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Oscillatory Behavior of Second Order Mixed Functional Nonlinear Differential Equations with Superlinear Neutral Terms

[1]Dr. R. Rama, [2]S. Chithambara Bharathy, [3]Dr. R. Sridevi
[1]*Quaid-E-Millath Govt College for Women,Chennai-600002, India.*
[2]*Quaid-E-Millath Govt College for Women,Chennai-600002, India.*
[3]*Meenakshi College for Women,Chennai-600024, India.*

Abstract: This paper finds some new oscillation conditions of second order mixed functional nonlinear differential equations with superlinear neutral terms of the form

$$\left(b(\phi)\, w'(\phi)\right)' - p(\phi)\, \mathrm{f}\Big(v\big(\xi(\phi)\big)\Big) = 0, \qquad \phi \geq \phi_0 > 0,$$

where $w(\phi) = v(\phi) + q_1(\phi)\, v^{\theta}\big(\chi(\phi)\big) - q_2(\phi)\, v^{\theta}\big(\mu(\phi)\big)$, θ is the ratio of odd positive integers with $\theta > 1$. Moreover, $\chi(\phi) \leq \phi \leq \mu(\phi)$ and $\xi(\phi)$ is a mixed type deviating argument. The results obtained here extend, simplify, and generalize existing ones in the literature. Examples are given to demonstrate the results.

***Keywords**:* Delay, Advanced, Oscillation, Superlinear neutral terms.

1. Introduction

The present paper is focused on the oscillatory behavior of solutions to the second order mixed functional nonlinear differential equations with superlinear neutral terms of the following form

$$\left(b(\phi)\, w'(\phi)\right)' - p(\phi)\, \mathrm{f}\Big(v\big(\xi(\phi)\big)\Big) = 0, \qquad \phi \geq \phi_0 > 0, \tag{1.1}$$

where $w(\phi) = v(\phi) + q_1(\phi)\, v^{\theta}\big(\chi(\phi)\big) - q_2(\phi)\, v^{\theta}\big(\mu(\phi)\big)$ and b, q_1, q_2, p, χ, μ, ξ are continuous real-valued functions on $[\phi_0, \infty)$.

Without further mention, throughout, the following hypotheses hold:

H1: θ is the ratio of odd positive integers with $\theta > 1$;

H2: b $\in C^1([\phi_0, \infty), R^+)$ and

$$\mathrm{I}(\phi) = \int_{\phi_0}^{\phi} \frac{1}{b(s)}\, ds \longrightarrow \infty \text{ as } \phi \longrightarrow \infty; \tag{1.2}$$

H3: p, q_1, $q_2 \in C([\phi_0, \infty), [0, \infty))$ and q is a positive continuous real-valued function with $0 < q_2 \leq \mathrm{q} < 1$;

H4: f $\in C(R, R)$ and, $\exists\, k > 0 \ni \mathrm{f(x)} \geq k\, x^{\alpha}$, $\forall$ x $\neq 0$, where α is the ratio of odd positive integers;

H5: $\chi, \mu \in C([\phi_0, \infty), R), \chi(\phi)) \leq \phi \leq \mu(\phi)$, χ *and* μ are strictly increasing functions and $\lim_{\phi \to \infty} \chi(\phi) = \lim_{\phi \to \infty} \mu(\phi) = \infty$;

H6: $\xi \in C^1([\phi_0, \infty), R)$, $\xi'(\phi) > 0$ and $\lim_{\phi \to \infty} \xi(\phi) = \infty$.

It is worth noting that $\xi(\phi)$ is of mixed type which means that its delayed part

$D_{\xi} = \{\phi \in [\phi_0, \infty) : \xi(\phi) < \phi\}$

and its advanced part

$A_{\xi} = \{\phi \in [\phi_0, \infty): \xi(\phi) > \phi\}$

are both unbounded subsets of $[\phi_0, \infty)$.

By a solution of (1.1), we mean a function $v(\phi) \in C([\phi_v, \infty), R)$, with w, $b(\phi)w'(\phi) \in C^1([\phi_v, \infty), R)$ that satisfies the differential equation (1.1) on $[\phi_v, \infty)$ where $\phi_v \geq \phi_0$. Without further mention, we will assume throughout that the solutions that satisfy

$\sup\{|v(\phi)|: \phi \geq T\} > 0, \forall\, T \geq \phi_v$.

