A Study on Minimum Efficient Dominating Extended Energy of Some Graphs

Yogalakshmi. Y[1], Sreeja. S[2], Mary. U[3]

[1,2]Assistant Professor, Department of Mathematics, PSGR Krishnammal College for Women, Coimbatore, Tamilnadu, India.

[3]Assoicate Professor and Head(Former), PG and Research Department of Mathematics, Nirmala College for women, Coimbatore, Tamilnadu, India.

Abstract:- By the inspiration of the work extended adjacency matrix[11], the concept of minimum efficient dominating extended matrix is established. In this manuscript, minimum efficient dominating extended matrix $\left(A_{ex}^{eD}\right)$ is calculated using the degree of the nodes and hence energy is obtained by the sum of the latent values of the matrix $\left(A_{ex}^{eD}\right)$. The minimum efficient dominating extended energy $\left(E_{ex}^{eD}\right)$ for complete graph, crown graph, friendship graph, cycle graph and path graph are estimated. Also the assets and restrictions of minimum efficient dominating extended energy are also calculated.

Keywords: Determinantal equation, Minimum efficient dominating set, Minimum efficient dominating extended matrix, Minimum efficient dominatind extended energy, Latent values.

1. Introduction

The concept of energy of a graph was introduced by I. Gutman [3] in the year 1978. Let *G* be a graph with *p* nodes and *q* lines. Let $A = \left(x_{ij}\right)$ be the adjacency matrix of the graph. The latent values $\mu_1, \mu_2, \ldots, \mu_p$ of *A*, assumed in non increasing order, are the latent values of the graph *G*. As *A* is real symmetric, the latent values of *G* are real with sum equal to zero. The energy *E(G)* of *G* is defined to be the sum of the absolute values of the latent values of *G*.

i.e., $$E(G) = \sum_{i=1}^{p} \left|\mu_i\right|$$

1.1 Dominating Set:

Let G be a simple graph of order *p* and size *q*. A subset D of V is called a **dominating set** of G if every node of V – D is adjacent to some node in D.[7]

1.2 Extended Energy:

Extended matrix of G is the $p \times p$ matrix defined by $A_{ex}(G) = \left(x_{ij}\right)$ where

$$x_{ij} = \begin{cases} \dfrac{1}{2}\left(\dfrac{d_i}{d_j} + \dfrac{d_j}{d_i}\right) & if\ a_i a_j \in E(G) \\ 0 & Otherwise \end{cases}$$

The latent values of $A_{ex}(G)$ are called extended latent values. The **extended energy** of G is described as $E_{ex} = \sum_{i=1}^{p} \left|\mu_i\right|$ where $\mu_1, \mu_2, \ldots, \mu_p$ are the latent values of $A_{ex}(G)$.[11]

1.3 Minimum Efficient Dominating Extended Energy:

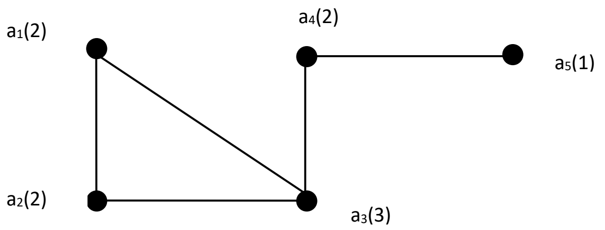
Let ψ be a simple graph of order p and size q. A node set D in ψ is an minimum efficient dominating set (EDS) for ψ if for every node $a \in N - D$, there is exactly one $d \in D$ dominating a. The minimum efficient dominating extended matrix of ψ is a $p \times p$ matrix $A_{ex}^{eD}(\psi) = (x_{ij})$ where,

$$x_{ij} = \begin{cases} \frac{1}{2}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right) & if\ a_i a_j \in E(\psi) \\ 1 & if\ i = j \,\&\, a_i \in EDS \\ 0 & Otherwise \end{cases}$$

The minimum efficient dominating extended latent values of the graph ψ are the latent values of $A_{ex}^{eD}(\psi)$. The minimum efficient dominating extended energy of ψ is defined as $E_{ex}^{eD}(\psi) = \sum_{i=1}^{p} |\mu_i|$.

