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Triangular Snake Graphs on HMC Labeling

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Abstract: This paper explores HMC labeling for a family of triangular snake graphs such as triangular snake (TS), double triangular snake (DTS), alternate triangular snake (ATS), and double alternate triangular snake (DATS) graphs.

Keywords: IoT-based, Real-time, Self-adjusting, Water quality monitoring, Wireless

1. Introduction

In this article, we focus on finite, undirected, and simple graphs. Denoted by G (V, E) and |V(G)|= a and |E(G)|= b. The concept of cordial labeling, pioneered by Cahit in 1987 [1], laid the foundation for subsequent research. Mean cordial labeling, introduced by Ponraj, M. Sivakumar, and Murugesan Sundaram [2]. Additionally, Geometric mean cordial labeling, proposed by K. Chithralakshmi and K. Nagarajan [3], contributed to the exploration of graph labeling techniques. Harmonic mean cordial (HMC) labeling, introduced by J. Gowri and J. Jayapriya, demonstrated its applicability to path, star, and Bistar graphs [4]. Pratik V. Shah and Dharamvirsinh B. Parmar proved that triangular snake graphs admit integer cordial labeling [5]. S.K. Vaidya and N.B. Vyas showed that alternate snake graphs admit product cordial labeling [6]. The mean cordiality of double triangular snake, alternate triangular snake, and double alternate triangular snake graphs was established by R. Ponraj and S. Sathish Narayanan [7]. Harmonic mean labeling of double triangular snakes was proven by C. Jayasekaran, S. Sandhya, and C. David Raj [8]. In this paper, we explore the HMC labeling behavior of triangular snake, double triangular snake, alternate triangular snake and double alternate triangular snake graphs.

2. Preliminaries

Definition 2.1

A simple graph G = (V, E) is said to be HMC labeling if there exist a function $f:V \rightarrow \{1,2\}$ such that the induced edge function g: E→ {1, 2} defined by g(uv) = $\left|\frac{2f(u)f(v)}{f(u)+f(v)}\right|, f(u), f(v) \neq 0$for each edge and $|v_f(i) - v_f(j)| \le 1$, $|e_g(i) - e_g(j)| \le 1$ where $v_f(x)$- means the quantity of vertices labelled with x, $e_g(x)$ - means the quantity of edges labeled with x, where $x \in \{1,2\}$ respectively.[4]

Definition 2.2

A snake graph is a plane graph made up of a series of tiles arranged so that, beginning with the first tile, each tile is positioned to the right of or above the previous tile. Each tile in this case is a square with four vertices and four edges, and there is exactly one edge shared by any two neighboring tiles.

Definition 2.3

Every edge of a path can be replaced by C_3 is called triangular snake graph TS_m. [7]

Definition 2.4

A double-triangular snake graph by connecting u_t and u_{t+1} with two new vertices, v_t and w_t, 1≤t≤m-1, the network DTS_m is created from the path $u_1u_2... u_m$.[8]

Definition 2.5

An alternate triangular snake ATS_m is obtained from a path $u_1u_2 \ldots u_m$ by joining u_t and u_{t+1} (alternatively) to new vertex v_t. That is every alternate edge of a path is replaced by C_3.[7]

Definition 2.6

Two alternate triangular snakes sharing a common path make up a double alternate triangular snake $DATS_m$. That is, by linking u_t and u_{t+1} (alternatively) to new vertex v_t and w_t, a double alternate triangular snake can be formed from a path $u_1, u_2 \ldots u_m$.

3. Main Results

Theorem 3.1:

TS_m is an HMC graph, $m \geq 2$.

Proof:

Let the m vertices be $u_1, u_2, \ldots u_m$. Join u_t and u_{t+1} to a new vertex v_t for $1 \leq t \leq m-1$. As a result, TS_m has a total of a = 2m-1 vertices and b = 3m−3 edges. For the value of m, there are two possibilities.

State 1: m is even.

Triangle count is odd, when m is even.

Consider an *f*: v → {1, 2} as follows.

$$f(u_t) = \begin{cases} 1, & 1 \leq t < \frac{m}{2} + 1 \\ 2, & \frac{m}{2} + 1 \leq t < m+1 \end{cases}$$

$$f(v_t) = \begin{cases} 2, & \frac{m}{2} \leq t < m \\ 1, & \text{otherwise} \end{cases}$$

State 2: m is odd.