A solution $v(\phi)$ of (1.1) is called oscillatory if it has arbitrarily large zeros on $[\phi_v, \infty)$, that is, $\forall\, \phi_1 \in [\phi_v, \infty)\ \exists\ \phi_2 \geq \phi_1 \ni v(\phi_2) = 0$; if not, it is called nonoscillatory, that is, if it is eventually positive or eventually negative. If every solution to (1.1) is oscillatory, then (1.1) is called oscillatory.

The set W of all nonoscillatory solutions of (1.1) is the union

$$W = \cup_{j=1}^{j=4} W_j,$$

where

$W_1 : w(\phi) > 0$ and $w'(\phi) < 0$; $W_2 : w(\phi) > 0$ and $w'(\phi) > 0$;

$W_3 : w(\phi) < 0$ and $w'(\phi) < 0$; $W_4 : w(\phi) < 0$ and $w'(\phi) > 0$.

We consider the situation that $W = \emptyset$ for (1.1), i.e., every nontrivial solution to (1.1) is oscillatory.

2. Main results

Lemma 2.1 [7]. If X, Y > 0, then

$$X^{\lambda} + (\lambda - 1)Y^{\lambda} - \lambda X Y^{\lambda-1} \geq 0, \qquad \text{for } \lambda > 1, \tag{2.1}$$

and

$$X^{\lambda} - (1 - \lambda)Y^{\lambda} - \lambda X Y^{\lambda-1} \leq 0, \qquad \text{for } 0 < \lambda < 1, \tag{2.2}$$

where equalities hold if, and only if, X = Y.

For convenience, we utilize

$B_1(\phi) := (\theta - 1)\, \theta^{\frac{\theta}{1-\theta}}\, q^{\frac{\theta}{\theta-1}}(\phi)\, q_1^{\frac{1}{1-\theta}}(\phi)$,

$B_2(\phi) := (\theta - 1)\, \theta^{\frac{\theta}{1-\theta}}\, q^{\frac{\theta}{\theta-1}}(\phi)\, q_2^{\frac{1}{1-\theta}}(\phi)$,

and

$B_3(\phi) := 1 - \left(\frac{B_2(\phi) - B_1(\phi)}{c_1\, I(\phi)}\right)$,

$B_4(\phi) := 1 - \left(\frac{B_2(\phi) - B_1(\phi)}{c_2}\right)$,

$c_1 < 0$, $c_2 > 0$ are constants.

Theorem 2.1. Assume $q(\phi)$ is such that

$$\lim_{\phi \to \infty} [B_2(\phi) - B_1(\phi)] = 0. \tag{2.3}$$

Moreover, assume the following condition:

$$k \int_{\phi_0}^{\infty} \frac{1}{b(u)} \left(\int_u^{\infty} p(s)\, ds\right) du = \infty. \tag{2.4}$$

If $\exists$ $\{\phi_k\}$, $\{s_k\}$with ϕ_k, $s_k \longrightarrow \infty$ as $k \longrightarrow \infty \ni \phi_k \in D_\xi$ and $s_k \in A_\xi$ respectively,

$$\limsup_{k \longrightarrow \infty} \int_{\xi(\phi_k)}^{\phi_k} k\ n_1\, p(s)\, [I(\xi(\phi_k)) - I(\xi(s))]\, ds > 1, \tag{2.5}$$

and

$$\limsup_{k \longrightarrow \infty} \int_{s_k}^{\xi(s_k)} k\ n_2\, p(s)\, [I(\xi(s)) - I(\xi(s_k))]\, ds > 1, \tag{2.6}$$

$\forall$ k = 1, 2, 3, …, and $k > 0$, where n_1 and n_2 are constants, then (1.1) is oscillatory.

