = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 3 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

The characteristic polynomial of the Laplacian minimum pendant dominating matrix $L_{pe}(G)$ of a graph G:

$$f_n(G, \lambda) = \lambda^7 - 13\,\lambda^6 + 59\,\lambda^5 - 106\,\lambda^4 + 44\,\lambda^3 + 63\,\lambda^2 - 44\lambda - 4 = 0$$

The Laplacian minimum pendant dominating eigenvalues of a graph G as follows:

$\lambda_1 = -0.5597,\ \lambda_2 = -0.1842,\ \lambda_3 = 0.9583, \lambda_4 = 2,\ \lambda_5 = 2.1960,\ \lambda_6 = 4.1923,\ \lambda_7 = 4.3973$

The average degree of the graph G is

$\frac{2m}{n} = \frac{2(8)}{7} = \frac{16}{7} \approx 2.2857$

Based on the calculations with the eigenvalues and the average degree, the Laplacian minimum pendant dominating energy of the graph G is indeed approximately $LE_{pe}G) \approx 10.0263$

Therefore, we conclude that the Laplacian minimum pendant dominating energy of a graph G is influenced by the minimum pendant dominating set of G.

4. The Laplacian minimum pendant dominating energy of some standard graphs

Theorem 4.1. The Laplacian minimum pendant dominating energy of a complete graph K_n for $n > 2$ is

$$(n-3) + \sqrt{n^2 - 2n + 9}$$

Proof: The Laplacian minimum pendant dominating energy for a complete graph K_n with the minimum pendant dominating set $D = \{v_1, v_2\}$, follow these steps:

$$A_{pe}(K_n) = \begin{bmatrix} 1 & 1 & 1 & \cdots\cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots\cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots\cdots & 1 & 1 \\ . & . & . & \cdots\cdots & . & . \\ . & . & . & \cdots\cdots & . & . \\ . & . & . & \cdots\cdots & . & . \\ . & . & . & \cdots\cdots & . & . \\ 1 & 1 & 1 & \cdots\cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots\cdots & 1 & 0 \end{bmatrix} \quad \text{and} \quad D(K_n) = \begin{bmatrix} n-1 & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & n-1 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 0 & n-1 & \cdots\cdots & 0 & 0 \\ . & . & . & \cdots\cdots & . & . \\ . & . & . & \cdots\cdots & . & . \\ . & . & . & \cdots\cdots & . & . \\ . & . & . & \cdots\cdots & . & . \\ 0 & 0 & 0 & \cdots\cdots & n-1 & 0 \\ 0 & 0 & 0 & \cdots\cdots & 0 & n-1 \end{bmatrix}$$

$$L_{pe}(K_n) = D(K_n) - A_{pe}(K_n) = \begin{bmatrix} n-2 & -1 & -1 & \cdots\cdots & -1 & -1 \\ -1 & n-2 & 1 & \cdots\cdots & -1 & -1 \\ -1 & -1 & n-1 & \cdots\cdots & -1 & -1 \\ . & . & . & \cdots\cdots & . & . \\ . & . & . & \cdots\cdots & . & . \\ . & . & . & \cdots\cdots & . & . \\ . & . & . & \cdots\cdots & . & . \\ -1 & -1 & -1 & \cdots\cdots & n-1 & -1 \\ -1 & -1 & -1 & \cdots\cdots & -1 & n-1 \end{bmatrix}$$

The characteristic polynomial of the Laplacian minimum pendant dominating matrix $LE_{pe}(K_n)$ for a complete graph K_n is

$$f_n(K_n, \lambda) = [\lambda - (n-1)]\,(\lambda - n)^{n-3}\{\lambda^2 - (n-1)\lambda - 2\} = 0$$

Thus, the eigenvalues of $LE_{pe}(K_n)$ are

$\lambda = n-1$ {1 time}, $\lambda = n$ {(n-3) times}, $\lambda = \frac{(n-1)\pm\sqrt{(n^2-2n+9)}}{2}$ {one time each}

