Generalized Almost Matrix Summability of Fourier Series and Its Conjugate Series

Suresh Kumar Sahani[1], and Gyan Das[2*]

[1*]Department of Mathematics, MIT Campus, (T.U.), Janakpurdham, Nepal
[2]Department of Mathematics, Research Scholar, Dr. C.V. Raman University, Bilaspur (C.G.) Email: sureshkumarsahani35@gmail.com[1] and gyangendle@gmail.com[2] *

Abstract
The object of this paper is to give three specific theorems which implies authors [1] and [14] theorems on Nörlund summability of conjugate derived Fourier series and a generalization of a theorem of Iyengar on the harmonic summability of Fourier series respectively and deduce several known and new results from the theorems.

Introduction:

Let $\zeta(z)$ be a 2π-periodic and Lebesgue integral function of z in the interval $(-\pi,\pi)$. The Fourier series of the function $\zeta(z) \sim \frac{a_o}{2} + \sum_{\alpha=1}^{\infty}(a_\alpha \cos \alpha z + b_\alpha \sin \alpha z)$ (1)

The conjugate series of the Fourier series (1) is given by

$$\sum_{\alpha=1}^{\infty}\left(b_\alpha \cos\alpha z - a_\alpha \sin\alpha z\right). \tag{2}$$

We shall use the following notations:-

$$\xi(z)=\zeta(y+z)+\zeta(y-z)-2\zeta(y),$$

$$\chi(z)=\zeta(y+z)-\zeta(y-z),$$

$$\tau=\left[\frac{1}{z}\right], \tau \text{ is integral part of } \frac{1}{z}.$$

Let $\{U_l\}$ denote the sequence of l^{th} partial sum of the infinite series $\sum \nu_l$.

Let $\{p_l\}$ be sequence of constants, real or complex and $P_l = p_0 + p_1 + p_2 + ... + p_l$.

Definition 1 "(see [2])":

A bounded sequence $\{U_l\}$ is said to be almost convergent to a finite limit U if

$$\lim_{l\to\infty}\frac{1}{l+1}\sum_{g=r}^{l+r} U_g = U \tag{3}$$

uniformly with respect to r.

Definition 2 "(see [3])":

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Definition 2 "(see [3])":

Let $(\wedge) \equiv (\gamma_{\ell,g})\ (\ell = 0,1,2,...,\ g = 0,1,2,...,\gamma_{\ell,o} = 1)$ (4)

be infinite triangular matrix of real or complex numbers. We define sequence to sequence transformation

$$t_1 = \sum_{r=0}^{\ell} \gamma_{\ell,g} U_g$$

$$= \sum_{r=0}^{\ell} \Delta\gamma_{\ell,r} U_r ,$$

where $\Delta\gamma_{\ell,r} = \gamma_{\ell,r} - \gamma_{\ell,r+1}$.

It $t_\ell \to U$ as $\ell \to \infty$, then the infinite series $\sum \nu_\ell$ is summable by triangular means to the value U.

Definition 3 "(see [2])":

An infinite series $\sum \nu_\ell$ with the sequence of partial sums $\{U_\ell\}$ is said to be almost Riesz summable to U, provided

$$t_{\ell,g} = \frac{1}{P_\ell} \sum_{g=0}^{\ell} p_g U_{g,r} \to U,\ \text{as}\ \ell \to \infty \text{ uniformly with respect to } r, \tag{5}$$

where $U_{g,r} = \sum_{\delta=r}^{g+r} U_\delta$ (6)

and $\{p_1\}$ is a sequence of constants such that $P_\ell > 0$ and $P_\ell = p_0 + p_1 + p_2 + ... + p_\ell$.

Definition 4 "(see [3] and [6])":

An infinite series $\sum V_\ell$ with the sequence of partial sums $\{U_\ell\}$ is said to be almost summable to U,

if

$$T_{\ell,r} = \sum_{g=0}^{1} \Delta\gamma_{\ell,g} .\ U_{g,r} \to U \tag{7}$$

as $\ell \to \infty$ uniformly with respect to r, where $\{U_{g,r}\}$ is defined (6) and matrix summability mean t_ℓ is regular.

The regularity conditions for $\wedge$ summability method are easily mean to be (see [4], [6])

(i) $\lim_{\ell\to\infty} \gamma_{\ell,r} = 0$ for every fixed value of g,

(ii) $\sum_{g=0}^{1} |\gamma_{\ell,g}| \le M$ independent of ℓ and

(iii) $\lim_{\ell\to\infty} \sum_{g=0}^{\ell} \gamma_{\ell,g} = 1$.

We use the following notations:

$$N_{\ell,r}(z)=\sum_{g=0}^{\ell}\left[\gamma_{\ell,g}\left\{\sin\left(r+\frac{g+1}{2}\right)z\ \frac{\sin\frac{g+1}{2}z}{(g+1)\sin^2\frac{z}{2}}\right\}\right]$$

and

$$\overline{N}_{\ell,r}(z)=\sum_{g=0}^{\ell}\gamma_{\ell,g}\left\{\cos\left(r+\frac{g+1}{2}\right)z\ \frac{\sin\frac{g+1}{2}z}{(g+1)\sin^2\frac{z}{2}}\right\}.$$

2. Known theorems:

Minute study on Nörlund summability of Fourier series and its conjugate series was made by author in 1980 as mentioned in the research work (see [1]) and they proved the following theorems:

Theorem A:

Let $\gamma(z)$ and $\mu(z)$ are two functions.