Proof. Assume, on the contrary, that $v(\phi)$ is a nonoscillatory solution of (1.1). We can assume, without losing generality, that $v(\phi)$ is an eventually positive solution of (1.1) because, if $v(\phi)$ is a solution of (1.1), then $-\ v(\phi)$ is also a solution of (1.1). Hence, we get $v(\phi) > 0$, $v(\chi(\phi)) > 0$, $v(\mu(\phi)) > 0$, $v(\xi(\phi)) > 0$, for large sufficient ϕ. Then the following cases arise:

Case I. Suppose that $w(\phi) \in W_1$. By the definition of $w(\phi)$, we have

$$\begin{aligned} v(\phi) &= w(\phi) - q_1(\phi)\, v^\theta(\chi(\phi)) + q_2(\phi)\, v^\theta(\mu(\phi)) \\ &= w(\phi) + \Big(q(\phi)\, v(\chi(\phi)) - q_1(\phi)\, v^\theta(\chi(\phi)) \Big) - \Big(q(\phi)\, v(\mu(\phi)) - q_2(\phi)\, v^\theta(\mu(\phi)) \Big). \end{aligned} \tag{2.7}$$

Put $\lambda = \theta > 1$, $X = q_1^{\frac{1}{\theta}}(\phi)\, v(\chi(\phi))$ and $Y = \left(\frac{1}{\theta}\, q(\phi)\, q_1^{\frac{-1}{\theta}}(\phi) \right)^{\frac{1}{\theta - 1}}$ in (2.1), we get

$$q(\phi)\, v(\chi(\phi)) - q_1(\phi)\, v^\theta(\chi(\phi)) \le (\theta - 1)\, \theta^{\frac{\theta}{1-\theta}}\, q^{\frac{\theta}{\theta-1}}(\phi)\, q_1^{\frac{1}{1-\theta}}(\phi) := B_1(\phi). \tag{2.8}$$

Put $\lambda = \theta > 1$, $X = q_2^{\frac{1}{\theta}}(\phi)\, v(\mu(\phi))$ and $Y = \left(\frac{1}{\theta}\, q(\phi)\, q_2^{\frac{-1}{\theta}}(\phi) \right)^{\frac{1}{\theta - 1}}$ in (2.1), we get

$$q(\phi)\, v(\mu(\phi)) - q_2(\phi)\, v^\theta(\mu(\phi)) \le (\theta - 1)\, \theta^{\frac{\theta}{1-\theta}}\, q^{\frac{\theta}{\theta-1}}(\phi)\, q_2^{\frac{1}{1-\theta}}(\phi) := B_2(\phi). \tag{2.9}$$

Using (2.8) and (2.9) in (2.7) we get

$$v(\phi) \ge \left[1 - \left(\frac{B_2(\phi) - B_1(\phi)}{w(\phi)} \right) \right] w(\phi). \tag{2.10}$$

From (1.1),

$$\big(b(\phi)\, w'(\phi) \big)' = p(\phi)\, \mathrm{f}\Big(v(\xi(\phi)) \Big) \ge 0, \tag{2.11}$$

therefore $\mathrm{b}(\phi) w'(\phi)$ is increasing for $\phi \ge \phi_0$. It is easy to obtain

$$\begin{aligned} w(\phi) &= \int_{\phi_0}^{\phi} \frac{b(s)\, w'(s)}{b(s)}\, ds \\ &\ge b(\phi_0)\, w'(\phi_0) \int_{\phi_0}^{\phi} \frac{1}{b(s)}\, ds \\ &= c_1\, I(\phi), \end{aligned}$$

where $c_1 = \mathrm{b}(\phi_0)\, w'(\phi_0) < 0$ is a constant. Hence

$$v(\phi) \ \ge\ \left[1 - \left(\frac{B_2(\phi) - B_1(\phi)}{c_1\, I(\phi)} \right) \right] w(\phi) := B_3(\phi)\, w(\phi). \tag{2.12}$$

Now, $\exists$ a constant $n_1 \ge 1 \ni$

$$v(\phi) \ge n_1\, w(\phi). \tag{2.13}$$

Taking into account the fact that $\xi'(\phi) > 0$, it is easy to see that $\phi_k \in D_\xi$ implies that $(\xi(\phi_k), \phi_k) \subset D_\xi$.