Triangle count is even, when m is odd.

Consider an *f*: v → {1, 2} as follows.

$$f(u_t) = \begin{cases} 1, & 1 \leq t < \frac{m+1}{2} \\ 2, & \frac{m+1}{2} \leq t \leq m \end{cases}$$

$$f(v_t) = \begin{cases} 1, & 1 \leq t < \frac{m+1}{2} \\ 2, & \frac{m+1}{2} \leq t < m \end{cases}$$

State No.	Value of m	Triangle State	Vertex State	Edge State
1	m is even	Odd	$v_f(1) = m - 1$ $v_f(2) = m$	$e_f(1) = \frac{3m-2}{2}$; $e_f(2) = \frac{3m-4}{2}$
2	m is odd	Even	$v_f(1) = m - 1$ $v_f(2) = m$	$e_f(1) = \frac{3m-3}{2}$; $e_f(2) = \frac{3m-3}{2}$

Thus, in both State, we get $|v_f(1) - v_f(2)| \leq 1$ and $|e_f(1) - e_f(2)| \leq 1$. Triangular snake graph T (S_m) is hence an HMC graph.

Theorem 3.2:

DTS_m graph is an HMC graph.

Proof:

For $1 \leq t < m-1$, let $u_1, u_2, \ldots u_m$ be the m vertices. Join u_t and u_{t+1} to a new vertex, v_t and w_t. Because of this, DTS_m has a total of a = 3m-2 vertices and b = 5m−5 edges. For a given value of m, there are two possibilities.

State 1: m is odd.

The count of double triangles has an even value, when m is odd.

Consider f: v → {1, 2} as follows.

$$f\left(u_{\frac{m+1}{2}}\right) = 2$$

$$f(u_t) = \begin{cases} 1, & 1 \leq t < \frac{m+1}{2} \\ 2, & \frac{m+1}{2} < t \leq m \end{cases}$$

$$f(v_t) = \begin{cases} 1, & 1 \leq t < \frac{m+1}{2} \\ 2, & \frac{m+1}{2} \leq t < m \end{cases}$$

$$f(w_t) = \begin{cases} 1, & 1 \leq t < \frac{m+1}{2} \\ 2, & \frac{m+1}{2} \leq t < m \end{cases}$$

State 2: m is even.

Double triangles count is odd, when m is even.

Consider an f: v → {1, 2} as follows.

$$f(u_t) = \begin{cases} 1, & 1 \leq t < \frac{m+2}{2} \\ 2, & \frac{m+2}{2} < t \leq m \end{cases}$$

$$f(v_t) = \begin{cases} 1, & t \leq \frac{m}{2} \\ 2, & t > \frac{m}{2} \end{cases}$$

$$f(w_t) = \begin{cases} 1, & t \leq \frac{m}{2} - 1 \\ 2, & t > \frac{m}{2} - 1 \end{cases}$$

State No.	Value of m	Triangle State	Vertex State	Edge State
1	m is odd	Even	$v_f(1) = \frac{3m-3}{2}$; $v_f(2) = \frac{3m-1}{2}$	$e_f(1) = 5m - 5$ $e_f(2) = 5m - 5$
2	m is even	Odd	$v_f(1) = \frac{3m-2}{2}$; $v_f(2) = \frac{3m-2}{2}$	$e_f(1) = \frac{5m-2}{2}$ $e_f(2) = \frac{5m-8}{2}$

In state 1, we have $|v_f(1) - v_f(2)| \le 1$, $|e_f(1) - e_f(2)| \le 1$. Thus, *f* satisfies both vertex and edge conditions. Hence DTS_m is HMC graph.

In state 2, only vertex condition is satisfied. Hence it is a contradiction to HMC graph.

Thus, we conclude that DTS_m is HMC only when m is odd.

