Inverse Double Domination in Graphs

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Abstract

A vertex set $S \subseteq V(G)$ is a double – dominating set, if each vertex $v \in V\backslash S, v$ is adjacent to at least two vertices in S. The minimum cardinality of a double dominating set of G, denoted by $\gamma_{dd}(G)$ is called the double domination number of G.

A double dominating set S of a graph $G = (V, E)$ is an inverse double dominating set, if $V - S$ contains a double dominating set say S' of G. Then S' is called an inverse double dominating set. The inverse double domination number $\gamma_{dd}{}^{-1}(G)$ of G is the minimum cardinality of an inverse double dominating set.

In this paper, we obtain exact value of $\gamma_{dd}{}^{-1}(G)$ for some standard graphs and also, we establish some general results on this parameters.

Keywords: graph, domination number, double domination number, inverse domination number.

I Introduction

All graphs considered are undirected finite simple graph only and we refer in the two books by Haynes, Hedetniemi and Slater [2,3]. Let $G = (V, E)$ be a graph, the open neighborhood $N(v)$ of a vertex $v \in V(G)$ is $\{u: uv \in E(G)\}$ and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. The $deg(v)$ (or $d(v) = |N(v)|$) of v is the number of edges incident with v in G. Since G is simple undirected graph. $\delta(G)$ *and* $\Delta(G)$ is the minimum and maximum degrees of a graph G.

A set $S \subseteq V(G)$ is a dominating set of a graph G, if each vertex $v \in V(G), |N[v] \cap S| \geq 1$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in a graph G.

A set S' is an inverse dominating set of G if $S'$is a dominating set of $G$and $S' \subseteq V/S$ for some dominating set S. The inverse domination number of G, denoted $\gamma^{-1}(G)$, is the minimum cardinality among all inverse dominating set of G. An inverse dominating set of $G$of cardinality $\gamma^{-1}(G)$ we call $\gamma^{-1}(G)$ – set. If S' is a $\gamma^{-1}(G)$ – set and S is a $\gamma(G)$ – set.

In communication network, let S denote the set of transmitting stations so that every station not belonging to S has a link with at least one station in S. If this set of stations fails, then one has to find another disjoint such set of stations. This leads to define the domatic number $d(G)$ of G is the maximum number of disjoint dominating set in G. This concept was defined by Cockayne and Hedetniemi. In Kulli and Sigarkanti, consider the problem of selecting two disjoint sets of transmitting stations $S_1(S_2)$ so that every station not belonging to $S_1(S_2)$ has a link with at least one station in $S_1(S_2)$, where $|S_1|$ and $|S_1 \cup S_2|$ are minimum among all the pairs of disjoint transmitting stations. This leads us to define the inverse domination number.

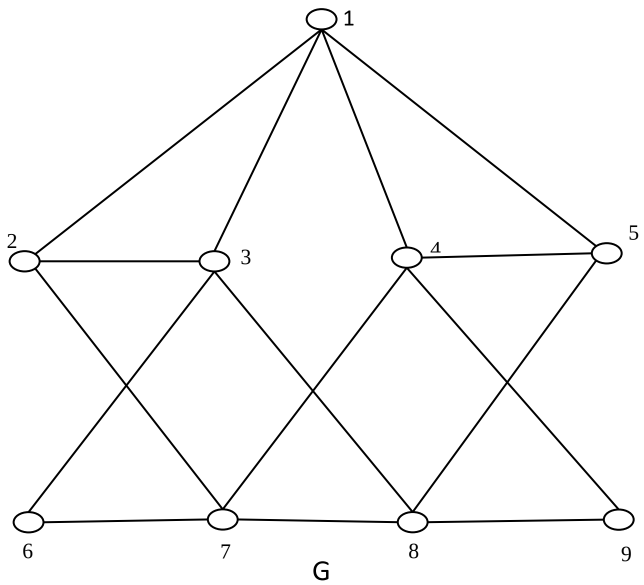
Ii Inverse Double Dominating Sets

Definition 2.1 A dominating set is said to be double dominating set, if every vertex in $V - S$ is adjacent to at least two vertices in S. The minimum cardinality taken over all, the minimal double dominating set is called double domination number and is denoted by $\gamma_{dd}(G)$.