From (1.1), we have

$$\big(b(\phi)\, w'(\phi)\big)' \geq\ k\ \mathrm{p}(\phi)\ v^{\alpha}\big(\xi(\phi)\big) \geq\ k\ \mathrm{p}(\phi)\ v\big(\xi(\phi)\big).$$

Now integrating (1.1) from $\xi(\phi_k)$ to ϕ_k, and using (2.13), we get

$$-\, b\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big) \geq\ k\ \int_{\xi(\phi_k)}^{\phi_k}\ p(s)\, v\big(\xi(s)\big)\ ds$$

$$\geq\ k\, n_1 \int_{\xi(\phi_k)}^{\phi_k}\ p(s)\, w\big(\xi(s)\big)\ ds. \qquad (2.14)$$

For s $\in (\xi(\phi_k), \phi_k)$, we get

$$w\big(\xi(s)\big)\ \geq\ \int_{\xi(s)}^{\xi(\phi_k)}\ \frac{-\,b(u)\, w'(u)}{b(u)}\, du$$

$$\geq -\, b\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big)\ \int_{\xi(s)}^{\xi(\phi_k)}\ \frac{1}{b(u)}\, du$$

$$= -\, b\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big)\, \big[I\big(\xi(\phi_k)\big) - I\big(\xi(s)\big)\big],$$

therefore, in view of (2.14) implies

$$-\, b\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big) \geq\ k\ \ n_1 \int_{\xi(\phi_k)}^{\phi_k}\ p(s)\, w\big(\xi(s)\big)\ ds,$$

that is

$$-\, b\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big) \geq -\, k\, n_1\, b\big(\xi(\phi_k)\big)\, w'\big(\xi(\phi_k)\big) \int_{\xi(\phi_k)}^{\phi_k} p(s)\, \big[I\big(\xi(\phi_k)\big) - I\big(\xi(s)\big)\big] ds,$$

that is

$$1 \geq \int_{\xi(\phi_k)}^{\phi_k}\ k\, n_1\, p(s) \big[I\big(\xi(\phi_k)\big) - I\big(\xi(s)\big)\big] ds.$$

Taking limit supremum as $k\ \longrightarrow\ \infty$, we get a contradiction to (2.5), and hence, $W_1\ =\ \emptyset$, that is, (1.1) is oscillatory.

Case II. Suppose that $w(\phi) \in W_2$. Since $w(\phi)$ is increasing, ∃ a constant $c_2\ >\ 0$ ∋ $w(\phi) \geq c_2$ for large sufficient ϕ, and so, writing (2.10) as

$$v(\phi)\ \geq\ \left[1 - \left(\frac{B_2(\phi) - B_1(\phi)}{c_2}\right)\right]\ w(\phi) :=\, B_4(\phi)\, w(\phi). \qquad (2.15)$$

Now, ∃ a positive constant $n_2 \in (0,1)$ ∋

$$v(\phi) \geq n_2\ w\ (\phi). \qquad (2.16)$$

There exists $\{s_k\}$ ∋ $s_k \in A_{\xi}$, and given that $\xi(\phi)$ is increasing, which implies that $\big(s_k, \xi(s_k)\big) \subset A_{\xi}$.

From (1.1), we have

$$\big(b(\phi)\, w'(\phi)\big)' \geq k\ \mathrm{p}(\phi)\ \ v^{\alpha}\big(\xi(\phi)\big) \geq k\ \mathrm{p}(\phi)\ \ v\big(\xi(\phi)\big). \qquad (2.17)$$

Now, integrating (2.17) from s_k to $\xi(s_k)$ and using $(\mathrm{b}(\phi)\ w'(\ \phi))' > 0$ and (2.16), we get

$$b\big(\xi(s_k)\big)\, w'\big(\xi(s_k)\big) \geq\ k\ \ \int_{s_k}^{\xi(s_k)}\ \ p(s) v\big(\xi(s)\big)\ \ ds$$

$$\geq\ k\ \ n_2 \int_{s_k}^{\xi(s_k)}\ \ p(s) w\big(\xi(s)\big)\ ds. \qquad (2.18)$$