The average degree of a graph K_n is

$\frac{2m}{n} = \frac{2\frac{n(n-1)}{2}}{n} = n-1$

The Laplacian minimum pendant dominating energy $LE_{pe}(K_n)$ of a complete graph K_n is

$$LE_{pe}(K_n) = |(n-1)-(n-1)| + |n-(n-1)|(n-3) + \left|\frac{(n-1)+\sqrt{(n^2-2n+9)}}{2} - (n-1)\right| + \left|\frac{(n-1) - \sqrt{(n^2-2n+9)}}{2} - (n-1)\right|$$

$LE_{pe}(K_n) = (n-3) + \left|\frac{-n+1+\sqrt{(n^2-2n+9)}}{2}\right| + \left|\frac{-n+1-\sqrt{(n^2-2n+9)}}{2}\right|$

$$\therefore LE_{pe}(K_n) = (n-3) + \sqrt{n^2-2n+9}$$

Theorem 4.2. The Laplacian minimum pendant dominating energy of a star graph K_n for $n \geq 4$ is atmost $\frac{2n^2-4n+4}{n}$

Proof: For the star graph $K_{1,n-1}$ where v_1 is the centre and the remaining vertices $v_2, v_3, \dots, v_n$ are leaves and given that $D = \{v_1, v_2\}$ is the minimum pendant dominating set. The Laplacian minimum pendant dominating energy $LE_{pe}(K_{1,n-1})$ of this graph is

$$A_{pe}(K_{1,n-1}) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \text{ and } D(K_{1,n-1}) = \begin{bmatrix} n-1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

$$L_{pe}(K_{1,n-1}) = D(K_{1,n-1}) - A_{pe}(K_{1,n-1}) = \begin{bmatrix} n-2 & -1 & -1 & \dots & -1 & -1 \\ -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ -1 & 0 & 0 & \dots & 1 & 0 \\ -1 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

The characteristic polynomial of the Laplacian minimum pendant dominating matrix $LE_{pe}(K_{1,n-1})$ for a complete graph $K_{1,n-1}$ is

$f_n(K_{1,n-1}, \lambda) = (\lambda - 1)^{n-3}(\lambda^3 - (n-1)\lambda^2 - \lambda + 1)$

Now,

$$f_n(K_{1,n-1}, \lambda) = (\lambda - 1)^{n-3}(\lambda^3 - (n-1)\lambda^2 - \lambda + 1)$$
$$f_n(K_{1,n-1}, \lambda) \leq (\lambda - 1)^{n-3}(\lambda^3 - (n-1)\lambda^2 - \lambda + (n-1))$$

$\leq (\lambda - 1)^{n-3}(\lambda^2 - 1)(\lambda - (n-1))$

$= (\lambda - 1)^{n-3}(\lambda + 1)(\lambda - 1)(\lambda - (n-1))$

$f_n(K_{1,n-1},\lambda) \leq (\lambda-1)^{n-3}(\lambda+1)(\lambda-1)(\lambda-(n-1))$

Thus, the eigenvalues of $LE_{pe}(K_{1,n-1})$ are

$\lambda = 1$ {(n-3) times}, $\lambda = -1$ {1 time}, $\lambda = 1$ {1 time}, $\lambda = (n-1)$ {1 time}

The average degree of a graph $K_{1,n-1}$ is

$\frac{2m}{n} = \frac{2(n-1)}{n}$

The Laplacian minimum pendant dominating energy $LE_{pe}(K_{1,n-1})$ of a complete graph $K_{1,n-1}$is

$$LE_{pe}(K_{1,n-1}) \leq \left|1-\frac{2(n-1)}{n}\right|(n-3)+\left|-1-\frac{2(n-1)}{n}\right|+\left|1-\frac{2(n-1)}{n}\right|+\left|(n-1)-\frac{2(n-1)}{n}\right|$$

$$LE_{pe}(K_{1,n-1}) = \frac{(n-2)(n-3)}{n}+\frac{(3n-2)}{n}+\frac{(n-1)}{n}+\frac{(n^2-3n+2)}{n}$$

$$LE_{pe}(K_{1,n-1}) \leq \frac{2n^2-4n+4}{n}$$

$$\therefore LE_{pe}(K_{1,n-1}) \leq \frac{2n^2-4n+4}{n}$$

Theorem 4.3. The Laplacian minimum pendant dominating energy of a bi-star graph $B(m,n)$ for $n \geq 2$ is $\frac{2n(n-1)}{(n+1)}+\sqrt{n^2+4}+\sqrt{n^2+4n}$