If $\int_0^z|\xi(u)|du=O\left[\frac{\gamma\left(\frac{1}{z}\right).p_\tau}{\mu(P_\tau)}\right]$, as $z\to 0$.

and $\gamma(\ell)P_\ell=O[\mu(p_\ell)]$, as $\ell\to 0$, then the Fourier series (1) of $\zeta(z)$ at $z=y$ is summable (N,p_ℓ) to $\zeta(z)$ where $\{p_\ell\}$ is a real non-negative and non-increasing sequence such that $P_\ell\to\infty$, as $\ell\to\infty$.

Theorem B:

Let the sequence $\{p_n\}$ and the functions $\gamma(z)$ be the same in theorem A.

Then if,

$$\int_0^z|\chi(u)|du=O\left[\frac{\gamma\left(\frac{1}{z}\right).p_\tau}{\mu(P_\tau)}\right],\text{ as } z\to+0,$$

then the series (2) of (1) is summable $(\overline{N},p_\tau)$ to $\frac{1}{2\pi}\int_0^\pi\chi(u)\cot\frac{z}{2}dz$ of every point when this integral exists.

Author [17] has proved the following theorem on a generalization of a theorem of Iyengar on the harmonic summability of Fourier series by using positive decreasing sequence of constants.

Theorem C: If (N,P_ℓ) be a regular Nörlund method defined by a real, non-negative monotonic, non-increasing sequence of coefficients $\{P_\ell\}$ such that

$P_\ell = p_0 + p_1 + p_2 + ... + p_\ell \to 0$, as $\ell \to \infty$ and $\log \ell = O(P_\ell)$, as $\ell \to \infty$,

then if $\xi(z) = \int_0^z |\xi(u)| du$

$= \left[\frac{z}{p_\tau} \right]$, as $z \to +0$

then the series (1) is summable (N, P_ℓ) to $\zeta(z)$ at a point $z = y$.

3. Main theorems:

Recently published research works contend the view point more or less identical to the concept regarding absolute summability factors of infinite series and Fourier series "(see [5, 7, 8, 9-11])". Several phenomenological research works are sought to have very identical reflection on the concept of the matrix summability of Fourier series (see [12, 13, 15-22]). The motivating factor in such marvelous studies led me to the point that the degree of approximation of a function in more generalized as particular cases is necessary to be studied closely (see[22-35]). Our utmost effort goes on extending Tripathi and Singh theorems "(see [1])" by using following theorems:

Theorem 1:

Let $\{p_\ell\}$ is positively monotonic decreasing sequence of constant such that non-vanishing ℓ^{th} partial sum $P_\ell \to \infty$ as $\ell \to \infty$. Let $\gamma(z), \mu(z)$ and $\frac{z\gamma(z)}{\mu(z)}$ increase monotonically with z, if for $0 < \delta < 1$,

$$\gamma(\ell + r).P_{\ell+r} = O\left[\mu(P_{\ell+r})^\delta\right], \text{ as } \ell \to \infty \tag{8}$$

uniformly with respect to r,

$$\int_0^z |\xi(u)| du = O\left[\frac{\gamma\left(\frac{1}{z}\right) . p_\tau}{\left[\mu(P_\tau)\right]^\delta}\right], \text{ as } z \to +0$$

and

$$\int_{\frac{1}{\ell+r}}^{\left(\frac{1}{\ell+r}\right)^v} \frac{|\xi(u)|}{u} du = O(1), \text{ as } \ell \to \infty \tag{9}$$

for $0 < v < 1$ uniformly w.r.to r, then the Fourier series (1) of the function $\zeta(z)$ is summable $(\wedge)$ to the sum $\zeta(z)$ at a point $z = y$ in the given interval.

Theorem 2:

Let $\{p_\ell\}$ is positively monotonic decreasing sequence of constants such that its non-vanishing ℓ^{th}

Partial sum $P_\ell \to \infty$, as $\ell \to \infty$. Let $\gamma(z), \mu(z)$ and $\frac{z.\gamma(z)}{\mu(z)}$ increase monotonically with z and satisfying the following conditions:

$\gamma(\ell+r)P_{\ell+r} = O\left[\{\mu(P_{\ell+r})\}^{\delta}\right], 0<\delta<1$, as $\ell \to \infty$ uniformly with respect to r and if

$$\int_0^z |\chi(u)|du = O\left[\frac{\gamma\left(\frac{1}{z}\right)p_\tau}{\mu(P_\tau)^{\delta}}\right] \text{ as } z \to +0 \tag{10}$$

and

$$\int_{\frac{1}{\ell+r}}^{\frac{1}{(\ell+r)^{\nu}}} \frac{|\chi(u)|}{u}du = O(1), \tag{11}$$

as $\ell \to \infty$, $0<\nu<1$ uniformly w.r.to r, then the series (2) of (1) of the given function $\zeta(z)$ is almost summable $(\wedge)$ to the sum

$$\frac{1}{2\pi}\int_0^{\pi} \chi(u)\cot\frac{z}{2}dz \tag{12}$$

at every point $z = y$, whenever the integral exists.