Theorem 3.3:

Under the following circumstances, the alternative triangular snake ATS_m is Hmc labeling for m >3:

a) In the event that the triangle terminates with u_m, m≡ 0, 2 (mod 4) and begins with u_1.

b) In the event that the triangle terminates with u_{m-1}, m≡ 0, 2(mod 4) and begins with u_2.

c) In the event that the triangle begins at u_1 and finishes at u_{m-1}, m ≡ 1(mod 4)

d) In the event that the triangle begins with u_2 and finishes with u_m, m ≡ 1(mod 4)

Proof:

State (a): The triangle's origin is u_1, and its destination is u_m. In the present case $a = \frac{3m}{2}$, $b = 2m - 1$,

Sub State (i): If both the count of triangles and m are even, then

$$m \equiv 0(\text{mod}4)$$
$$m=4p,\ p=1, 2, 3, \ldots$$

First p triangles should be given label 1, and the subsequent p triangles should be given label 2. The vertex labeling *f* above satisfies the HMC criterion, as shown the Tabulate below,

	1	2
$v_f(t)$	m−p	m−p
$e_f(t)$	M	m−1

Tabulate: 1

Sub State (ii): If the count of triangles is odd and m is even,

$$m \equiv 2(\text{mod}4)$$
$$m=4p+2,\ p=1, 2, 3, \ldots$$

Label first p triangles with label 1 and the last p triangles with label 2. The vertex u_{2p+1} and v_{p+1} of the $(p+1)^{th}$ triangle should be assigned the label 1. The aforementioned vertex labeling *f* is shown in Tabulate 2 to satisfy the HMC criterion.

	1	2
$v_f(t)$	m−p−1	m−p
$e_f(t)$	M	m−1

Tabulate: 2

Thus, ATS_m is HMC if the triangle begins from u_1 and finishes with u_m and m ≡ 0,2 (mod4).

State (b): The triangle Begins from u_2 and finishes with u_{m-1}. Here, $a = \frac{3m-2}{2}$, $b = 2m - 3$

Sub State (i): If both the count of triangles and m are even, then

$$m \equiv 2(\text{mod}4)$$
$$m=4p+2,\ p=1, 2, 3, \ldots$$

For first p triangles, label them 1, and for the subsequent p triangles, label them 2, respectively. Lastly, give pendent vertices u_1 and u_m, correspondingly, the labels 1 and 2. Tabulate 3 below confirms that the vertex labeling *f* mentioned above meets the HMC requirements.

	1	2
$v_f(t)$	m−p−1	m−p−1
$e_f(t)$	m−1	m−2

Tabulate: 3

Sub State (ii): If the count of triangles is odd and m is even,

$$m \equiv 0(\mathrm{mod}4)$$
$$m=4p,\ p=1, 2, 3, \dots$$

Provide first (p-1) triangles label 1, and last (p-1) triangles label 2. Allocate label 1 to vertex u_{2t} in the p^{th} triangle, and label 2 to vertices u_{2p+1} and v_p. Lastly, give the pendent vertices, u_1 and u_m, the labels 1 and 2, accordingly.

	1	2
$v_f(t)$	m−p−1	m−p
$e_f(t)$	m−1	m−2

Tabulate: 4

Thus, ATS_m is Hmc if the triangle begins from u_2 and finishes with u_{m-1} & $m \equiv 0,2$ (mod4).

State (c): The triangle begins from u_1 and finishes with u_{m-1}. In this state $a = \frac{3m-1}{2}, b = 2m - 2$

Sub State (i): If the count of triangles is even and m is odd,

$$m \equiv 1(\mathrm{mod}4)$$
$$m=4p+1,\ p=1, 2, 3, \dots$$

Label 1 should be applied to first p triangles, and Label 2 should be applied to subsequent p triangles. Lastly, give pendent vertex u_m the label 2. The vertex labeling *f* above satisfies the HMC criterion, as shown in Tabulate 5.

	1	2
$v_f(t)$	m−p−1	m−p
$e_f(t)$	m−1	m−1

Tabulate: 5

Sub state (ii): If both the count of triangles and m are odd, then,

$$m \equiv 3(\mathrm{mod}4)$$
$$m=4p+3,\ p=1, 2, 3, \dots$$

As in Sub state (ii) of state (a), label a vertices. At last, give the pendent vertex u_m the label 1. In the event that *f* is a Hmc, then *f* cannot satisfy the edge requirement since $v_f(1) = v_f(2)$. It is thus in conflict with HMC labeling. Therefore, ATS_m is HMC for $m \equiv 1(\mathrm{mod}4)$ & ATS_m is not an HMC for $m \equiv 3(\mathrm{mod}4)$ if the triangle begins at u_1 and finishes at u_{m-1}.