For s $\in \big(s_k, \xi(s_k)\big)$, we get

$$w\big(\xi(s)\big) \geq\ \int_{\xi(s_k)}^{\xi(s)}\ \frac{b(u)\, w'(u)}{b(u)}\ du$$

$$\geq b(\xi(s_k))\, w'(\xi(s_k)) \int_{\xi(s_k)}^{\xi(s)} \frac{1}{b(u)}\, du$$

$$= b(\xi(s_k))\, w'(\xi(s_k)) \left[I(\xi(s)) - I(\xi(s_k))\right] \tag{2.19}$$

and so, taking (2.18) into account, we get

$$b(\xi(s_k))\, w'(\xi(s_k)) \geq k\, n_2 \int_{s_k}^{\xi(s_k)} p(s) w(\xi(s))\, ds,$$

that is

$$b(\xi(s_k))\, w'(\xi(s_k)) \geq k\, n_2\, b(\xi(s_k))\, w'(\xi(s_k)) \int_{s_k}^{\xi(s_k)} p(s) \left[I(\xi(s)) - I(\xi(s_k))\right] ds,$$

that is

$$1 \geq \int_{s_k}^{\xi(s_k)} k\, n_2\, p(s)\left[I(\xi(s)) - I(\xi(s_k))\right] ds.$$

Taking limit supremum as $k \longrightarrow \infty$, we get a contradiction to (2.6), and hence, $W_2 = \emptyset$, that is, (1.1) is oscillatory.

Case III. Suppose that $w(\phi) \in W_3$. Here $w(\phi)$ satisfies

either

$$\lim_{\phi \longrightarrow \infty} w(\phi) = -\infty \tag{2.20}$$

or

$$\lim_{\phi \longrightarrow \infty} w(\phi) = k_1 < 0. \tag{2.21}$$

We claim that (2.20) is valid. Otherwise, from the definition of $w(\phi)$, we get

$$v(\phi) \geq \left(\frac{-\, w(\mu^{-1}(\phi))}{q}\right)^{\frac{1}{\theta}}, \qquad \phi \geq \phi_1.$$

It is obvious that $v(\phi)$ is bounded and $\exists$ a constant $M_1 \ni v(\phi) \geq M_1 > 0$ $\forall \phi \geq \phi_2 \geq \phi_1$. Using (2.17), we get

$$\left(b(\phi)\, w'(\phi)\right)' \geq M_1\, k\, \mathrm{p}(\phi), \qquad \phi \geq \phi_2. \tag{2.22}$$

Integrating (2.22) from ϕ to u and then put u $\longrightarrow \infty$, we have

$$-\, b(\phi)\, w'(\phi) \geq M_1\, k \int_{\phi}^{\infty} p(s)\, ds.$$

Now, integrating this from ϕ_2 to ϕ and then put $\phi \longrightarrow \infty$, we obtain

$$\lim_{\phi \longrightarrow \infty} w(\phi) \leq -\, M_1\, k \int_{\phi_2}^{\infty} \frac{1}{b(u)} \left(\int_u^{\infty} p(s)\, ds\right) du.$$

This contradicts with (2.21) from (2.4). Hence (2.20) is valid and $W_3 = \emptyset$.

Case IV. Suppose that $w(\phi) \in W_4$. Since $\mathrm{b}(\phi) w'(\phi)$ is positive and increasing, $\exists$ a constant $M_2 > 0 \ni$

$$\mathrm{b}(\phi)\, w'(\phi) \geq M_2, \;\; \forall\, \phi \geq \phi_1. \tag{2.23}$$

Integrating (2.23) from ϕ_1 to ϕ and taking $\phi \longrightarrow \infty$ yields

$$\lim_{\phi \longrightarrow \infty} w(\phi) \geq w(\phi_1) + M_2 \int_{\phi_1}^{\infty} \frac{1}{b(s)}\, ds,$$

which is impossible due to (1.2). Thus $W_4 = \emptyset$, and completes the proof. ■

Our next thought is the further development of Theorem 2.1. Let us state the required lemmas as follows in order to accomplish this.

Lemma 2.2. Suppose there exists $\{\phi_k\}$, $\phi_k \longrightarrow \infty$ as $k \longrightarrow \infty$, such that $\phi_k \in D_\xi$. Let $\exists\ \gamma > 0 \ni$

$$k\, n_1 \left[I(\phi) - I(\xi(\phi))\right] I(\phi)\, b(\phi)\, p(\phi) \geq \gamma, \; on \; \left(\xi(\xi(\phi_k)), \xi(\phi_k)\right), \; \forall\ \mathrm{k} = 1, 2, 3, \dots \quad (2.24)$$

If $v(\phi)$ is a positive solution of (1.1) $\ni$ $w(\phi) \in W_1$, then $-I^{\gamma}(\phi)\, b(\phi)\, w'(\phi)$ is decreasing on $\left(\xi(\xi(\phi_k)), \xi(\phi_k)\right)$.