The objects of this chapter is to generalize the Patti's [4] results in two ways for almost Nörlund summability of Fourier series in the following way from:

Theorem 3:

Let us suppose that $\{p_\ell\}$ is real monotonic non-decreasing and $\{\lambda_{\ell,g}\}$ be positive non-decreasing function of z such that $\left\{\frac{\lambda_{\ell,g}}{g+1}\right\}$ is non-increasing sequence of constants such that $P_\ell \to 0$ as $\ell \to \infty$ and

$$\sum_{g=0}^{\ell} \frac{\lambda_{\ell,g}}{g+1} p_{\ell-g} = O\left(\frac{P_\ell}{\ell}\right),$$

$$\lambda(\ell+r)\log(\ell+r) = O(P_{\ell+r}),$$

$$\int_0^z |\zeta(v)|dv = 0\left(\frac{p_\tau}{P_\tau}\right) \tag{13}$$

and $$\int_{\frac{1}{\ell+r+1}}^{\frac{1}{(\ell+r+1)^{\delta}}} \frac{|\xi(v)|}{v^2}dv = O(\ell), \text{ as } \ell \to \infty \tag{14}$$

Uniformly with respect to r then the Fourier series (1) of the function $\zeta(z)$ is summable (N, p_ℓ) at a point $z = y$ in the given interval.

To prove theorems, we follow series a of lemmas:

Lemma 2.1 "(see [2])":

If $0 < z < \frac{1}{\ell + r}$, then $N_{\ell,r}(z) = O(\ell + r)$.

Lemma 2.2 "(see [2])":

If $\frac{1}{\ell + r} < z < \ell < \pi$, then $N_{\ell,r}(z) = O\left(\frac{1}{z}\right)$.

Lemma 2.3 "(see [2])":

If $\frac{1}{\ell + r} < z < \pi$, then $\overline{N}_{\ell,r(z)} = O\left(\frac{1}{z}\right)$.

Lemma 2.4 "(see [2])":

For $\frac{1}{\ell + r} < z < \pi$,

$$N_{\ell,r}(z) = O\left(\frac{1}{\ell z^2}\right)$$

Proof of Lemma 1:

For $0 < z < \frac{1}{\ell + r}$,

$$\left|N_{\ell,r}(z)\right| = \left|\sum_{g=0}^{\ell} \gamma_{\ell,g} \frac{\sin\left(r + \frac{g+1}{2}\right)z \, \sin\frac{g+1}{2}z}{(g+1)\sin^2\frac{z}{2}}\right|$$

$$= \sum_{g=0}^{\ell} \gamma_{\ell,g} \left|\frac{(2r+g+1)\sin\frac{z}{2}\,(g+1)\sin\frac{z}{2}}{(g+1)\sin^2\frac{z}{2}}\right|$$

$$= \sum_{g=0}^{\ell} \gamma_{\ell,g}(2r+g+1)$$

$$= (2r+\ell+1)\sum_{g=0}^{\ell} \gamma_{\ell,g}$$

$$= (2r+2\ell)1$$

$$= O(\ell + r)$$

Proof of Lemma 2:

For $\frac{1}{\ell + r} < z < \ell < \pi$,

$$|N_{\ell,r}(z)| = \left| \sum_{g=0}^{\ell} \gamma_{\ell,g} \frac{\sin\left(r + \frac{g+1}{2}\right) z . \sin \frac{g+1}{2} z}{(g+1)\sin^2 \frac{z}{2}} \right|$$

$$\leq \sum_{g=0}^{\ell} \gamma_{\ell,g} \frac{\left|\sin\left(r + \frac{g+1}{2}\right) z\right| \left|(g+1)\sin \frac{z}{2}\right|}{(g+1)\sin^2 \frac{z}{2}}$$

$$\leq \sum_{g=0}^{\ell} \gamma_{\ell,g} \frac{1}{\sin \frac{z}{2}} \text{ (expanding } \sin\left(\frac{g+1}{2}\right)\frac{z}{2} \text{ in powers of } \sin\frac{z}{2}\text{)}$$

$$= \sum_{g=0}^{\ell} \gamma_{\ell,g} \left|\frac{\pi}{z}\right| \quad \left(\because \sin z \geq \frac{z}{\pi}\right)$$

$$= \left|\frac{\pi}{z}\right|$$

$$= O\left(\frac{1}{z}\right).$$

Proof of the lemma 3:

For $\frac{1}{\ell + r} < z < \pi$,

$$|\overline{N}_{\ell,r}(z)| = \left| \sum_{g=0}^{\ell} \gamma_{\ell,g} \frac{\cos\left[r + \frac{g+1}{2}\right] z \sin \frac{g+1}{2} z}{(g+1)\sin^2 \frac{z}{2}} \right|$$

$$= \sum_{g=0}^{\ell} \gamma_{\ell,g} \frac{\left|\cos\left[r + \frac{g+1}{2}\right] z\right| \left|(g+1)\sin \frac{z}{2}\right|}{(g+1)\sin^2 \frac{z}{2}}$$

$$=\sum_{g=0}^{\ell}\gamma_{\ell,g}\frac{\left|\cos\left[r+\frac{g+1}{2}\right]z\right|}{\sin\frac{z}{2}}$$

$$=\frac{\pi}{z}\sum_{g=0}^{\ell}\gamma_{\ell,g}$$

$$=\frac{\pi}{z}$$

$$=O\left(\frac{1}{z}\right).$$

$$N_{\ell,r}\left(z\right)=O\left(\frac{1}{\ell z^{2}}\right)$$

Proof of lemma 4:

