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Study of Generalized B-Curvature Tensor in Sasakian Manifold

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Abstract:- The objective of this paper is to study some curvature properties of generalized B-curvature tensor on Sasakian manifold. Here first, we describe certain vanishing properties of generalized B-curvature tensor on Sasakian manifold and obtained several interesting results. Next, we formulate $\varphi - B$ semi-symmetric condition on Sasakian manifold. Again, we discussed the generalized B pseudo-symmetric condition on Sasakian manifold. We also characterized generalized $B - \varphi-$ recurrent Sasakian manifold. Further, we deal with Sasakian manifold satisfying $B((L, M)Q)P = 0$, $R(\xi, P)P = 0$ condition. In the last section we derived an example which satisfies the theorem.

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1. Introduction

Let M be a (2n+1)- dimensional contact metric manifold with contact metric structure (φ, ξ, η, g). M is said to have a Sasakian structure or normal contact metric structure if its contact metric structure is normal. A Sasakian manifold, also known as a normal contact metric manifold.

The Sasakian structure [1],[2], which is defined on an odd dimensional manifold, is the closest possible analogue of the kaehler geometry of even dimension [18],[19]. The notion of Sasakian structure was introduced by Sasaki (1960) who considered it as a special kind of contact geometry. There was not much activity in this field after the mid 1970s, until the advent of string theory. The Sasakian and Sasakian Einstein structures appear in physics in the context of the string theory. Sasakian manifold is also studied by Boyer and Galicki (2008), Ghosh and Sharma [10], Goldlinski (2000), Shah (2012), Yano and Kon [15] and others (see [6],[11],[12],[13] , [14],[15]). On the other hand, in the analogous way of kaehler manifold, Matsumoto and chuman [5] introduced the notion of C-Bochner curvature tensor on a Sasakian manifold.

Definition 1.1. In [17],authors Shaikh and Kundu proved the equivalency of various geometric structures obtained by the same curvature restriction on different curvature tensors. For this purpose, they have introduced a special type of (0,4) tensor field, called B-curvature tensor and further they studied generalized B- curvature tensor tensor on a Riemannian manifold and is given by

$$B(U, V)X = a_0 R(U, V)X + a_1[S(V, X)U - S(U, X)V + g(V, X)QU - g(U, X)QV] + 2a_2 r[g(V, X)U - g(U, X)V], \quad (1.1)$$

where a_0, a_1, a_2 are scalars. The generalized B-curvature tensor includes the structures of quasi-conformal, Weyl conformal, conharmonic and concircular curvature tensors:

(i) The quasi-conformal curvature tensor C^* [9] if $a_0 = a$, $a_1 = b$ and $a_2 = -\frac{1}{n}\left[\frac{a}{n-1} + 2b\right]$.

(ii) The Weyl-conformal curvature tensor $C^{\sim}$ [8] if $a_0 = 1$, $a_1 = -\frac{1}{n-2}$ and $a_2 = -\frac{1}{n}\left[-\frac{1}{2(n-1)(n-2)}\right]$.

(iii) The concircular curavture tensor C [7] if $a_0 = 1$, $a_1 = 0$ and $a_2 = -\frac{1}{2n(n-1)}$.

(iv) The conharmonic curvature tensor P[3] if $a_0 = 1$, $a_1 = \frac{-1}{n-2}$ and $a_2 = 0$.

The authors extensively studied the properties of generalized *B*-curvature tensor on the various manifolds [4, 7, 21]. In this paper, we have studied some special properties of generalized *B*-curvature tensor on Sasakian manifold.

1. Preliminaries

In this section, we briefly recall some general definitions and condition of Sasakian manifold which is needed throughout this study:

Let *M* be a (2n + 1) dimensional connected almost metric manifold [20] with an almost contact metric structure (φ, ξ, η, g) where φ is a (1,1) tensor field, ξ is a covarient vector field, η is a 1- form and g is a compatible Riemannian [16] metric such that

(2.1) $\varphi^2 P = P + \eta(P)\xi,$

(2.2) $g(P, \xi) = \eta(P),$

(2.3) $\eta(\xi) = 1, \varphi\xi = 0, \eta o \varphi = 0, \eta(\varphi P) = 0,$

(2.4) $g(\varphi P, Q) = -g(P, \varphi Q),$

(2.5) $g(\varphi P, \varphi Q) = g(P, Q) - \eta(P)\eta(Q),$

for all P, Q ∈ ζ(M).