2. Illustration

Let ψ be the graph and $N = \{a_1, a_2, a_3, a_4, a_5\}$ be its node set. Its minimum efficient dominating set $EDS = \{a_1, a_5\}$.



$$A_{ex}^{eD}(\psi) = \begin{bmatrix} 1 & \frac{1}{2}\left(\frac{2}{2}+\frac{2}{2}\right) & \frac{1}{2}\left(\frac{2}{3}+\frac{3}{2}\right) & 0 & 0 \\ \frac{1}{2}\left(\frac{2}{2}+\frac{2}{2}\right) & 0 & \frac{1}{2}\left(\frac{2}{3}+\frac{3}{2}\right) & 0 & 0 \\ \frac{1}{2}\left(\frac{2}{3}+\frac{3}{2}\right) & \frac{1}{2}\left(\frac{2}{3}+\frac{3}{2}\right) & 0 & \frac{1}{2}\left(\frac{2}{3}+\frac{3}{2}\right) & 0 \\ 0 & 0 & \frac{1}{2}\left(\frac{2}{3}+\frac{3}{2}\right) & 0 & \frac{1}{2}\left(\frac{2}{1}+\frac{1}{2}\right) \\ 0 & 0 & 0 & \frac{1}{2}\left(\frac{2}{1}+\frac{1}{2}\right) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & \frac{13}{12} & 0 & 0 \\ 1 & 0 & \frac{13}{12} & 0 & 0 \\ \frac{13}{12} & \frac{13}{12} & 0 & \frac{13}{12} & 0 \\ 0 & 0 & \frac{13}{12} & 0 & \frac{5}{4} \\ 0 & 0 & 0 & \frac{5}{4} & 1 \end{bmatrix}$$

The minimum efficient dominating extended latent values are

$$\mu_1 = -1.7555 \qquad \mu_2 = -0.6519 \qquad \mu_3 = -0.1174$$

$$\mu_4 = 1.8099 \qquad \mu_5 = 2.7149$$

The minimum efficient dominating extended energy, $E_{ex}^{eD}(\psi) = \sum_{i=1}^{p} |\mu_i| = 7.0496$

3. Assets of Miniumum Efficient Dominating Extended Latent Values

Theorem 3.1:

Let ψ = (*p*, *q*) be a simple graph. Let $\mu_1, \mu_2, \ldots, \mu_p$ be the latent values of $A_{ex}^{eD}(\psi)$.

Then,

(i) $\sum_{i=1}^{p} \mu_i = |EDS|$ (ii) $\sum_{i=1}^{p} \mu_i^2 = |EDS| + \frac{1}{2} \sum_{i<j} \left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right)^2$

Proof:

(i) The totality of the latent values of $A_{ex}^{eD}(\psi)$ is the trace of $A_{ex}^{eD}(\psi)$

$$\therefore \sum_{i=1}^{p} \mu_i = \sum_{i=1}^{p} x_{ii} = |EDS|$$

(ii) The totality of squares of the latent values of $A_{ex}^{eD}(\psi)$ is the trace of $(A_{ex}^{eD}(\psi))^2$

$$\therefore \sum_{i=1}^{p} \mu_i^2 = \sum_{i=1}^{p} \sum_{j=1}^{p} x_{ij} x_{ji}$$

$$= \sum_{i=1}^{p} (x_{ii})^2 + \sum_{i \neq j} x_{ij} x_{ji}$$

$$= \sum_{i=1}^{p} (x_{ii})^2 + 2 \sum_{i<j} (x_{ij})^2$$

$$= |EDS| + 2\sum_{i<j}\left[\frac{1}{2}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)\right]^2$$

$$= |EDS| + \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)^2$$

Theorem 3.2:

Let ψ = (*p*, *q*) be a simple graph. Let EDS be the minimum efficient dominating set of ψ. Let $g_x(\psi,\mu_x) = y_0\mu_x^p + y_1\mu_x^{p-1} + y_2\mu_x^{p-2} + \ldots + y_p$ be the determinantal equation of ψ. Then,

(i) $y_0 = 1$

(ii) $y_1 = -|EDS|$

(iii) $y_2 = \binom{|EDS|}{2} - \sum_{i<j}\left[\frac{1}{2}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)\right]^2$

Proof:

(i) From the definition of $g_x(\psi,\mu_x) = \det(\mu_x I - A_{ex}^{eD}(\psi))$, follows that $y_0 = 1$.