Proof: For the bi-star graph $B(m,n)$ with the vertex set $\{v_1, v_2, \ldots, v_{2n+2}\}$ and given that $D = \{v_1, v_2\}$ is the minimum pendant dominating set. The Laplacian minimum pendant dominating energy $LE_{pe}(B(m,n))$ of this graph is

$$A_{pe}(B(n,n)) = \begin{bmatrix} 1 & 1 & 0 & \ldots\ldots & 1 & 1 \\ 1 & 1 & 1 & \ldots\ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots\ldots & 0 & 0 \\ . & . & . & \ldots\ldots & . & . \\ . & . & . & \ldots\ldots & . & . \\ . & . & . & \ldots\ldots & . & . \\ . & . & . & \ldots\ldots & . & . \\ 1 & 0 & 0 & \ldots\ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots\ldots & 0 & 0 \end{bmatrix} \text{ and } D(B(n,n)) = \begin{bmatrix} n+1 & 0 & 0 & \ldots\ldots & 0 & 0 \\ 0 & n+1 & 0 & \ldots\ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots\ldots & 0 & 0 \\ . & . & . & \ldots\ldots & . & . \\ . & . & . & \ldots\ldots & . & . \\ . & . & . & \ldots\ldots & . & . \\ . & . & . & \ldots\ldots & . & . \\ 0 & 0 & 0 & \ldots\ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots\ldots & 0 & 1 \end{bmatrix}$$

$$L_{pe}(B(n,n)) = D(B(n,n)) - A_{pe}(B(n,n)) = \begin{bmatrix} n & -1 & 0 & \ldots\ldots & -1 & -1 \\ -1 & n & -1 & \ldots\ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots\ldots & 0 & 0 \\ . & . & . & \ldots\ldots & . & . \\ . & . & . & \ldots\ldots & . & . \\ . & . & . & \ldots\ldots & . & . \\ . & . & . & \ldots\ldots & . & . \\ -1 & 0 & 0 & \ldots\ldots & 1 & 0 \\ -1 & 0 & 0 & \ldots\ldots & 0 & 1 \end{bmatrix}$$

The characteristic polynomial of $LE_{pe}(B(n,n))$ is

$f_n(B(n,n),\lambda) = (\lambda-1)^{2(n-1)}(\lambda^2-n\lambda-1)(\lambda^2-(n+2)\lambda+1)$

Thus, the eigenvalues of $LE_{pe}(B(n,n))$ are

$\lambda = 1$ $\{2(n-1)$ times$\}$, $\lambda = \frac{n\pm\sqrt{n^2+4}}{2}$ {1 time}, $\lambda = \frac{(n+2)\pm\sqrt{n^2+4n}}{2}$ {1 time}

The average degree of a graph $B(n,n)$ is

$$\frac{2m}{n}=\frac{2(2n+1)}{2n+2}=\frac{2n+1}{n+1}$$

The Laplacian minimum pendant dominating energy $LE_{pe}(B(n,n))$ of a bi-star graph $B(n,n)$ is

$$L_{pe}(B(n,n))=\left|1-\frac{(2n+1)}{n+1}\right|2(n-1)+\left|\frac{n\pm\sqrt{n^2+4}}{2}-\frac{(2n+1)}{n+1}\right|+\left|\frac{(n+2)\pm\sqrt{n^2+4n}}{2}-\frac{(2n+1)}{n+1}\right|$$

$$L_{pe}(B(n,n))=\frac{2n(n-1)}{n+1}+\sqrt{n^2+4}+\sqrt{n^2+4n}$$

$$\therefore L_{pe}(B(n,n))=\frac{2n(n-1)}{n+1}+\sqrt{n^2+4}+\sqrt{n^2+4n}$$

Theorem 4.4. The Laplacian minimum pendant dominating energy of a complete bipartite graph $K_{n,n}$ for $n \geq 2$ is atleast $(n+1)+\sqrt{n^2+2n-3}$