If Sasakian manifold M satisfies

(2.6) $\nabla P\, \xi = -\varphi P, (\nabla P\, \eta)Q = g(P, \varphi Q),$

where ∇ denotes the Riemannian connection in *M*. In a Sasakian manifold the following relations hold:

(2.7) $R(P,Q)R = g(Q,R)P - g(P,R)Q,$

(2.8) $R(\xi,P)Q = (\nabla P\, \varphi)Q = g(P,Q)\xi - \eta(Q)P$

(2.9) $R(P,Q)\xi = \eta(Q)P - \eta(P)Q,$

(2.10) $R(P,\xi)Q = \eta(Q)P - g(P,Q)\xi,$

(2.11) $\eta(R(P,Q)R) = g(Q,R)\eta(P) - g(P,R)\eta(Q),$

(2.12) $S(P,\xi) = 2n\eta(P), QP = 2nP, Q\xi = 2n\xi$

(2.13) $S(\varphi P, \varphi Q) = S(P,Q) - 2n\eta(P)\eta(Q)$

where R is a Riemannian curvature, S is the Ricci tensor and *Q* is the Ricci operator given by

$S(P,Q) = g(QP,Q)$ for all P, Q ∈ ζ(M).

Definition 2.1. A Sasakian manifold is said to be η− Einstein manifold if it's Ricci tensor is of the form

(2.14) $S(L,M) = \alpha 1 g(L,M) + \alpha 2 \eta(L)\eta(M).$

where α_1, α_2 are smooth function on *M*. If $\alpha_2 = 0$, then M is an η- Einstein manifold.

With the help of equation (2.2),(2.3),(2.8),(2.9),(2.10),(2.12) in Sasakian manifold and the generalized *B*-curvature tensor satisfies the following conditions from (1.1):

(2.15) $B(L,M)\xi = (a_0 + 2na_1 + 2a_2 r)[\eta(M)L - \eta(L)M] + a_1[\eta(M)QL - \eta(L)QM],$

(2.16) $B(\xi,M)\xi = (a_0 + 2na_1 + 2a_2 r)[\eta(M)\xi - M] + a_1[2n\eta(M)\xi - QM],$

(2.17) $B(L, \xi)\xi = (a_0 + 2na_1 cxxv + 2a_2 r)\,[L - \eta(L)\xi] + a_1\,[QL - 2n\eta(L)\xi]$

(2.18) $\eta(B(L, M)\xi) = \eta(B(\xi, M)\xi) = \eta(B(L, \xi)\xi) = 0.$

Using these condition, we shall prove some important results of Sasakian manifold in the following sections.

3. Sasakian manifold admitting some vanishing properties of generalized *B* - curvature tensor

Definition 3.1. A Sasakian manifold is said to be generalized *B*- flat if *B*(L, M).P = 0, for any vector fields L,M,P on *M*.

By virtue of Definition 3.1 in (1.1), we get

(3.1) $a_0 R(L, M)P + a_1[S(M, P)L - S(L, P)M + g(M, P)QL - g(L, P)QM] + 2a_2 r[g(M, P)L - g(L, P)M] = 0.$

Now taking an account of (3.1), we have

(3.2) $a_0 g(R(L, M)P, N) + a_1[S(M, P)g(L, N) - S(L, P)g(M, N) + g(M, P)S(L, N) - g(L, P)S(M, N)] + 2a_2 r[g(M, P)g(L, N) - g(L, P)g(M, N)] = 0.$

Contracting above equation over L and N, which imply as

(3.3) $a_0 S(M, P) + a_1(n - 2)S(M, P) + [a_1 r + 2a_2 r(n - 1)]\,g(M, P) = 0$

Or

(3.4) $S(M, P) = -\frac{a_1 r + 2a_2 r(n-1)}{a_0 + a_1(n-2)}\, g(M, P)).$

Hence, we state the following:

Theorem 3.2. If B(L, M)P = 0, an n-dimensional Sasakian manifold, is linearly independent of a0 and a1, then it is an Einstein manifold.