(ii) Since the sum of diagonal elements of $A_{ex}^{eD}(\psi)$ is equal to $|EDS|$. Thus,

$$(-1)^1 y_1 = |EDS|$$

$$y_1 = -|EDS|$$

(iii) The sum of the determinants of all 2×2 principal sub matrices of $A_{ex}^{eD}(\psi) = (-1)^2 y_2$

$$\therefore y_2 = \sum_{1\le i\le j\le p}\begin{vmatrix} x_{ii} & x_{ij} \\ x_{ji} & x_{jj}\end{vmatrix}$$

$$= \sum_{1\le i\le j\le p}(x_{ii}x_{jj} - x_{ij}x_{ji})$$

$$= \sum_{1\le i\le j\le p}x_{ii}x_{jj} - \sum_{1\le i\le j\le p}x_{ij}x_{ji}$$

$$= \sum_{1\le i\le j\le p}x_{ii}x_{jj} - \sum_{i<j}x_{ij}^2$$

$$= \binom{|EDS|}{2} - \sum_{i<j}\left[\frac{1}{2}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)\right]^2$$

4. Minimum Efficient Dominating Extended Energy Of Complete Graph, Crown Graph And Friendship Graph

Theorem – 4.1:

For $p \geq 2$, the minimum efficient dominating extended energy of Complete graph K_p is $(p-2)+\sqrt{p^2-2p+5}$.

Proof:

K_p is a Complete graph with node collection $N=\{a_1, a_2, ..., a_p\}$. The minimum efficient dominating set $EDS=\{a_1\}$.

$$A_{ex}^{eD}(K_p)=\begin{bmatrix} 1 & \frac{1}{2}\left[\frac{p-1}{p-1}+\frac{p-1}{p-1}\right] & \cdots & \frac{1}{2}\left[\frac{p-1}{p-1}+\frac{p-1}{p-1}\right] \\ \frac{1}{2}\left[\frac{p-1}{p-1}+\frac{p-1}{p-1}\right] & 0 & \cdots & \frac{1}{2}\left[\frac{p-1}{p-1}+\frac{p-1}{p-1}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\left[\frac{p-1}{p-1}+\frac{p-1}{p-1}\right] & \frac{1}{2}\left[\frac{p-1}{p-1}+\frac{p-1}{p-1}\right] & \cdots & 0 \end{bmatrix}$$

The determinantal equation is

$$(\mu+1)^{p-2}(\mu^2-(p-1)\mu-1)=0$$

The minimum efficient dominating extended latent values are

$$\mu=-1[(p-2)times]$$

$$\mu=\frac{(p-1)\pm\sqrt{p^2-2p+5}}{2}$$

The minimum efficient dominating extended energy,

$$E_{ex}^{eD}(K_p)=\sum_{i=1}^{p}|\mu_i|$$

$$=|-1|(p-2)+\left|\frac{(p-1)+\sqrt{p^2-2p+5}}{2}\right|+\left|\frac{(p-1)-\sqrt{p^2-2p+5}}{2}\right|$$

$$=p-2+\sqrt{p^2-2p+5}$$

Theorem 4.2:

For $p \geq 2$, the minimum efficient dominating extended energy of a Crown graph S_p^0 is $2(p-2)+\sqrt{p^2+2p-3}+\sqrt{p^2-2p+5}$.

Proof:

Let the nodes of a crown graph S_p^0 be $\{a_1, a_2, \dots, a_p, b_1, b_2, \dots\ b_P\}$. The minimum efficient dominating set $EDS = \{a_1, b_1\}$

$$A_{ex}^{eD}\left(S_p^0\right)=\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \frac{1}{2}\left(\frac{p-1}{p-1}+\frac{p-1}{p-1}\right) & \cdots & \frac{1}{2}\left(\frac{p-1}{p-1}+\frac{p-1}{p-1}\right) \\ 0 & 0 & \cdots & 0 & \frac{1}{2}\left(\frac{p-1}{p-1}+\frac{p-1}{p-1}\right) & 0 & \cdots & \frac{1}{2}\left(\frac{p-1}{p-1}+\frac{p-1}{p-1}\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{2}\left(\frac{p-1}{p-1}+\frac{p-1}{p-1}\right) & \frac{1}{2}\left(\frac{p-1}{p-1}+\frac{p-1}{p-1}\right) & \cdots & 0 \\ 0 & \frac{1}{2}\left(\frac{p-1}{p-1}+\frac{p-1}{p-1}\right) & \cdots & \frac{1}{2}\left(\frac{p-1}{p-1}+\frac{p-1}{p-1}\right) & 1 & 0 & \cdots & 0 \\ \frac{1}{2}\left(\frac{p-1}{p-1}+\frac{p-1}{p-1}\right) & 0 & \cdots & \frac{1}{2}\left(\frac{p-1}{p-1}+\frac{p-1}{p-1}\right) & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\left(\frac{p-1}{p-1}+\frac{p-1}{p-1}\right) & \frac{1}{2}\left(\frac{p-1}{p-1}+\frac{p-1}{p-1}\right) & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