Proof: For the complete bipartite graph $K_{n,n}$ with the vertex set $V(G)=\{v_1,v_2,\ldots,v_{2n}\}$ and given that $D=\{v_1,v_n\}$ is the minimum pendant dominating set. The Laplacian minimum pendant dominating energy $LE_{pe}(K_{n,n})$ of this graph is

$$A_{pe}(K_{n,n})=\begin{bmatrix}1&0&0&\ldots\ldots&1&1\\0&0&0&\ldots\ldots&1&1\\0&0&0&\ldots\ldots&1&1\\.&.&.&\ldots\ldots&.&.\\.&.&.&\ldots\ldots&.&.\\.&.&.&\ldots\ldots&.&.\\.&.&.&\ldots\ldots&.&.\\1&1&1&\ldots\ldots&0&0\\1&1&1&\ldots\ldots&0&1\end{bmatrix}\quad\text{and}\quad D(K_{n,n})=\begin{bmatrix}n&0&0&\ldots\ldots&0&0\\0&n&0&\ldots\ldots&0&0\\0&0&n&\ldots\ldots&0&0\\.&.&.&\ldots\ldots&.&.\\.&.&.&\ldots\ldots&.&.\\.&.&.&\ldots\ldots&.&.\\.&.&.&\ldots\ldots&.&.\\0&0&0&\ldots\ldots&n&0\\0&0&0&\ldots\ldots&0&n\end{bmatrix}$$

$$L_{pe}(K_{n,n})=D(K_{n,n})-A_{pe}(K_{n,n})=\begin{bmatrix}n-1&0&0&\ldots\ldots&-1&-1\\0&n&0&\ldots\ldots&-1&-1\\0&0&n&\ldots\ldots&-1&-1\\.&.&.&\ldots\ldots&.&.\\.&.&.&\ldots\ldots&.&.\\.&.&.&\ldots\ldots&.&.\\.&.&.&\ldots\ldots&.&.\\-1&-1&-1&\ldots\ldots&n&0\\-1&-1&-1&\ldots\ldots&0&n-1\end{bmatrix}$$

The characteristic polynomial of the Laplacian minimum pendant dominating matrix $LE_{pe}(K_{n,n})$ for a complete graph $K_{n,n}$ is

$$f_n(K_{n,n},\lambda)=(\lambda-n)^{2n-4}(\lambda^2-(n-1)\lambda-1)\left(\lambda^2-(3n-1)\lambda+(2n^2-2n+1)\right)$$

Thus, the eigenvalues of $LE_{pe}(K_{n,n})$ are

$\lambda = n\ \{(2n-4)\text{ times}\}$, $\lambda=\frac{(n-1)\pm\sqrt{n^2-2n+5}}{2}\ \{1\text{ time}\}$, $\lambda=\frac{(3n-1)\pm\sqrt{n^2+2n-3}}{2}\ \{1\text{ time}\}$

The average degree of a graph $K_{n,n}$ is

$$\frac{2m}{n}=\frac{2n^2}{2n}=n$$

The Laplacian minimum pendant dominating energy $LE_{pe}(K_{n,n})$ of a bi-star graph $K_{n,n}$ is

$$L_{pe}(K_{n,n})=|n-n|(2n-4)+\left|\frac{(n-1)\pm\sqrt{n^2-2n+5}}{2}-n\right|+\left|\frac{(3n-1)\pm\sqrt{n^2+2n-3}}{2}-n\right|$$

$$L_{pe}(K_{n,n}) = 0 + \left|\frac{(-n-1) \pm \sqrt{n^2 - 2n + 5}}{2}\right| + \left|\frac{(n-1) \pm \sqrt{n^2 + 2n - 3}}{2}\right|$$

$$\therefore L_{pe}(K_{n,n}) = (n+1) + \sqrt{n^2 + 2n - 3}$$

5. Properties of the Laplacian minimum pendant dominating eigen values of a graph

Theorem 5.1. Let D be a minimum pendant dominating set of G and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $LE_{pe}(G)$ then