Corollary 3.3. If the condition B(L, M)P = 0, holds then the generalized *B*- curvature tensor converted to Concircular curvature tensor as scalars $a_0 = 1$, $a_1 = 0$ and $a_2 = -\frac{1}{2n(n-1)}$.

4. Sasakian manifold admitting *B*(L, M)ξ = 0

Definition 1. A Sasakian manifold is said to be generalized ξ − *B*− flat if *B*(L, M)ξ = 0, for any vector fields L,M on *M*. It follows from equation (1.1) that

(4.1) $a_0 R(L, M)\xi + a_1[S(M, \xi)L - S(L, \xi)M + g(M, \xi)QL - g(L, \xi)QM] + 2a_2 r[g(M, \xi)L - g(L, \xi)M] = 0.$

Applying (2.2),(2.9) and (2.12) in (4.1), we have

(4.2) $a_0\,[\eta(M)L - \eta(L)M] + a_1[2n\eta(M)L - 2n\eta(L)M + \eta(M)QL - \eta(L)QM] + 2a_2 r[\eta(M)L - \eta(L)M] = 0,$

(4.3) $(a_0 + 2na_1 + 2a_2 r)\,[\eta(M)L - \eta(L)M] + a_1[\eta(M)QQL - \eta(L)QM] = 0.$

Contracting with N, we have

(4.4) $(a_0 + 2na_1 + 2a_2 r)[\eta(M)g(L, N) - \eta(L)g(M, N)] + a_1[\eta(M)g(QL, N) - \eta(L)g(QM, N)] = 0.$

Put L = N = e_i, we obtain

(4.5) $\eta(M)[(a_0 + 2na_1 + 2a_2 r)(n - 1)a_1(r - 2n)] = 0.$

Since η(M) 6= 0, so

(4.6) . $$r = -\frac{(a_0(n-1) + 2na_1(n-2))}{2a_2(n-1) + a_1}$$

Now we state the following theorem:

Theorem 4.1. If a Sasakian manifold *M* admits *B*(L, M)ξ = 0,is a constant scalar curvature on condition, then the scalar a_1 and a_2 are linearly independent to each other.

5. Sasakian manifold admitting g(*B*(φL, φM) φP, φN) = 0

In this section, we study the Sasakian manifold admitting g(*B*(φL, φM)φP, φN) = 0, condition:

From equation (1.1), we have

(5.1) $$a_0 g(R(\varphi L, \varphi M)\varphi P, \varphi N) + a_1[S(\varphi M, \varphi P)\, g(\varphi L, \varphi N) - S(\varphi L, \varphi P)\, g(\varphi M, \varphi N) + g(\varphi M, \varphi P)\, S(\varphi L, \varphi N) - g(\varphi L, \varphi P)\, S(\varphi M, \varphi N)] + 2a_2 r[g(\varphi M, \varphi P)\, g(\varphi L, \varphi N) - g(\varphi L, \varphi P)\, g(\varphi M, \varphi N)] = 0,$$

On contracting above equation (5.1) over L and N, we have

(5.2) $$a_0 S(\varphi M, \varphi P) + a_1[S(\varphi M, \varphi P)(n-1) - S(\varphi M, \varphi L) + (r-2n)\, g(\varphi M, \varphi P) - S(\varphi M, \varphi P)] + 2a_2 r[(n-1)\, g(\varphi M, \varphi P) - g(\varphi M, \varphi P)] = 0,$$

or

(5.3) $$S(\varphi M, \varphi P) = -\frac{a_1(r-2n) + 2a_2 r(n-2)}{a_0 + a_1(n-3) g(\varphi M, \varphi P)}$$

using (2.5) and (2.13) in (5.3), we get

(5.4) $$S(M, P) = \alpha_1 g(M, P) + \alpha_2 \eta(M)\eta(P),$$

where $\alpha_1 = -\frac{(a_1(r-2n) + 2a_2 r(n-2))}{a_0 + a_1(n-3)}$ and $\alpha_2 = -\frac{(a_1(r-8n+2n^2) + 2a_2 r(n-2) + 2na_0)}{a_0 + a_1(n-3)}$

thus we state the following theorem:

Theorem 5.1. If a Sasakian manifold M admits g(*B*(φL, φM)φP, φN) = 0, is an η- Einstein manifold, then the scalar a_0 and a_1 are linearly independent to each other.