Proof. Since $-\mathrm{b}(\phi) w'(\phi)$ is decreasing, then it is easy to view that

$$\begin{aligned} w(\xi(\phi)) &\geq \int_{\xi(\phi)}^{\phi} \frac{-\,b(u)\, w'(u)}{b(u)}\, du \\ &\geq -\,b(\phi)\, w'(\phi) \int_{\xi(\phi)}^{\phi} \frac{1}{b(u)}\, du \\ &= -\,b(\phi)\, w'(\phi)\left[I(\phi) - I(\xi(\phi))\right]. \end{aligned}$$

From (2.13) and (2.17),

$$\left(b(\phi)\, w'(\phi)\right)' \geq k\, n_1\, p(\phi)\, b(\phi)\left(-w'(\phi)\right)\left[I(\phi) - I(\xi(\phi))\right].$$

If we confine $\phi \in (\xi(\phi_k), \phi_k) \subset D_\xi$, k =1, 2, 3, …, then from (2.24),

$$I(\phi)\left(b(\phi)\, w'(\phi)\right)' \geq \gamma\left(-w'(\phi)\right)$$

and hence

$$\begin{aligned} \left(-\, I^{\gamma}(\phi)\, b(\phi)\, w'(\phi)\right)' &\leq -\,\gamma\, I^{\gamma-1}(\phi)\, I'(\phi)\left(b(\phi)\, w'(\phi)\right) - I^{\gamma}(\phi)\left(b(\phi)\, w'(\phi)\right)' \\ &\leq 0 \end{aligned}$$

completes the proof.

Lemma 2.3. Suppose there exists $\{s_k\}$, $s_k \longrightarrow \infty$ as $k \longrightarrow \infty$, such that $s_k \in A_\xi$. Let $\exists\ \delta > 0 \ni$

$$k\, n_2 \left[I(\xi(\phi)) - I(\phi)\right] I(\phi)\, b(\phi)\, p(\phi) \geq \delta, \; on \; \left(\xi(s_k), \xi(\xi(s_k))\right), \; \forall\ \mathrm{k} = 1, 2, 3, \dots \quad (2.25)$$

If $v(\phi)$ is a positive solution of (1.1) $\ni$ $w(\phi) \in W_2$, then $I^{-\delta}(\phi)\, b(\phi)\, w'(\phi)$ is increasing on $\left(\xi(s_k), \xi(\xi(s_k))\right)$.

The proof is similar to that of Lemma 2.2.

Theorem 2.2. Suppose (2.4) holds and $\exists$ q(ϕ) $\ni$ (2.3) holds. Moreover, suppose $\exists$ $\{\phi_k\}$, $\{s_k\}$with $\phi_k, s_k \longrightarrow \infty$ as $k \longrightarrow \infty \ni \phi_k \in D_\xi$ and $s_k \in A_\xi$.

If

$$\limsup_{k \longrightarrow \infty} k\, n_1\, I^{\gamma}(\xi(\phi_k)) \int_{\xi(\phi_k)}^{\phi_k} p(s)\left[\frac{I^{1-\gamma}(\xi(\phi_k)) - I^{1-\gamma}(\xi(s))}{1-\gamma}\right] ds > 1 \quad (2.26)$$

and

$$\limsup_{k \longrightarrow \infty} k\, n_2\, I^{-\delta}(\xi(s_k)) \int_{s_k}^{\xi(s_k)} p(s)\left[\frac{I^{1+\delta}(\xi(s)) - I^{1+\delta}(\xi(s_k))}{1+\delta}\right] ds > 1 \quad (2.27)$$

where (2.24) and (2.25) define γ and δ, respectively, and n_1 and n_2 are constants, then (1.1) is oscillatory.