For $\frac{1}{\ell+r}z<\pi$,

$$\left|N_{\ell,r}\left(z\right)\right|=\left|\sum_{g=0}^{\ell}\gamma_{\ell,g}\frac{\sin\left(r+\frac{g+1}{2}\right)z\,\sin\frac{g+1}{2}z}{\left(g+1\right)\sin^{2}\frac{z}{2}}\right|$$

$$=\sum_{g=0}^{\ell}\gamma_{\ell,g}\frac{\left|\sin\left(r+\frac{g+1}{2}\right)z\right|\left|\sin\left(g+1\right)\frac{z}{2}\right|}{\left(g+1\right)\sin^{2}\frac{z}{2}}$$

$$=\sum_{g=0}^{\ell}\frac{\gamma_{\ell,g}}{g+1}.\frac{\pi.\pi}{z.z}$$

$$=\frac{\pi^{2}}{z^{2}}\sum_{g=0}^{\ell}\frac{\gamma_{\ell,g}}{g+1}$$

$$=\frac{\pi^{2}}{z^{2}}.\frac{1}{\ell}\ \text{(by Jordon's lemma)}$$

$$=O\left(\frac{1}{\ell z^{2}}\right).$$

Proof of the theorem 1:

The ℓ^{th} partial sum $U_\ell(y)$ of Fourier series (1) of the function $\zeta(z)$ at a point $z = y$ in the interval $(-\pi, \pi)$ is given by

$$U_\ell(y)-\zeta(y)=\frac{1}{2\pi}\int_0^\pi \xi(z).\frac{\sin\left(\ell+\frac{1}{2}\right)z}{\sin\frac{z}{2}}dz. \tag{15}$$

Therefore, following (9), we have almost matrix transformation $t_{\ell,r}$ of $\{U_\ell(y)\}$ of partial sum of (1) given by

$$t_{\ell,r}-\zeta(y)=\sum_{g=0}^{\ell}\gamma_{\ell,g}\{U_{g,r}-\zeta(y)\}$$

$$=\sum_{g=0}^{\ell}\gamma_{1,g}\left[\frac{1}{g+1}\sum_{r=1}^{g+r}U_r-\zeta(y)\right]$$

$$=\sum_{g=0}^{\ell}\gamma_{1,g}\left[\frac{1}{g+1}\sum_{r=1}^{g+r}U_r-\zeta(y)\right]$$

$$=\sum_{g=0}^{\ell}\gamma_{1,g}\frac{1}{g+1}\left[\sum_{r=1}^{g+r}\left\{\frac{1}{2\pi}\int_0^\pi \xi(z)\frac{\sin\left(r+\frac{1}{2}\right)z}{\sin\frac{z}{2}}dz\right\}\right]$$

$$=\sum_{g=0}^{\ell}\gamma_{1,g}\left[\frac{1}{2\pi(g+1)}\int_0^\pi\frac{\xi(z)}{\sin\frac{z}{2}}\left\{\sum_{r=1}^{g+r}\sin\left(r+\frac{1}{2}\right)z\right\}\right]dz$$

$$=\sum_{g=0}^{\ell}\gamma_{1,g}\frac{1}{2\pi}\int_0^\pi\xi(z)\left(\sin\left(r+\frac{g+1}{2}\right)z\right)\frac{\sin\left(\frac{g+1}{2}\right)z}{(g+1)\sin^2\frac{z}{2}}dz$$

$$=O\left[\int_0^\pi\xi(z)\left\{\sum_{g=0}^{\ell}\gamma_{1,g}\left(\sin\left(r+\frac{g+1}{2}\right)z\right)\frac{\sin\left(\frac{g+1}{2}\right)z}{(g+1)\sin^2\frac{z}{2}}dz\right\}\right]$$

$$=O\left[\int_0^\pi\xi(z)N_{\ell,r}(z)dz\right]$$

$$\therefore t_{\ell,r}-\zeta(y)=O\left[\int_0^\pi\xi(z)N_{\ell,r}(z)dz\right] \tag{16}$$

Now, we have to show that

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$$\int_0^\pi \xi(z) N_{\ell,r}(z) dz = O(1) \text{ as } \ell \to \infty \tag{17}$$

uniformly with respect to r.

For $0 < \nu < 1$,

$$I = \int_0^\pi \xi(z).N_{\ell,r}(z) dz$$

$$\left(\int_0^{\frac{1}{\ell+r}} + \int_{\frac{1}{\ell+r}}^{\frac{1}{(\ell+r)^\nu}} + \int_{\frac{1}{(\ell+r)^\nu}}^{\pi} \right) \xi(z).N_{\ell,r} z = (z) dz$$

$$= I_1 + I_2 + I_3 \text{ (say).} \tag{18}$$

For I_1,

$$|I_1| = O\left[\int_0^{\frac{1}{\ell+r}} |\xi(z)| |N_{\ell,r}(z)| dz \right]$$

$$= O(\ell + r) \int_0^{\frac{1}{\ell+r}} |\xi(z)| dz \text{, (using lemma 2.1)}$$

$$= O(\ell + r) o \left(\frac{\gamma(\ell + r).p_{\ell+r}}{\{\mu(P_{\ell+r})\}^\delta} \right)$$

$$= O(1) \text{, as } \ell \to \infty \text{ (using (8))} \tag{19}$$

uniformly with respect to r.

Also, for I_2,

$$|I_2| \le \int_{\frac{1}{\ell+r}}^{\frac{1}{(\ell+r)^\nu}} |\xi(z)| |N_{\ell,r}(z)| dz$$

$$= O\left[\int_{\frac{1}{\ell+r}}^{\frac{1}{(\ell+r)^\nu}} \frac{|\xi(z)|}{z} dz \right]$$

$$= O(1) \text{, as } \ell \to \infty \text{ (using (11))} \tag{20}$$

uniformly with respect to r.