State (d): The triangle begins from u_2 and finishes with u_m. In this state $a = \frac{3m-1}{2}, b = 2m - 2$

Sub State (i): If the count of triangles is even and m is odd,

$$m \equiv 1(\mathrm{mod}4)$$
$$m=4p+1,\ p=1, 2, 3, \dots$$

Label first p triangles with label 2 and the subsequent p triangles with label 1. Lastly, give the pendent vertex u_1 label 2. The vertex labeling *f* above satisfies the HMC criterion, as shown in Tabulate 6.

	1	2
$v_f(t)$	m−p−1	m−p
$e_f(t)$	m−1	m−1

Tabulate: 6

Sub State (ii): If both the count of triangles and m are odd, then

$$m \equiv 3(\text{mod}4)$$
$$m = 4p+3,\ p=1, 2, 3, \ldots$$

As in Sub state (ii) of State (a), assign the label to the vertices. In the end, give the pendent vertex u_1 label 1. Assuming that f is an HMC, the edge requirement cannot be satisfied by f since $v_f(1) = v_f(2)$. Therefore, it defies the HMC condition. As a result, if the triangle begins at u_2 and ends at u_m, ATS_m is an HMC for $m \equiv 1(\text{mod}4)$ and not an HMC for $m \equiv 3(\text{mod}4)$.

Theorem 3.4:

$DATS_m$ is an Hmc graph for m>3 under the following conditions:

a) The triangle begins from u_1 and finishes with u_m, $m \equiv 0(\text{mod } 4)$

b) The triangle begins from u_2 and finishes with

U_{m-1}, $m \equiv 2(\text{mod } 4)$

c) The triangle begins from u_1 and finishes with u_{m-1}, $m \equiv 1(\text{mod } 4)$

d) The triangle begins from u_2 and finishes with u_m, $m \equiv 1(\text{mod } 4)$

Proof:

State (a): If the count of double triangles is even and m is odd, then

$$m \equiv 0(\text{mod}4)$$
$$m=4p,\ p=1, 2, 3, \ldots$$

For first p double triangles, label 1; for next p double triangles, label 2. The above vertex labeling f satisfies the HMC criterion, as shown in Tabulate 7.

	1	2
$v_f(t)$	m	M
$e_f(t)$	6p	6p−1

Tabulate: 7

Sub State (ii): If the count of double triangles is odd and m is even, then

$$m \equiv 2(\text{mod}4)$$
$$m = 4p+2,\ p=1, 2, 3, \ldots$$

First p double triangles should be labeled 1 and the last p double triangles should be labeled 2. Assign the labels 1, 2, 1, 2 to the vertex u_{2p+1}, u_{2p+2}, v_{p+1}, and w_{p+1} in the $(p+1)^{th}$ triangle, accordingly. f meets the vertex requirement in this sub state (ii), but not the edge condition. As a result, the HMC condition is violated. As a result, $DATS_m$ is an HMC for $m \equiv 0$ (mod4) and is not an HMC for $m \equiv 2$ (mod4) if the triangle begins at u_1 and finishes at u_m.

State (b): The triangle begins from u_2 and finishes with u_{m-1}. In this state $a = 2m - 2,\ b = 3m - 5$

Sub state (i): If both m and double triangles are even, then

$$m \equiv 2(\text{mod}4)$$
$$m=4p+2,\ p=1, 2, 3, \ldots$$

For the first p double triangles, label 1; for the next p double triangles, label 2. Lastly, give pendent vertices u_1 and u_m, respectively, the labels 1 and 2. The vertex labeling f above satisfies the HMC requirements, as shown in Tabulate 8.

	1	2
$v_f(t)$	m−1	m−1

$e_f(t)$	6p+1	6p

Tabulate: 8

Sub state (ii): If the count of double triangles is odd and m is even, then

m ≡ 0(mod4)
m=4p, p=1, 2, 3, …

Give first (p-1) double triangles the label 1, and the last (p-1) double triangles the label 2. Allocate label 1 to vertices u_{2p} and v_p and label 2 to vertices u_{2p+1} and w_p in the p^{th} double triangle. At last, give the pendent vertices u_1 and u_m, respectively, the labels 1 and 2.

Here, *f* cannot meet the edges requirement since $|v_f(1)\text{-}v_f(2)|\leq 1$. Thus, it defies the HMC condition. As a result, $DATS_m$ is HMC for m ≡ 2 (mod4) and is not an HMC for m ≡ 0 (mod4) if the triangle begins at u_2 and finishes at u_{m-1}.