Remark 2.1 Every double dominating is a dominating set but converse need not be true.

Definition 2.2 A double dominating set S of a graph $G = (V, E)$ is an inverse double dominating set, if $V - S$ contains a double dominating set say S' of G. Then S' is called an inverse double dominating set. The inverse double domination number $\gamma_{dd}{}^{-1}(G)$ of G is the order of a smallest inverse double dominating set of G.

Example 2.1 Consider a graph G



G

{2, 5, 6, 9}, {3, 4, 7, 8}, {1, 3, 4, 7, 8} and {1, 2, 5, 6, 9} is a double dominating set and $\gamma_{dd}(G) = 4$.

{2, 5, 6, 9} , {3, 4, 7, 8}, {1, 3, 4, 7, 8}and {1, 2, 5, 6, 9} is an inverse double dominating set and $\gamma_{dd}{}^{-1}(G) = 4$.

Remark 2.2 Every inverse double dominating is a double dominating set but every double dominating set is not an inverse double dominating set.

Remark 2.3 An inverse double dominating set of a graph G may or may not be a minimal double dominating set.

Iii Main Results

First we obtain exact values of $\gamma_{dd}{}^{-1}(G)$ for some standard graphs.

Proposition 3.1. For any graph G,

$$\gamma_{dd}(G) \leq \gamma_{dd}{}^{-1}(G)\,. \qquad (1)$$

Proof: Clearly every inverse double dominating set of a graph G is a double dominating set of a graph G. Thus (1) holds.

Proposition 3.2. For any cycle C_p with $p \geq 5$ vertices

$$\gamma_{dd}{}^{-1}(C_p) = \lceil p/3 \rceil.$$

Proof: Let S be the minimal double dominating set of C_p,

$$\gamma_{dd}(C_p) = \lceil p/3 \rceil.$$

Then $S' = \{v_i \mid v_{i+1} \in S\}$ is an inverse double dominating set of C_p. Then

$$\gamma_{dd}{}^{-1}(C_p) = |S'| = \lceil p/3 \rceil.$$

Proposition 3.3. For any path P_p with $p \geq 5$ vertices

$$\gamma_{dd}{}^{-1}(P_p) = \lceil p/2 \rceil \text{ if } p \equiv 0(mod\ 3)$$

$$\gamma_{dd}{}^{-1}(P_p) = \lceil p/3 \rceil \text{ otherwise.}$$

Proof: Here we consider two cases.

Case 1: Suppose $p \equiv 0(mod\ 3)$. In this case there exist a unique minimum double dominating set $S = \{v_{3i-1}\ |1 \leq i \leq \frac{p}{3}\}$ where each $v_i \in V(P_p)$. Then $S' = \{v_i\ |v_{i+1} \in S\} \cup \{v_p\}$ is an inverse double dominating set. Thus

$$\gamma_{dd}{}^{-1}(P_p) = |S'| = |D| + 1 = \lceil p/3 \rceil + 1.$$

Case 2: Suppose $p \not\equiv 0(mod\ 3)$. In this case there exist a unique minimum double dominating setin which the end vertex v_p is included. Then $S' = \{v_i\ |v_{i+1} \in S\}$ is an inverse double dominating set. Thus

$$\gamma_{dd}{}^{-1}(P_p) = |S'| = |D| = \lceil p/3 \rceil.$$

Theorem 3.1. For any graph G,

$$\gamma_{dd}{}^{-1}(G) \leq \beta_0(G).$$

Proof: Let S be the minimal double dominating set of G. Let S' be the maximal independent set of $(V - S)$. Here we consider two cases.