The determinantal equation is

$$(\mu+1)^{p-2}(\mu-1)^{p-2}[\mu^2-(p-3)\mu-(2p-3)][\mu^2-(p-1)\mu-1=0$$

The minimum efficient dominating extended latent values are

$$\mu=-1\ [(p-2)\,times]$$

$$\mu=1\ [(p-2)\,times]$$

$$\mu=\frac{(p-3)\pm\sqrt{p^2+2p-3}}{2}$$

$$\mu=\frac{(p-1)\pm\sqrt{p^2-2p+5}}{2}$$

The minimum efficient dominating extended energy is

$$E_{ex}^{eD}(S_p^0)=\sum_{i=1}^{p}\left|\mu_i\right|$$

$$=|-1|(p-2)+|1|(n-2)+\left|\frac{(p-3)+\sqrt{p^2+2p-3}}{2}\right|+\left|\frac{(p-3)-\sqrt{p^2+2p-3}}{2}\right|$$

$$+\left|\frac{(p-1)+\sqrt{p^2-2p+5}}{2}\right|+\left|\frac{(p-1)-\sqrt{p^2-2p+5}}{2}\right|$$

$$=2(p-2)+\sqrt{p^2+2p-3}+\sqrt{p^2-2p+5}$$

Theorem 4.3:

For $p \geq 2$, the minimum efficient dominating extended energy of Friendship graph F_p is $(2p-1)+4\sqrt{p^2-2p+1}$.

Proof:

Let the nodes set of Friendship graph F_p be $\{a_1, a_2, ..., a_{2p+1}\}$. Then the minimum efficient dominating set $EDS = \{a_1\}$.

$$A_{ex}^{eD}\left(F_p\right)=\begin{bmatrix} 1 & \frac{1}{2}\left[\frac{2}{2p}+\frac{2p}{2}\right] & \frac{1}{2}\left[\frac{2}{2p}+\frac{2p}{2}\right] & \cdots & \frac{1}{2}\left[\frac{2}{2p}+\frac{2p}{2}\right] & \frac{1}{2}\left[\frac{2}{2p}+\frac{2p}{2}\right] \\ \frac{1}{2}\left[\frac{2}{2p}+\frac{2p}{2}\right] & 0 & \frac{1}{2}\left[\frac{2}{2}+\frac{2}{2}\right] & \cdots & 0 & 0 \\ \frac{1}{2}\left[\frac{2}{2p}+\frac{2p}{2}\right] & \frac{1}{2}\left[\frac{2}{2}+\frac{2}{2}\right] & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2}\left[\frac{2}{2p}+\frac{2p}{2}\right] & 0 & 0 & \cdots & 0 & \frac{1}{2}\left[\frac{2}{2}+\frac{2}{2}\right] \\ \frac{1}{2}\left[\frac{2}{2p}+\frac{2p}{2}\right] & 0 & 0 & \cdots & \frac{1}{2}\left[\frac{2}{2}+\frac{2}{2}\right] & 0 \end{bmatrix}$$

The determinantal equation is

$$(\mu+1)^p(\mu-1)^{p-1}[\mu^2-2\mu-(4p^2-8p+3)]=0$$

The minimum efficient extended dominating latent values are

$$\mu=-1\ [\,p\,times\,]$$

$$\mu=1\ [(p-1)\,times\,]$$

$$\mu=\frac{2\pm 4\sqrt{p^2-2p+1}}{2}$$

The minimum efficient extended dominating energy,

$$\begin{aligned} E_{ex}^{eD}(F_p) &= \sum_{i=1}^{p}\left|\mu_i\right| \\ &= |-1|(p)+|1|(p-1)+\left|\frac{2+4\sqrt{p^2-2p+1}}{2}\right|+\left|\frac{2-4\sqrt{p^2-2p+1}}{2}\right| \\ &= p+p-1+\frac{2}{2}+\frac{4\sqrt{p^2-2p+1}}{2}-\frac{2}{2}+\frac{4\sqrt{p^2-2p+1}}{2} \\ &= (2p-1)+4\sqrt{p^2-2p+1} \end{aligned}$$

4.4 SPECIAL CASE

Cycle graph

For $n \geq 3$, the minimum efficient dominating extended energy for C_p can be computed only when $p \equiv 0\,(mod\ 3)$.