Lastly, for I_3,

$$I_3 = \int_{\frac{1}{(\ell+r)^\nu}}^{\pi} \xi(z) N_{\ell,r}(z) dz$$

$$=\int_{\frac{1}{(\ell+r)^{v}}}^{\pi}\xi(z)\left\{\sum_{g=0}^{\ell}\gamma_{1,g}\sin\left(r+\frac{g+1}{2}\right)z\,\frac{\sin\left(g+\frac{1}{2}\right)z}{(g+1)\sin^{2}\frac{z}{2}}\right\}dz$$

$$=\int_{\frac{1}{(\ell+r)^{v}}}^{\pi}\xi(z)\left\{\sum_{g=0}^{\ell}\gamma_{1,g}\frac{\cos rz-\cos(r+g+1)z}{2(g+1)\sin^{2}\frac{z}{2}}\right\}dz$$

$$=\int_{\frac{1}{(\ell+r)^{v}}}^{\pi}\xi(z)\left\{\sum_{g=0}^{\ell}\gamma_{1,g}\frac{\cos rz}{2(g+1).\sin^{2}\frac{z}{2}}\right\}dz-\int_{\frac{1}{(\ell+r)^{v}}}^{\pi}\xi(z)\left\{\sum_{g=0}^{\ell}\gamma_{1,g}\frac{\cos(r+g+1)z}{2(g+1)\sin^{2}\frac{z}{2}}\right\}dz$$

$$=I_{3,1}-I_{3,2}. \tag{21}$$

Again, for $I_{3,1}$.

$$|I_{3,1}|=O\left[\sum_{g=0}^{\ell}\gamma_{1,g}\right]\frac{1}{2\sin^{2}\left(\frac{1}{1+r}\right)^{v}}\left|\int_{\frac{1}{(1+r)^{v}}}^{\beta}\xi(z)\cos rz\,dz\right|$$

$$=O(1),\text{ as }\ell\to\infty \tag{22}$$

uniformly with respect to r.

For $\frac{1}{(1+r)^{v}}<\beta\leq\eta<\pi$,

$$|I_{3,2}|=O\left(\sum_{g=0}^{\ell}\gamma_{1,g}\right)\frac{1}{2\sin^{2}\left(\frac{1}{1+r}\right)^{v}}\left|\int_{\frac{1}{(\ell+r)^{v}}}^{\beta}\xi(z)\cos(r+g+1)z\,dz\right|$$

$$=O\left(\sum_{g=0}^{\ell}\gamma_{1,g}\right)\frac{1}{2\sin^{2}\left(\frac{1}{(1+r)^{v}}\right)}\int_{\frac{1}{(1+r)^{v}}}^{\pi}\xi(z)dz$$

$$=O(1),\text{ as }\ell\to\infty \tag{23}$$

uniformly with respect to r.

Hence, from (22) and (23), we have

$$|I_3| = O(1) \text{ as } \ell \to \infty \tag{24}$$

uniformly with respect to r

Now, combining (18), (19), (20), (21), (22), (23) and (24), we get the required results (17).

Proof of the theorem 2:

Let $\overline{U}_\ell(y)$ denote ℓ^{th} partial sum of (2) or (1) at $z = y$ in $(-\pi, \pi)$.

Then

$$\overline{U}_\ell(y) = \frac{1}{2\pi}\int_0^\pi \chi(z)\left[\cos\frac{z}{2} - \cos\left(\ell+\frac{1}{2}\right)z\right]\frac{1}{\sin\frac{z}{2}}dz$$

$$\Rightarrow \qquad \overline{U}_\ell(y) - \frac{1}{2\pi}\int_0^\pi \chi(z)\cot\frac{z}{2}dz$$

$$= -\frac{1}{2\pi}\int_0^\pi \chi(z)\frac{\cos\left(\ell+\frac{1}{2}\right)}{\sin\frac{z}{2}}dz$$

$$= \frac{1}{g+1}\sum_{r=1}^{g+r}\overline{U}_\ell(y) - \frac{1}{2\pi}\int_0^\pi \chi(z)\cos\frac{z}{2}dz$$

$$= \frac{-1}{2\pi}\int_0^\pi \frac{\chi(z)}{\sin\frac{z}{2}}\sum_{r=1}^{g+r}\cos\left(r+\frac{1}{2}\right)z\,dz$$

$$= \frac{-1}{2\pi}\int_0^\pi \frac{\chi(z)\cos\left(r+\frac{g+1}{2}\right)z}{(g+1)\sin^2\frac{z}{2}}\sin\left(\frac{g+1}{2}\right)z.$$

Therefore, the following (7), the almost matrix transformation $t_{\ell,r}$ of the sequence $\left\{\overline{U}_\ell(y)\right\}$ of the partial sum of the series (2) will be given by

$$\bar{t}_{\ell,r} - \frac{1}{2\pi}\int_0^\pi \chi(z)\cot\frac{z}{2}dz = \sum_{g=0}^{\ell}\gamma_{g,r}\overline{U}_{g,r} - \frac{1}{2\pi}\int_0^\pi \chi(z)\cos\frac{z}{2}dz$$

$$= \sum_{g=0}^{\ell}\gamma_{1,g}\left[\overline{U}_{g,r} - \frac{1}{2\pi}\int_0^\pi \chi(z)\cot\frac{z}{2}dz\right]$$

$$= -\frac{1}{2\pi}\int_0^\pi \chi(z)\sum_{g=0}^{r}\gamma_{1,g}\frac{\cos\left(r+\frac{g+1}{2}\right)z\sin\left(\frac{g+1}{2}\right)z}{(g+1)\sin^2\frac{z}{2}}dz$$

$$= O\left[\int_0^{\pi} \chi(z)\overline{N}_{\ell,r}(z)dz\right] \tag{25}$$

Now, we have to show that

$$J = \int_0^{\pi} \chi(z)\overline{N}_{\ell,r}(z)dz$$

$$= O(1) \text{ as to } \infty \tag{26}$$

uniformly with respect to r.