6. Sasakian manifold satisfying the condition *B*((L, M).*Q*)P = 0.

In this section, we study the Sasakian manifold satisfying *B*((L, M).*Q*)P = 0, then we have

(6.1) $$B(L, M)QP - Q(B(L, M)P) = 0$$

putting M = ξ in above equation, we have

(6.2) $$B(L, \xi)QX - Q(B(L, \xi)P) = 0$$

from (1.1), we have

(6.3) $$a_0 R(L, \xi)QP + a_1[S(\xi, QP)L - S(L, QP)\xi + g(\xi, QP)QL - g(L, QP)Q\xi] + 2a_2 r[g(\xi, QP)L - g(L, QP)\xi] - Q[a_0 R(L, \xi)P + a_1[S(\xi, P)L - S(L, P)\xi + g(\xi, P)QL - g(L, P)Q\xi] + 2a_2 r[g(\xi, P)L - g(L, P)\xi]] = 0,$$

(6.4) $$(-a_0 - 2a_2 r)\, S(L, P)\, \xi - a_1 S^2(L, P)\, \xi + 2n[a_0 + 2na_1 + 2a_2 r]\, g(L, P)\, \xi = 0.$$

Taking inner product with ξ in (6.4), we have

(6.5) $$S^2(L, P) = \frac{[-a_0 - 2a_2 r]}{a_1} S(L, P) + 2n\frac{[a_0 + 2na_1 + 2a_2 r]}{a_1} g(L, P)$$

Therefore, the S^2 of the Ricci tensor S is the linear combination of the Ricci tensor and the metric tensor g. Here, the (0, 2)-tensor S^2 is defined by $S^2(L, P) = S(QL, P)$. Hence, we state the following:

Theorem 6.1. Let M be an n-dimensional Sasakian manifold is satisfying the condition $BQ = 0$. Then the S^2 of the Ricci tensor S is the linear combination of the Ricci tensor and the metric tensor g has the form $S^2(L, P) = \frac{[-a_0 - 2a_2 r]}{a_1} S(L,P) + \frac{2n[a_0 + 2na_1 + 2a_2 r]}{a_1} g(L, P)$.

7. Generalized $B - \varphi-$ recurrent Sasakian manifold

Definition 7.1. A Sasakian manifold is called a generalized B -recurrent manifold, if for every nonzero one form A satisfies

(7.1) $\varphi^2((\nabla_N B)(L, M)P) = A(W) B(L, M)P,$

for any vector fields L, M, P, N ∈ $T_P M$.

In view of (2.1), we have

(7.2) $-(\nabla_N B)(L, M)P + \eta((\nabla_N B)(L, M)P)\xi = A(W) g(B(L, M)P, Q)$

which yeild

(7.3) $-g((\nabla_N B)(L, M)P, Q) + \eta((\nabla_N B)(L, M)P)\eta(Q) = A(W) B(L, M)P.$

Taking an account of (1.1) in the above equation and then contracting over L and Q, we have

(7.4) $-a_1[(n-1)(\nabla_S)(M, P) + g(M, P)dr(N) - (\nabla_S)(M, P)] - 2a_2 dr(N)(n-1)g(M, P) + a_1[(\nabla_S)(M, P) - (\nabla_S)(\xi, P)\eta(M)] + 2a_2 dr(N)[g(M, P) - \eta(M)\eta(P)]$