Example:

The minimum efficient dominating extended energy of the Cycle graph C_3 is 3.8284

Solution:

The minimum efficient dominating set of the given graph is EDS = $\{a_1\}$.

$$A_{ex}^{eD} = \begin{bmatrix} 1 & \frac{1}{2}\left(\frac{2}{2}+\frac{2}{2}\right) & \frac{1}{2}\left(\frac{2}{2}+\frac{2}{2}\right) \\ \frac{1}{2}\left(\frac{2}{2}+\frac{2}{2}\right) & 0 & \frac{1}{2}\left(\frac{2}{2}+\frac{2}{2}\right) \\ \frac{1}{2}\left(\frac{2}{2}+\frac{2}{2}\right) & \frac{1}{2}\left(\frac{2}{2}+\frac{2}{2}\right) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The Determinantal equation is

$$(\mu+1)(\mu^2-2\mu-1)=0$$

The minimum efficient dominating extended latent values are

$$\mu_1 = -1 \qquad \mu_2 = -0.4142 \qquad \mu_3 = 2.4142$$

The minimum efficient dominating extended energy is

$$E_{ex}^{eD}(C_3) = |-1| + |-0.4142| + |2.4142|$$
$$= 3.8284$$

Path graph

For $n \geq 3$, the minimum efficient dominating extended energy for P_p can be computed only when $p \equiv 1,2\,(mod\ 3)$.

Example:

The minimum efficient dominating extended energy of the Path graph P_4 is 5.7016

Solution:

The minimum efficient dominating set of the Path graph EDS = $\{a_1, a_4\}$.

$$A_{ex}^{eD}(P_4) = \begin{bmatrix} 1 & \frac{1}{2}\left(\frac{1}{2}+\frac{2}{1}\right) & 0 & 0 \\ \frac{1}{2}\left(\frac{2}{1}+\frac{1}{2}\right) & 0 & \frac{1}{2}\left(\frac{2}{2}+\frac{2}{2}\right) & 0 \\ 0 & \frac{1}{2}\left(\frac{2}{2}+\frac{2}{2}\right) & 0 & \frac{1}{2}\left(\frac{1}{2}+\frac{2}{1}\right) \\ 0 & 0 & \frac{1}{2}\left(\frac{2}{1}+\frac{1}{2}\right) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{5}{4} & 0 & 0 \\ \frac{5}{4} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{5}{4} \\ 0 & 0 & \frac{5}{4} & 1 \end{bmatrix}$$

The minimum efficient dominating extended latent values are

$$\mu_1 = -1.6008 \qquad \mu_2 = -0.2500 \qquad \mu_3 = 1.6008 \qquad \mu_4 = 2.2500$$

The minimum efficient dominating extended energy is

$$E_{ex}^{eD}(P_4) = |-1.6008| + |-0.2500| + |1.6008| + |2.2500|$$
$$= 5.7016$$

5. Restrictions for Minimum Efficient Dominating Extended Energy

Theorem 5.1:

If $\mu_1(\psi)$ is the biggest minimum efficient dominating extended latent value of $A_{ex}^{eD}(\psi)$, then $\mu_1(\psi) \geq \dfrac{|EDS| + 2\sum_{i<j} \frac{1}{2}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)}{p}$.

Proof:

Let Y be some non zero vector. Then by [Adiga] we have $\mu_1(\psi) = \max_{Y \neq 0}\left\{\frac{Y' AY}{Y'Y}\right\}$

$\therefore \mu_1(\psi) \geq \dfrac{J' AJ}{J' J} \geq \dfrac{|EDS| + 2\sum_{i<j} \frac{1}{2}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)}{p}$ where J remains a unit vector [1,1,1,…,1].

Alike to Koolen and Moulton's [5] upper bound for energy of a graph, higher bounds for $E_{ex}^{eD}(\psi)$ is specified in next theorem.