(i) $\sum_{i=1}^{n} \lambda_i = 2|E| - |D|$

(ii) $\sum_{i=1}^{n} {\lambda_i}^2 = 2|E| + \sum_{i=1}^{n} (d_i - h_i)^2$ where $h_i = \begin{cases} 1, if\ v_i \in D \\ 0, if\ v_i \notin D \end{cases}$

Proof: (i) The formula provided

$$\sum_{i=1}^{n} d_i - |D| = 2|E| - |D|$$

where d_i is the degree of vertex i.

$|D|$ is the number of vertices in the pendant dominating set D.

$|E|$ is the number of edges in the graph G

represents the sum of the diagonal elements of a matrix $LE_{pe}(G)$ related to the graph G and its minimum pendant dominating set D.

Also, the sum of the eigenvalues of the matrix $LE_{pe}(G)$ is equal to the trace $LE_{pe}(G)$ which is based on the given context $2|E| - |D|$.

(ii) The result that the sum of the squares of the eigenvalues of $LE_{pe}(G)$ is equal to the trace of $LE_{pe}(G)^2$ is a direct application of a general property of matrices.

Therefore,

$$\sum_{i=1}^{n} {\lambda_i}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij} l_{ji} = \sum_{i=1}^{n} (l_{ij})^2 + \sum_{j=1}^{n} (l_{ji})^2$$

$$= 2\sum_{i<j}^{n} (l_{ij})^2 + \sum_{j=1}^{n} (l_{ji})^2$$

$$\sum_{i=1}^{n} {\lambda_i}^2 = 2|E| + \sum_{i=1}^{n} (d_i - h_i)^2 \text{ where } h_i = \begin{cases} 1, \text{if } v_i \in D \\ 0, \text{if } v_i \notin D \end{cases}$$

$$\sum_{i=1}^{n} {\lambda_i}^2 = 2N \text{ where } N = |E| + \frac{1}{2}\sum_{i=1}^{n} (d_i - h_i)^2$$

6. Upper and Lower bounds

Theorem 6.1. Given the graph G with n vertices and m edges, and D is a minimum pendant dominating set of G then $LE_{pe}(G) \le \sqrt{2Nn} + 2m$

Proof: Given a graph G with n vertices and m edges, and the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the Laplacian matrix $LE_{pe}(G)$. By using Cauchy's - Schwarz inequality we have,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Put $a_i = 1, b_i = \lambda_i$ then,

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq (n)(2N)$$

$$\therefore \left(\sum_{i=1}^{n} |\lambda_i|\right) \leq \sqrt{2Nn}$$

By Triangle inequality,

$$\left|\lambda_i - \tfrac{2m}{n}\right| \leq |\lambda_i| + \left|\tfrac{2m}{n}\right| \forall\, i = 1,2,\ldots,n$$

$$i.e., \left|\lambda_i - \frac{2m}{n}\right| \leq |\lambda_i| + \frac{2m}{n} \ \forall\, i$$

$$\left(\sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|\right) \leq \left(\sum_{i=1}^{n} \lambda_i\right)\left(\sum_{i=1}^{n} \frac{2m}{n}\right)$$

$$\leq \sqrt{2Nn} + 2m$$

$$\therefore LE_{pe}(G) \leq \sqrt{2Nn} + 2m$$

Theorem 6.2. Given the graph G with n vertices and m edges, and D is a minimum pendant dominating set of G then $LE_{pe}(G) \leq \sqrt{2Nn + 4m(|D| - m)}$