For $0 < \delta < 1$,

$$J = \int_0^{\pi} \chi(z)\overline{N}_{\ell,r}(z)dz$$

$$= \left(\int_0^{\frac{1}{\ell+r}} + \int_{\frac{1}{\ell+r}}^{\frac{1}{(\ell+r)^v}} + \int_{\frac{1}{(\ell+r)^v}}^{\pi}\right)\chi(z)\overline{N}_{\ell,r}(z)dz$$

$$= J_1 + J_2 + J_3. \tag{27}$$

For J_1,

$$|J_1| = O\left[\frac{1}{\int_0^{\ell+r}}|\chi(z)||\overline{N}_{\ell,r}(z)|dz\right]$$

$$= O(\ell+r)\frac{1}{\int_0^{\ell+r}}|\chi(z)|dz, \text{ (using lemma 2.3)}$$

$$= O(\ell+r).O\left[\frac{\gamma_{\ell,r}p_{\ell+r}}{\left[\mu(P_{\ell,r})\right]^{\delta}}\right]$$

$$= O(1), \text{ as } \ell \to \infty \tag{28}$$

uniformly with respect to r.

For J_2,

$$|J_2| = O\left[\int_{\frac{1}{\ell+r}}^{\frac{1}{(\ell+r)^v}}|\chi(z)||\overline{N}_{\ell,r}(z)|\right]dz$$

$$= O\left[\int_{\frac{1}{\ell+r}}^{\frac{1}{(\ell+r)^v}}\frac{|\chi(z)|}{z}dz\right], \text{ (using lemma 2.1)}$$

$$= O(1), \text{ as } \ell \to \infty \tag{29}$$

uniformly with respect to r.

Again, for J_3

$$J_3 = \int_{\frac{1}{(\ell+r)^v}}^{\pi} \sum_{g=0}^{\ell} \gamma_{\ell,g} \left\{ \frac{\cos\left(r+\frac{g+1}{2}\right)z - \sin\left(\frac{g+1}{2}\right)z}{(g+1)\sin^2\frac{z}{2}} \right\} dz$$

$$= \int_{\frac{1}{(\ell+r)^v}}^{\pi} \chi(z)\left(\sum_{g=0}^{\ell}\gamma_{\ell,g}\frac{\sin\left(r+\frac{g+1}{2}\right)z}{(g+1)\sin^2\frac{z}{2}}\right)dz - \int_{\frac{1}{(\ell+r)^v}}^{\pi}\chi(z)\sum_{g=0}^{\ell}\gamma_{\ell,g}\frac{\sin rz}{2(g+1)\sin^2\frac{z}{2}}dz$$

$$= J_{3,1} - J_{3,2}. \tag{30}$$

Applying second mean value theorem for $J_{3,2}$, we have

$$\left|J_{3,2}\right| = O\left(\sum_{g=0}^{\ell}\gamma_{\ell,g}\right)\frac{1}{2\sin^2\frac{1}{(\ell+r)^v}}\int_{\frac{1}{(\ell+r)^v}}^{\beta}\left|\xi(z)\right|\sin rz\,dz$$

$$= O(1), \text{ as } \ell \to \infty \tag{31}$$

uniformly with respect to r, where

$$\frac{1}{(\ell+r)^v} < \beta \le \eta < \pi.$$

Again, for $J_{3,1}$,

$$\left|J_{3,1}\right| = O\left(\sum_{g=0}^{\ell}\gamma_{\ell,g}\right)\frac{1}{2\sin^2\frac{1}{(\ell+r)^v}}\left|\int_{\frac{1}{(\ell+r)^v}}^{\pi}\xi(z)\sin(r+g+1)z\,dz\right|$$

$$= O\left(\sum_{g=0}^{\ell}\gamma_{\ell,g}\frac{1}{2\sin^2\left(\frac{1}{(\ell+r)^v}\right)}\right)\int_{\frac{1}{(\ell+r)^v}}^{\pi}\left|\xi(z)\right|dz$$

$$= O(1), \text{ as } \ell \to \infty \tag{32}$$

uniformly with respect to r

From (30), (31) and (32), we obtain

$$\left|J_3\right| = O(1), \text{ as } \ell \to \infty \tag{33}$$

uniformly with respect to r.

Now, combining (27), (28), (29) and (33), we get (26)

i.e. $J=\int_0^{\pi}\chi(z)\overline{N}_{\ell,r}(z)dz$

$=O(1)$, as $\ell\to\infty$ uniformly with respect to r.