$= A(W)[a_0(n-1) + a_1 r + 2a_2 r]g(M, P) + a_1(n-2)S(M, P).$

Putting M = P = ξ, in (7.4), we have

(7.5) $-a_1[(n-1)(\nabla_S)(\xi, \xi) + g(\xi, \xi)dr(N) - (\nabla_S)(\xi, \xi)] - 2a_2 dr(N)(n-1)g(\xi, \xi) + a_1[(\nabla_S)(\xi, \xi) - (\nabla_S)(\xi, \xi)\eta(\xi)] + 2a_2 dr(N)[g(\xi, \xi) - \eta(\xi)\eta(\xi)] = A(W)[a_0(n-1) + a_1 r + 2a_2 r] g(\xi, \xi) + a_1(n-2) S(\xi, \xi)$

(7.6) $A(W) = \frac{[4(1-n)a_2 - 1]dr(N)}{[a_0(n-1) + (a_1 + 2a_2)r] + a_1(n-2)r}$

Thus, we state the following theorem:

Theorem 7.2. In an n-dimensional generalized B -recurrent Sasakian manifold, the nonzero 1-form A is given by (7.6) provided that $[a_0(n-1) + (a_1 + 2a_2)r] + a_1(n-2)r$ 6= 0.

Further, if we assume that the scalar curvature of an n-dimensional generalized

$B - \varphi$ −recurrent manifold is constant, then drN = 0. Hence, Eq. (7.6) yields

(7.7) A(W) = 0.

Making use of (7.6) in (7.1), we get

(7.8) $\varphi^2((\nabla_W B)(L, M)P) = 0.$

Hence, we can state the following:

Theorem 7.3. The generalized $B - \varphi-$recurrent Sasakian manifold with constant scalar curvature r reduces to a generalized B locally φ− symmetric space.

8. Generalized B pseudo-symmetric Sasakian Manifold

Definition 8.1. A Sasakian manifold is said to be generalized B pseudo symmetric

if it's curvature tensor satisfies

(8.1) $(R(L, M) \cdot B)(P, Q)N = \underline{L}_B[((L \wedge M) \cdot B)(P, Q)N],$

holds on L_B = {x ∈ M : B not equal to 0 at x} , where $\underline{L}_B$ is some function on L_B.

from equation (8.1), we may write

(8.2) $(R(L, \xi)\cdot B)(P, Q)N = L_B[((L \wedge \xi)(B(P, Q)N)) - B((L \wedge \xi)P, Q)N - B(P,(L \wedge \xi)Q)N - B(P, Q)(L \wedge \xi)N].$

From left hand side of (8.2), we can easily obtained

(8.3) $[\eta(B(P, Q)N)L - g(L, B(P, Q)N)\xi - \eta(P)B(L, Q)N + g(L, P)B(\xi, Q)N - \eta(Q)B(P, L)N + g(L, Q)B(P, \xi)N - \eta(N)B(P, Q)L + g(L, N) B(P, Q)\xi].$

Similarly, right hand side of (8.2) gives,

(8.4) $L_B[\eta(B(P, Q)N)L - g(L, B(P, Q)N)\xi - \eta(P) B(L, Q)N + g(L, P) B(\xi, Q)N - \eta(Q) B(P, L)N + g(L, Q)B(P, \xi)N - \eta(N)B(P, Q)L + g(L, N)B(P, Q)\xi].$

Now the foregoing equation can takes the form,

(8.5) $(L_B - 1)[\eta(B(P, Q)N)L - g(L, B(P, Q)N)\xi - \eta(P)B(L, Q)N + g(L, P)B(\xi, Q)N - \eta(Q) B(P, L)N+ g(L, Q) B(P, \xi)N - \eta(N) B(P, Q)L + g(L, N)B(P, Q)\xi] = 0.$

Plugging Q = ξ in (8.5) and then by virtue of (1.1), we obtain either $L_B = 1$ or

(8.6) $B(P, L)N = -\eta(B(P, \xi)N)L + g(L, B(P, \xi)N)\xi + \eta(P)B(L, \xi)N - \eta(L)B(P, \xi)N + \eta(N)B(P, \xi)L - g(L, N)B(P, \xi)\xi.$

Making use of (1.1), (2.2), (2.3), (2.11) and (2.12), the equation (8.6) takes the Form