Proof: By using Cauchy's - Schwarz inequality we have,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Put $a_i = 1, b_i = \left|\lambda_i - \frac{2m}{n}\right|$ then,

$$\left(\sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \left|\lambda_i - \frac{2m}{n}\right|^2\right)$$

$$[LE_{pe}(G)]^2 = n\left[\sum_{i=1}^{n} \lambda_i^2 + \sum_{i=1}^{n} \frac{4m^2}{n^2} - \frac{4m}{n}\sum_{i=1}^{n} \lambda_i\right]$$

$$= n\left[2N + \frac{4m^2}{n^2} \cdot n - \frac{4m}{n}(2m - |D|)\right]$$

$$= n\left[2N + \frac{4m^2}{n} - \frac{8m^2}{n} + \frac{4m|D|}{n}\right]$$

$= 2Nn + 4m(|D| - m)$

$\therefore LE_{pe}(G) \leq \sqrt{2Nn + 4m(|D| - m)}$

Theorem 6.3. Given the graph G with n vertices and m edges, and D is a minimum pendant dominating set of G and if $D = \left|\det LE_{pe}(G)\right|$ then $LE_{pe}(G) \geq \sqrt{2N + n(n-1)D^{\frac{2}{n}}} - 2m$

Proof: Consider

$$\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)\cdot\left(\sum_{j=1}^{n}|\lambda_j|\right)$$

$$= \sum_{i=1}^{n}|\lambda_i|^2 + \sum_{i\neq j}|\lambda_i|\ |\lambda_j|$$

$$\therefore \sum_{i\neq j}|\lambda_i|\ |\lambda_j| = \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 - \sum_{i=1}^{n}|\lambda_i|^2 \quad \text{(Theorem 6.1.)}$$

Using AM-GM inequality for $n(n-1)$ non-negative terms shows that the arithmetic mean of these terms is atleast as large as their geometric mean and thus it follows that:

$$\frac{\sum_{i\neq j}|\lambda_i|\ |\lambda_j|}{n(n-1)} \geq \left[\prod_{i\neq j}|\lambda_i|\ |\lambda_j|\right]^{\frac{1}{n(n-1)}}$$

$$i.e., \sum_{i\neq j}|\lambda_i|\ |\lambda_j| \geq n(n-1)\left[\prod_{i\neq j}|\lambda_i|\ |\lambda_j|\right]^{\frac{1}{n(n-1)}}$$

Using Theorem.6.1 we get,

$$\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 - \sum_{i=1}^{n}|\lambda_i|^2 \geq n(n-1)\left[\prod_{i=1}^{n}|\lambda_i|^{2(n-1)}\right]^{\frac{1}{n(n-1)}}$$

$$\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 - 2N \geq n(n-1)\left[\prod_{i=1}^{n}|\lambda_i|\right]^{\frac{2}{n}}$$

$$\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 \geq 2N + n(n-1)\left[\prod_{i=1}^{n}|\lambda_i|\right]^{\frac{2}{n}}$$

$$\therefore \sum_{i=1}^{n}|\lambda_i| \geq \sqrt{2N + n(n-1)D^{\frac{2}{n}}} \qquad \text{(Theorem 6.2.)}$$

W.K.T

$$|\lambda_i| - \left|\frac{2m}{n}\right| \leq \left|\lambda_i - \frac{2m}{n}\right| \forall\ i$$

$$\sum_{i=1}^{n}|\lambda_i| - \sum_{i=1}^{n}\frac{2m}{n} \leq \sum_{i=1}^{n}\left|\lambda_i - \frac{2m}{n}\right|$$

$$\sum_{i=1}^{n} |\lambda_i| - 2m \leq LE_{pe}(G)$$

$$LE_{pe}(G) \geq \sum_{i=1}^{n} |\lambda_i| - 2m$$

$$\geq \sqrt{2N + n(n-1)D^{\frac{2}{n}}} - 2m \quad (\text{Theorem } 6.2.)$$

$$\therefore LE_{pe}(G) \geq \sqrt{2N + n(n-1)D^{\frac{2}{n}}} - 2m$$

7. References

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