Theorem 5.2:

If $|EDS|+2\sum_{i<j}\frac{1}{2}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)\geq p$ then

$$E_{ex}^{eD}(\psi)\leq\frac{|EDS|+2\sum_{i<j}\frac{1}{2}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)}{p}+\sqrt{(p-1)\left[|EDS|+\frac{1}{2}\left(\sum_{i<j}\left[\frac{d_i}{d_j}+\frac{d_j}{d_i}\right]\right)^2-\left(\frac{|EDS|+2\sum_{i<j}\frac{1}{2}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)}{p}\right)^2\right]}$$

Proof:

Cauchy – Schwartz variation stands

$$\left(\sum_{i=2}^{p}x_iy_i\right)^2\leq\left(\sum_{i=2}^{p}x_i^2\right)\left(\sum_{i=2}^{p}y_i^2\right)$$

Put $x_i=1, y_i=|\mu_i|$ then $\left(\sum_{i=2}^{p}|\mu_i|\right)^2\leq\left(\sum_{i=2}^{p}1\right)\left(\sum_{i=2}^{p}\mu_i^2\right)$

$$\Rightarrow\left[E_{ex}^{eD}(\psi)-\mu_1\right]^2\leq(p-1)\left[|EDS|+\frac{1}{2}\sum_{i<j}\left[\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)\right]^2-\mu_1^2\right]$$

$$\Rightarrow E_{ex}^{eD}(\psi)\leq\mu_1+\sqrt{(p-1)\left[|EDS|+\frac{1}{2}\sum_{i<j}\left[\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)\right]^2-\mu_1^2\right]}$$

$$Let\ f(\alpha)=\alpha+\sqrt{(p-1)\left[|EDS|+\frac{1}{2}\sum_{i<j}\left[\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)\right]^2-\alpha^2\right]}$$

For decreasing function $f(\alpha)\leq 0$

$$\Rightarrow 1-\frac{\alpha(p-1)}{\sqrt{(p-1)\left[|EDS|+\frac{1}{2}\sum_{i<j}\left[\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)\right]^2-\alpha^2\right]}}\leq 0$$

$$\Rightarrow\alpha\geq\sqrt{\frac{|EDS|+\frac{1}{2}\sum_{i<j}\left[\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)\right]^2}{p}}$$

$\because \left[|EDS| + 2\sum_{i<j} \frac{1}{2}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right) \right] \geq p$, we have

$$\sqrt{\frac{|EDS| + 2\sum_{i<j} \frac{1}{2}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right)}{p}} \leq \frac{|EDS| + 2\sum_{i<j} \frac{1}{2}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right)}{p} \leq \mu_1$$

From theorem 1,

$$\therefore f(\mu_1) \leq f\left[\frac{|EDS| + 2\sum_{i<j} \frac{1}{2}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right)}{p} \right]$$

$$(i.e)\, E_{ex}^{eD}(\psi) \leq f(\mu_1) \leq f\left[\frac{|EDS| + 2\sum_{i<j} \frac{1}{2}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right)}{p} \right]$$

$$(i.e)\, E_{ex}^{eD}(\psi) \leq f\left[\frac{|EDS| + 2\sum_{i<j} \frac{1}{2}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right)}{p} \right]$$

$$E_{ex}^{eD}(\psi) \leq \frac{|EDS| + 2\sum_{i<j} \frac{1}{2}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right)}{p} + \sqrt{(p-1)\left[|EDS| + \frac{1}{2}\left(\sum_{i<j}\left[\frac{d_i}{d_j} + \frac{d_j}{d_i} \right] \right)^2 - \left(\frac{|EDS| + 2\sum_{i<j} \frac{1}{2}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right)}{p} \right)^2 \right]}$$

Milovanic bounds[8] for extended distance energy are proved below.