(8.7) $B(P, L)N = (a_0 + 2na_1 + 2a_2 r)\{-g(P, N)L + g(L, N)P\} + a_1[-S(P, N)L + 2ng(L, P)\eta(N)\xi - \eta(P)S(L, N)\xi - \eta(W)S(P, L)\xi + 2ng(L, N)P + 2n\eta(P)g(L, N)\xi].$

Contracting the equation (8.7) with respect to P, we obtain

(8.8) $S(L, N) = 2ng(L, N).$

Again contracting (8.8) over L and N, we get

(8.9) $r = 2n(2n + 1).$

Theorem 8.2. In a generalized *B* pseudo-symmetric Sasakian manifold, either $L_B = 1$ or the manifold reduces to Einstein manifold with constant scalar curvature $2n^2$.

9. Sasakian Manifold Satisfying *R*(ξ, P) · *B* = 0 Condition

In this section, we study the Sasakian manifold satisfying $R(\xi, P)\cdot B = 0$. Then, we have

(9.1) $R(\xi, P) B(L, M)N - B(R(\xi, P)L, M)N - B(L, R(\xi, P)M) N - B(L, M) R(\xi, P) N = 0.$

We fetch the equation (2.8) into (9.1) to achieve

(9.2) $g(P, B(L, M) N)\xi - \eta(B(L, M)N)P - g(P, L) B(\xi, M)N + \eta(L) B(P, M)N - g(P, M) B(L, \xi)N + \eta(M) B(L, P) N - g(P, N) B(L, M)\xi + \eta(N) B(L, M)P = 0.$

By taking inner product with ξ in (9.2), we have

(9.3) $-g(P, B(L, M)N) - \eta(B(L, M)N)\eta(P) - g(P, L)\eta(B(\xi, M)N) + \eta(L)\eta(B(P, M)N) - g(P, M)\eta(B(L, \xi)N) + \eta(M)\eta(B(L, P)N) - g(P, N)\eta(B(L, M)\xi) + \eta(N)\eta(B(L, M)P) = 0.$

By virtue of (1.1) and on simplification, we acquire

(9.4) $-a_0 g(P, R(L, M)N) + a_1[-2ng(M, N)g(P, L) + 2ng(L, N)g(P, M) + 2ng(P, L)\eta(N)\eta(M) - 2ng(P, M)\eta(N)\eta(L) + S(M, P)\eta(L)\eta(N) - S(L, P)\eta(M)\eta(N)] - 2a_2 r\{g(M, N)g(P, L) - g(L, N)g(P, M)\} = 0.$

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal frame field at any point of the manifold Mn.

If we put $L = P = e_i$ in (9.4) and taking summation over i, $1 \leq i \leq n$, we have

(9.5) $$S(M, N) = -\frac{2(n-1)[a_1 n + a_2 r]}{a_0} g(M, N) + \frac{(2n^2 - r)a_1}{a_0} \eta(M)\eta(N).$$

Again contracting over M and N, we infer

(9.6) $$r = \frac{2n^2(2-n)}{[a_0 + a_1 + 2n(n-1)a_2]}$$

.

As a result, we reach the following determination:

Theorem 9.1. If a Sasakian manifold satisfying $R(\xi, P) \cdot B = 0$ condition, then the manifold is an η-Einstein manifold with constant scalar curvature of the form $\frac{2n^2(2-n)}{[a_0 + a_1 + 2n(n-1)a_2]}$ provided that a0, a1 and a2 are linearly independent to each others.

10. Example of a three-dimensional Sasakian manifold

In this section, we construct an example of a three-dimensional Sasakian manifold. Consider the three dimensional manifold $M = \{(p, q, r) \in \mathbb{R}^3 : r \neq 0\}$ where (p, q, r) are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$e_1 = e^r\left(\frac{\partial}{\partial p} + \frac{\partial}{\partial q}\right),\ e_2 = e^r\left(\frac{\partial}{\partial q} - \frac{\partial}{\partial p}\right),\ e^3 = -\frac{\partial}{\partial r},$$

which are linearly independent at each point of the manifold *M*.