Proof of the theorem 3: The ℓ^{th} partial sum $u_\ell(y)$ of Fourier series (1) of the function $\zeta(z)$ at a point $z=y$ in the interval $(-\pi,\pi)$ is given by

$$v_\ell(y)-\zeta(y)=\frac{1}{2\pi}\int_0^{\pi}\xi(z)\frac{\sin\left(\ell+\frac{1}{2}\right)z}{\sin\frac{z}{2}}dz \tag{34}$$

Now,

$$t_{\ell,r}-\zeta(y)=\sum_{g=0}^{\ell}\lambda_{\ell,g}\frac{1}{g+1}\left[\sum_{r=1}^{g+r}\left\{\frac{1}{2\pi}\int_0^{\pi}\frac{\xi(z)\sin\left(r+\frac{1}{2}\right)z}{\sin\frac{z}{2}}dz\right\}\right]$$

$$=\sum_{g=0}^{\ell}\lambda_{\ell,g}\left[\frac{1}{2\pi(g+1)}\int_0^{\pi}\frac{\xi(z)}{\sin\frac{z}{2}}\left\{\sum_{r=1}^{g+r}\sin\left(r+\frac{1}{2}\right)z\right\}dz\right]$$

$$=\sum_{g=0}^{\ell}\lambda_{\ell,g}\frac{1}{2\pi}\int_0^{\pi}\xi(z)\sin\left(r+\frac{1}{2}\right)z.\frac{\sin\frac{g+1}{2}z}{(g+1)\sin^2\frac{z}{2}}dz$$

$$=O\left[\int_0^{\pi}\xi(z)\left\{\sum_{g=0}^{\ell}\lambda_{\ell,g}\sin\left(r+\frac{g+1}{2}\right)z\frac{\sin\left(\frac{g+1}{2}\right)z}{(g+1)\sin^2\frac{z}{2}}dz\right\}\right]$$

$$=O\left[\int_0^{\pi}\xi(z)N_{\ell,r}(z)dz\right] \tag{35}$$

we have to show that $\int_0^{\pi}\xi(z)N_{\ell,r}(z)dz=O(1)$ as $\ell\to\infty$ (36)

uniformly with respect to r.

For $0<\delta<1$, we have

$$\int_0^{\pi} \xi(z) N_{\ell,r}(z)\,dz$$

$$= \left(\int_0^{\frac{1}{\ell+r}} + \int_{\frac{1}{\ell+r}}^{\frac{1}{(\ell+r)^{\delta}}} + \int_{\frac{1}{(\ell+r)^{\delta}}}^{\pi} \right) \xi(z) N_{\ell,r}(z)\,dz$$

$$= K_1 + K_2 + K_3 \qquad \text{say} \tag{37}$$

For K_1,

$$|K_1| = O\left[\int_0^{\frac{1}{\ell+r}} |\xi(z)| |N_{\ell,r}(z)| dz \right]$$

$$= O(\ell + r)\left[\int_0^{\frac{1}{\ell+r}} |\xi(z)\,dz| \right] \text{ (using lemma 2.1)}$$

$$= O(\ell + r).O\left(N_{(\ell+r)} \frac{P_{\ell+r}}{P_{\ell+r}} \right)$$

$$= O(1) \text{ as } \ell \to \infty \tag{38}$$

For K_2,

$$|K_2| = O\left[\int_{\frac{1}{\ell+r}}^{\frac{1}{(\ell+r)^{\delta}}} |\xi(z)| |N_{\ell,r}(z)| dz \right]$$

$$\le \int_{\frac{1}{\ell+r}}^{\frac{1}{(\ell+r)^{\delta}}} |\xi(z)| |N_{\ell,r}(z)| dz$$

$$= O\left[\int_{\frac{1}{\ell+r}}^{\frac{1}{(\ell+r)^{\delta}}} \frac{|\xi(z)|}{\ell^2 z^2} dz \right] \text{ (using lemma 2.4)}$$

$$= O(1) \text{ as } \ell \to \infty \text{ (using (34))} \tag{39}$$

Lastly, we have

$$K_3 = \int_{\frac{1}{(\ell+r)^{\delta}}}^{\pi} \xi(z) N_{\ell,r}(z)\,dz$$

$$= \int_{\frac{1}{(\ell+r)^{\delta}}}^{\pi} \xi(z) \left\{ \sum_{g=0}^{\ell} \lambda_{\ell,g} \sin\left(r + \frac{g+1}{2} \right) z. \frac{\sin\left(g + \frac{1}{2} \right) z}{(g+1)\sin^2 \frac{z}{2}} \right\} dz$$

$$=\int_{\frac{1}{(\ell+r)^{\delta}}}^{\pi}\xi(z)\left\{\sum_{g=0}^{\ell}\lambda_{\ell,g}\frac{\cos rz-\cos(r+g+1)z}{2(g+1)\sin^{2}\frac{z}{2}}\right\}dz$$

$$=\int_{\frac{1}{(\ell+r)^{\delta}}}^{\pi}\xi(z)\left\{\sum_{g=0}^{\ell}\lambda_{\ell,g}\frac{\cos rz}{z(g+1)\sin^{2}\frac{z}{2}}\right\}dz-\int_{\frac{1}{(\ell+r)}}^{\pi}\xi(z)\left\{\sum_{g=0}^{\ell}\lambda_{\ell,g}\frac{\cos(r+g+1)}{2(g+1)\sin^{2}\frac{z}{2}}\right\}dz$$

$$=I_{3,1}-I_{3,2}. \tag{40}$$

again for $I_{3,1}$,

$$|I_{3,1}|=O\left[\sum_{g=0}^{\ell}\lambda_{\ell,g}\right]\frac{1}{2\sin^{2}\left(\frac{1}{\ell+r}\right)^{\delta}}\cdot\left|\int_{\frac{1}{(\ell+r)^{\delta}}}^{\beta}\xi(z)\cos rz dz\right|$$

$=O(1)$, as $\ell\to\infty$ uniformly with respectively r. (41)