State (c): The triangle begins from u_1 and finishes with u_{m-1}. In this state $a = 2m - 1, b = 3m\text{-}3,$

Sub State (i): When m is odd, and number of double triangles is even. then

m ≡ 1(mod4)
m=4p+1, , p=1, 2, 3, …

Initial p double triangles should be labeled 1 and the following p double triangles should be labeled 2. Lastly, give the pendent vertex u_m the label 2. The vertex labeling *f* above satisfies the HMC criterion, as shown in Tabulate 9.

	1	2
$v_f(t)$	m−1	m
$e_f(t)$	6p	6p

Tabulate: 9

Sub state (ii): If both m and double triangles are odd, then

m ≡ 3(mod4)
m=4p+3

First p double triangles should be labeled 1 and the last p double triangles should be labeled 2. Assign label 1 to vertices u_{2p+1}, v_{p+1} in the $(p+1)^{th}$ double triangle, and label 2 to vertices u_{2p+2}, w_p. Lastly, give the pendent vertex u_m label 1 or 2. In this instance, *f* cannot meet the edges requirement since $|v(1)\text{-}v_f(2)|\leq 1$. As a result, the HMC condition is violated. As a result, $DAT(S_m)$ is an HMC for m ≡ 1 (mod4) and is not an HMC for m ≡ 3 (mod4) if the triangle begins at u_1 and finishes at u_{m-1}.

State (d): The triangle begins from u_2 and finishes with u_m. In this state $a = 2m - 1, b = 3m - 3$

Sub state (i): The count of double triangles is even and m is odd,

Label the first p double triangles with label 2 and the subsequent p double triangles with label 1. Lastly, give the pendent vertex u_1 the label 2. The above vertex labeling *f* meets the Hmc criterion, as shown in Tabulate 10.

	1	2
$v_f(t)$	m−1	m
$e_f(t)$	6p	6p

Tabulate: 10

Sub state (ii): If both m and double triangles are odd, then

m ≡ 3(mod4)
m=4p+3

Label the vertices in accordance with Sub state (ii) of State C. Lastly, give the pendent vertex u_1 a label of 1 or 2. Here, f cannot meet the edges requirement since $|v_f(1)-v_f(2)|\leq 1$. Thus, it defies the HMC condition. Therefore, $DATS_m$ an Hmc for m ≡1(mod4) and not an HMC for m≡3(mod4) if the triangles begin at u_2 and terminate at u_m.

4. Conclusion

Our findings indicate that a family of triangular snake graphs, which includes the Triangular Snake TS, Double Triangular Snake DTS, Alternate Triangular Snake ATS and Double Alternate Triangular Snake graphs DATS satisfy HMC labeling, contingent on the value of m and the quantity of triangles included in the graph.

References

[1] Gallian J. A “A Dynamic Survey of graph labeling”, ‘Electronic Journal of combinatorics’ (2019) 3.7 pp79-95.
[2] Ponraj. R and Sivakumar. M, Sundaram. M, “Mean cordial labeling of graphs”, Open Journal of Discrete Mathematics, Vol. 2 No. 4: 145–148.
[3] Chithralakshmi. K, Nagarajan. K “Geometric mean cordial labeling of graphs”, ‘International Journal of mathematics and soft computing’, Vol. 7, No. 1, (2017) pp75–87.
[4] Gowri. J and Jayapriya. J “Hmc Labeling of Certain Types of Graphs”, ‘Turkish Journal of Computer and Mathematics Education’, Vol.12 No.10(2021),3913-3915.
[5] Pratik V. Shah, Dharamvirsinh B. Parmar, "Integer Cordial labeling of triangular snake graph "International Journal of Scientific Research and Reviews "2019,8(1),3118-3126, ISSN:2279-0543.
[6] S.K. Vaidya and N.B. Vyas "Product Cordial Labeling for alternate snake graphs", 'Malaya Journal of Matematik' 2(3) (2014) 188-196, ISSN: 2319-3786.
[7] R. Ponraj and S. Sathish Narayanan, "Mean Cordiality of some snake graphs ", 'Palestine Journal of Mathematics ' vol.4(2) (2015) 439-445.
[8] C. Jayasekaran, S. S. Sandhya and C. David Raj, "Harmonic Mean Labeling on Double Triangular Snake Graphs", 'International Journal of Mathematics Research', ISSN 0976-5840, vol.5 Number 2 (2013) pp 251-256, International Research Publication House, http://www.irphouse.com