Case 1: Suppose $V - S - S' = \emptyset$. Then $V - S = S'$ is an independent inverse double dominating set of G. Thus

$$\gamma_{dd}{}^{-1}(G) \leq |V - S| = |S'| \leq \beta_0(G).$$

Case 1: Suppose $V - S - S' \neq \emptyset$. Then every vertex in $V - S - S'$ is adjacent to at least two vertex in S'. If every vertex in S is adjacent to at least two vertex in S', then S' is an inverse double dominating set of G. Otherwise, $D \subset S$ be a set of vertices in D such that no vertex of D' is adjacent to the vertex of S. Since D is a minimum double dominating set, every vertex in D' must be adjacent to at least two vertex in $V - S - S'$. Let $D'' \subset V - S - S'$, be such that every vertex of D' is adjacent to at least two vertex in D''. Clearly $|D''| \leq |D'|$ and $S' \cup D''$ is an inverse double dominating set. Thus

$$\gamma_{dd}{}^{-1}(G) \leq |S' \cup D''| \leq |S' \cup D'| \leq \beta_0(G).$$

We now give a sufficient condition for which

$$\gamma_{dd}(G) = \gamma_{dd}{}^{-1}(G).$$

Theorem 3.2. Let S be a minimum double dominating set of G. If for every vertex $v \in S$, the induced subgraph $\langle N[v] \rangle$ is a complete graph of order at least four, then

$$\gamma_{dd}(G) = \gamma_{dd}{}^{-1}(G).$$

Proof: Let $S = \{u_1, u_2, \ldots, u_n\}$ be a minimum double dominating set of G. Let $v_1, v_2, \ldots, v_n$ be the vertices adjacent to $u_1, u_2, \ldots, u_n$ respectively. By the assumption, for each vertex $u_i \in S$, the graph $\langle N[u_i] \rangle$ is complete. Then $N[u_i] \subset N[v_i]$. Therefore $V(G) = N[u_1] \cup N[u_2] \cup \ldots \cup N[u_n] \subset N[v_1] \cup N[v_2] \cup \ldots \cup N[v_n] = V(G)$. Therefore $\{v_1, v_2, \ldots, v_n\}$ is an inverse double dominating set of G. Thus

$$\gamma_{dd}{}^{-1}(G) = |S| = \gamma_{dd}(G).$$

Theorem 3.3. Let τ denote the family of minimum double dominating set of G. If for every minimum double dominating set $S \in \tau$, $V - S$ is independent, then

$$\gamma_{dd}(G) + \gamma_{dd}{}^{-1}(G) = p.$$

Proof: Since for every minimum double dominating set S, $V - S$ is independent, therefore $V - S$ itself is a minimum inverse double dominating set of G. Then

$$\gamma_{dd}(G) + \gamma_{dd}{}^{-1}(G) = p.$$

Theorem 3.4. Let T be a tree such that every non – end vertex is adjacent to at least one end vertex. Then

$$\gamma_{dd}(T) + \gamma_{dd}{}^{-1}(T) = p.$$

Proof: Let T be a tree as in the Statement. If every non – end vertex of T is adjacent to at least two end vertices, then the set of all non – end vertices is a minimum double dominating set and the set of all end vertices is a minimum inverse double dominating set. Suppose there are non – end vertices which are adjacent to exactly one end vertex. Let S and S' denote the minimum double dominating and inverse double dominating sets respectively. Let u be a non – end vertex adjacent to exactly two end vertex v. If $u \in S$, then $v \in S'$, and $u \in S'$, then $v \in S$. In any case

$$|S| + |S'| = p$$

Thus

$$\gamma_{dd}(T) + \gamma_{dd}{}^{-1}(T) = p.$$

Theorem 3.4. Let T be a (p, q) graph with $\gamma_{dd}(G) = \gamma_{dd}{}^{-1}(G)$. Then $q \geq 2p - 3\gamma_{dd}(G)$.

Proof: Let S and S' be minimum double dominating and inverse double dominating sets of G respectively. Then

$$q \geq |V(G) - S - S'|2 + |S|$$
$$= \left(p - 2\gamma_{dd}(G)\right)2 + \gamma_{dd}(G)$$
$$= 2p - 3\gamma_{dd}(G).$$

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