For $\frac{1}{(\ell+r)^{\delta}}<\beta\leq\eta<\pi$,

$$|I_{3,2}|=O\left(\sum_{g=0}^{\ell}\lambda_{\ell,g}\right)\frac{1}{2\sin^{2}\left(\frac{1}{(\ell+r)^{\delta}}\right)}\left|\int_{\frac{1}{(\ell+r)^{\delta}}}^{\beta}\xi(z)\cos(r+g+1)z\,dz\right|$$

$$=O\left(\sum_{g=0}^{\ell}\lambda_{\ell,g}\right)\frac{1}{2\sin^{2}\frac{1}{(\ell+r)^{\delta}}}\int_{\frac{1}{(\ell+r)^{\delta}}}^{\pi}\xi(z)dz$$

$=O(1)$, as $\ell\to\infty$ (42)

uniformly w.r.t. to r.

Combining (37), (38), (39), (40), (41) and (42), we get (36)

Particular cases:

(i) $(C,1)$ means when $\lambda_{\ell,g}=\frac{1}{\ell+1}$

(ii) Harmonic means when

$$\lambda_{\ell,g}=\frac{1}{(\ell-g+1)\log\ell}$$

(iii) (C,β) means when

$$\lambda_{\ell,g}=\frac{\begin{pmatrix}\ell-g+\beta+1\\ \beta-1\end{pmatrix}}{\begin{pmatrix}\ell+\beta\\ \beta\end{pmatrix}},\ (0\le\beta\le1)$$

(iv) (H,p) means, when

$$\lambda_{\ell,g}=\frac{1}{\log^{p-1}(g+1)}\prod_{r=0}^{p-1}\log^{r}(g+1)$$

(v) Nörlund means, when

$\lambda_{\ell,g}=\dfrac{p_{\ell-g}}{p_\ell}$, where $\{p_\ell\}$ is non-negative monotonic decreasing sequence such that

$p_\ell=\sum_{g=0}^{\ell}p_g\to\infty$, as $\ell\to\infty$.

(vi) Riesz means $(\overline{N},p_\ell)$, when

$\lambda_{\ell,g}=\dfrac{p_g}{P_g}$ where $\{p_\ell\}$ is a non-negative, increasing sequence such that

$P_\ell=\sum_{g=0}^{\ell}p_g\to\infty$, as $\ell\to\infty$.

(vii) Generalized Nörlund means (N,p,q), when

$\lambda_{\ell,g}=\dfrac{p_{\ell-g}q_g}{R_\ell}$ where $\{p_\ell\}$ is a positive monotonic decreasing sequence and $\{q_g\}$ is positive monotonic increasing sequence such that $R_\ell=\sum_{g=0}^{\ell}p_{\ell-g}q_g\to\infty$, as $\ell\to\infty$.

An application:

Let $\zeta(z)$ be a $2\pi-$periodic and Lebesgue integrable function of z in the interval $(-\pi,\pi)$. Then the trigonometric Fourier series of the function $\zeta(z)$ is given by

$$\zeta(z)\sim\frac{a_o}{2}+\sum_{\alpha=1}^{\infty}(a_\alpha\cos\alpha z+b_\alpha\sin\alpha z)$$

where $a_0=\dfrac{1}{\pi}\int_{-\pi}^{\pi}\zeta(z)\,dz$,

$$a_\alpha = \frac{1}{\pi}\int_{-\pi}^{\pi} \zeta(z)\cos\alpha z\, dz\,,$$

$$b_\alpha = \frac{1}{\pi}\int_{-\pi}^{\pi} \zeta(z)\sin\alpha z\, dz\,,$$

and

$$\xi(z) = \frac{\zeta(y+z)+\zeta(y-z)}{2}, \tag{43}$$

and

$$\xi_\alpha(z) = \frac{\alpha}{z^\alpha}\int_0^z (z-\alpha)^{\alpha-1}\xi(v)\,dv\,,\ (\alpha > 0) \tag{44}$$

If $\alpha = 1$ then

$$\xi_1(z) = \int_0^z \xi(v)\,dv \tag{45}$$

It is clear that if $\xi(z) \in BV(0,\pi)$ then $z_\alpha(y) = O(1)$ where $z_\alpha(y)$ is $(C,1)$ mean of the sequence $\langle \alpha M_\alpha(z) \rangle$ (see [23]). By using this fact, we obtain the specific results of Özarslan [3], Yildiz [7] and Bor [24].

Conclusion

If $\{p_\ell\}$ is positively monotonic decreasing sequence such that $P_\ell \to \infty, \ell \to \infty$ and $\gamma(z), \mu(z), \frac{z\gamma(z)}{\mu(z)}$ increase with z satisfying the Theorem 'A' and 'B', along with the lemmas (2.1) to (2.4) are satisfied, then the series (1) and (2) of (1) are respectively summable $(\wedge)$ to the sum $\zeta(z)$ and $\frac{1}{2\pi}\int_0^\pi \chi(z)\cot\frac{z}{2}dz$, under the conditions (8), (9) and (10). Thus our results generalizes the results of [1].

Similarly, if $\{p_\ell\}$ is real monotonic non-decreasing and $\{\lambda_{\ell,g}\}$ be positive non-decreasing of z such that $\left\{\frac{\lambda_{\ell,g}}{g+1}\right\}$ is non-increasing satisfying the theorem (c) along with the lemmas (2.1) to (2.4) are satisfied, then the series (1) is summable to the sum under the conditions (13) and (14).

Thus, our result generalizes the result of Patti [14].

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