Let the Lorentzian metric g defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0,$$

Let η be the 1-form defined by $\eta(P) = g(P, e_3)$ for any $P \in \chi(M^3)$ which satisfies the relation

$$\eta(e_3) = 1.$$

Let φ be the (1, 1)-tensor field defined by $\varphi(e_1) = -e_2$, $\varphi(e_2) = -e_1$, $\varphi(e_3) = 0$.

Then we have

$$\varphi^2(R) = -R + \eta(R)e_3,$$

$$g(\varphi R, \varphi N) = g(R, N) - \eta(R)\eta(N),$$

for any $R, N \in \chi(M^3)$.

Thus for $e_3 = \xi$, M^3 (φ, ξ, η, g) defines an almost contact metric structure on *M*.

Now, we have

$$[e_1, e_3] = e_1,\ [e_1, e_2] = 0,\ [e_2, e_3] = e_2.$$

The Riemannian connection ∇ of the metric g is given by the Koszuls formula which is

(10.1) $$2g(\nabla_P Q, R) = P\, g(Q, R) + Qg(R, P) - Rg(P, Q) - g(P, [Q, R]) - g(Q, [P, R]) + g(R, [P, Q]).$$

Taking $e_3 = \xi$ and using Koszul's formula, we get the following

$$\nabla_{e1} e_1 = e_1,\ \nabla_{e2} e_1 = 0,\ \nabla_{e3} e_1 = 0,$$

$$\nabla_{e1} e_2 = 0,\ \nabla_{e2} e_2 = e_2,\ \nabla_{e3} e_2 = 0,$$

$\nabla_{e1} e_3 = e_1$, $\nabla_{e2} e_3 = e_2$, $\nabla_{e3} e_3 = 0$.

These result shows that the manifold satisfies

$$\nabla_P \xi = -\varphi P$$

for $\xi = e_3$. Hence the manifold under consideration is a Sasakian manifold of dimension three.

The components of curvature tensor are given as follows:

$$R(e_1, e_2)\, e_1 = -e_2,\ R(e_2, e_3)\, e_1 = 0,\ R(e_1, e_3)\, e_1 = -e_3,$$

$$R(e_1, e_2)\, e_2 = e_1,\ R(e_2, e_3)\, e_2 = -e_3,\ R(e_1, e_3)\, e_2 = 0,$$

$$R(e_1, e_2)e_3 = 0,\ R(e_2, e_3)e_3 = e_2,\ R(e_1, e_3)e_3 = e_1.$$

Using the curvature tensor formulas mentioned above, it can be concluded that

(10.2) $$R(L, M)N = g(M, N)L - g(L, N)M.$$

So, the manifold is of constant curvature. From (8.8) it follows that

(10.3) $$S(M, N) = 2ng(M, N).$$

From above we can determine the Ricci tensor with respect to the Levi-Civita connection ∇:

$$S(e_1, e_1) = 2,\ S(e_2, e_2) = 2,\ S(e_3, e_3) = 2.$$

The scalar curvature with respect to the Levi-Civita connection ∇ given by

$$r = \sum_{i=1}^{3} S(e_i, e_i) = 6.$$

From the above value of scalar curvature, the theorem from section 8 is verified.

11. conclusions

The purpose of this article to investigate the curvature properties of generalized *B*-curvature tensor on Sasakian manifold. First of all we studied the flatness properties of generalized *B*−curvature tensor. Specially, we consider generalized B flat manifold, generalized $\xi - B$ flat and generalized $\varphi - B$ flat Sasakian manifold and we have that the manifold converted to an η Einstein, space of constant scalar curvature and η -Einstein manifold, respectively. Further, if an n-dimensional Sasakian manifold satisfying the condition $B.Q = 0$, then the S^2 of the Ricci tensor S is the linear combination of the Ricci tensor and the metric tensor g. Also, this paper presents if the generalized B −φ− recurrent Sasakian manifold with constant scalar curvature r is reduces to a generalized *B*− locally φ− symmetric space. Sasakian manifold satisfying $B((L, M)Q)P = 0$, $R(\xi, P)P = 0$ condition. In the last section we derived an example which satisfies the theorem.

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