Theorem 5.3:

Let $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_p|$ be a decreasing directive of extended distance latent values of $A_{ex}^{eD}(\psi)$,

then $E_{ex}^{eD}(\psi) \geq \sqrt{p\left[|EDS| + \frac{1}{2}\sum_{i<j}\left[\frac{d_i}{d_j} + \frac{d_j}{d_i} \right]^2 \right] - \eta(p)\left(|\mu_1| - |\mu_p|\right)^2}$

Whenever $\eta(p) = p\left[\frac{p}{2} \right]\left[1 - \frac{1}{p}\left[\frac{p}{2} \right] \right]$

Proof:

Let r, r_1, r_2, … , r_p, R and s, s_1, s_2, … , s_p, S be real quantities such that $r \le r_i \le R$ and $s \le s_i \le S \ \forall\ i = 1, 2, \cdots, p$. Then the subsequent variation is effective.

$$\left| p\sum_{i=1}^{p} r_i s_i - \sum_{i=1}^{p} r_i \sum_{i=1}^{p} s_i \right| \le \eta(p)(R-r)(S-s) \quad \text{where} \quad \eta(p) = p\left[\frac{p}{2}\right]\left[1-\frac{1}{p}\left[\frac{p}{2}\right]\right] \quad \text{and}$$

equivalence clutches if and only if $r_1 = r_2 = \ldots = r_p$ and $s_1 = s_2 = \ldots = s_p$. If $r_i = |\mu_i|,\ s_i = |\mu_i|,\ r = s = |\mu_p|\ and\ R = S = |\mu_1|$, then $\left| p\sum_{i=1}^{p} |\mu_i|^2 - \left(\sum_{i=1}^{p} |\mu_i|\right)^2 \right| \le \eta(p)\left(|\mu_1| - |\mu_p|\right)^2$

However $\sum_{i=1}^{p} |\mu_i|^2 = |EDS| + \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2$ and $E_{ex}^{eD}(\psi) \le \sqrt{p\left[|EDS| + \frac{1}{2}\sum_{i<j}\left[\frac{d_i}{d_j} + \frac{d_j}{d_i}\right]^2\right]}$ then

the overhead variation turns out to be

$$p\left[|EDS| + \frac{1}{2}\sum_{i<j}\left[\frac{d_i}{d_j} + \frac{d_j}{d_i}\right]^2\right] - \left(E_{ex}^{eD}(\psi)\right)^2 \le \eta(p)\left(|\mu_1| - |\mu_p|\right)^2$$

$$(i.e)\ E_{ex}^{eD}(\psi) \ge \sqrt{p\left[|EDS| + \frac{1}{2}\sum_{i<j}\left[\frac{d_i}{d_j} + \frac{d_j}{d_i}\right]^2\right] - \eta(p)\left(|\mu_1| - |\mu_p|\right)^2}$$

Theorem 5.4:

Let $|\mu_1| \ge |\mu_2| \ge \cdots \ge |\mu_p| > 0$ remain a decreasing directive of latent values of $A_{ex}^{eD}(\psi)$ then

$$E_{ex}^{eD}(\psi) \ge \frac{|EDS| + \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 + p\,|\mu_1|\,|\mu_p|}{\left(|\mu_1| + |\mu_p|\right)}.$$

Proof:

Let $f_i \ne 0, g_i, h\ and\ H$ stand actual numbers nourishing $h\ f_i \le g_i \le H\ f_i$, then the succeeding variation grips [Theorem 2, [Milovanic]].

$$\sum_{i=1}^{p} g_i^2 + h\,H\sum_{i=1}^{p} f_i \le (h+H)\sum_{i=1}^{p} f_i g_i$$

Put $g_i = |\mu_i|,\ f_i = 1,\ h = |\mu_p|\ and\ H = |\mu_1|$ then

$$\sum_{i=1}^{p} |\mu_i|^2 + |\mu_1|\,|\mu_p|\sum_{i=1}^{p} 1 \le \left(|\mu_1| + |\mu_p|\right)\sum_{i=1}^{p} |\mu_i|$$

$$|EDS| + \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 + p\,|\mu_1|\,|\mu_p| \le \left(|\mu_1| + |\mu_p|\right) E_{ex}^{eD}(\psi)$$

$$E_{ex}^{eD}(\psi) \ge \frac{|EDS| + \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 + p\,|\mu_1|\,|\mu_p|}{\left(|\mu_1| + |\mu_p|\right)}$$

Hence the proof.

Conclusion

In this paper, the minimum efficient dominating extended energy of complete graph, crown graph and friendship graph are estimated. As the special case, the minimum efficient dominating extended energy of cycle graph C_3 and path graph P_4 are discussed. Also the assets and restrictions of minimum efficient dominating extended energy are derived.

Extended adjacency matrices are used to calculate the boiling point of the alkanes [11]. This application can be extended for efficient dominating extended energy in future.

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