Proof. Assume, on the contrary, that $v(\phi)$ is an eventually positive solution of (1.1). Then the following cases arise:

Case I. Suppose that $w(\phi) \in W_1$. By lemma 2.2, the function $-I^{\gamma}(\phi)b(\phi)w'(\phi)$ is decreasing on $(\xi(\xi(\phi_k)), \xi(\phi_k))$. Thus, for s ∈ $(\xi(\phi_k), \phi_k)$, we obtain

$$w(\xi(s)) \geq \int_{\xi(s)}^{\xi(\phi_k)} \frac{-b(u)\, I^{\gamma}(u)\, w'(u)}{b(u)\, I^{\gamma}(u)}\, du$$

$$\geq -\, b(\xi(\phi_k))\, I^{\gamma}(\xi(\phi_k))\, w'(\xi(\phi_k)) \int_{\xi(s)}^{\xi(\phi_k)} \frac{1}{b(u)\, I^{\gamma}(u)}\, du$$

$$\geq -\, b(\xi(\phi_k))\, I^{\gamma}(\xi(\phi_k))\, w'(\xi(\phi_k)) \left[\frac{I^{1-\gamma}(\xi(\phi_k)) - I^{1-\gamma}(\xi(s))}{1-\gamma}\right].$$

Using the above inequality in (2.14), we get

$$-\, b(\xi(\phi_k))\, w'(\xi(\phi_k)) \geq$$
$$k n_1 \left(-\, b(\xi(\phi_k))\, I^{\gamma}(\xi(\phi_k))\, w'(\xi(\phi_k))\right) \int_{\xi(\phi_k)}^{\phi_k} p(s) \left[\frac{I^{1-\gamma}(\xi(\phi_k)) - I^{1-\gamma}(\xi(s))}{1-\gamma}\right] ds,$$

that is,

$$1 \geq k\, n_1\, I^{\gamma}(\xi(\phi_k)) \int_{\xi(\phi_k)}^{\phi_k} p(s) \left[\frac{I^{1-\gamma}(\xi(\phi_k)) - I^{1-\gamma}(\xi(s))}{1-\gamma}\right] ds,$$

which contradicts the condition (2.26), and hence $W_1 = \emptyset$.

Case II. Suppose that $w(\phi) \in W_2$. By lemma 2.3, the function $I^{-\delta}(\phi)\, b(\phi)\, w'(\phi)$ is increasing on $(\xi(s_k), \xi(\xi(s_k)))$. Thus, for s ∈ $(s_k, \xi(s_k))$, we obtain

$$w(\xi(s)) \geq \int_{\xi(s_k)}^{\xi(s)} \frac{b(u)\, I^{-\delta}(u)\, w'(u)}{b(u)\, I^{-\delta}(u)}\, du$$

$$\geq b(\xi(s_k))\, I^{-\delta}(\xi(s_k))\, w'(\xi(s_k)) \int_{\xi(s_k)}^{\xi(s)} \frac{1}{b(u)\, I^{-\delta}(u)}\, du$$

$$\geq b(\xi(s_k))\, I^{-\delta}(\xi(s_k))\, w'(\xi(s_k)) \left[\frac{I^{1+\delta}(\xi(s)) - I^{1+\delta}(\xi(s_k))}{1+\delta}\right].$$

Using the last inequality in (2.18), we get

$$b(\xi(s_k))\; w'(\xi(s_k)) \geq$$
$$k\, n_2 \left(b(\xi(s_k))\, I^{-\delta}(\xi(s_k))\, w'(\xi(s_k))\right) \int_{s_k}^{\xi(s_k)} p(s) \left[\frac{I^{1+\delta}(\xi(s)) - I^{1+\delta}(\xi(s_k))}{1+\delta}\right] ds,$$

that is,

$$1 \geq k\, n_2\, I^{-\delta}(\xi(s_k)) \int_{s_k}^{\xi(s_k)} p(s) \left[\frac{I^{1+\delta}(\xi(s)) - I^{1+\delta}(\xi(s_k))}{1+\delta}\right] ds,$$

which contradicts the condition (2.27), and hence $W_2 = \emptyset$.

The remaining two cases, Cases III and IV, are identical to those in Theorem 2.1, and this completes the proof.■

3. Examples

This section illustrates the application of main results.

Example 3.1. Consider the second order differential equation of the form

$$\left(v(\phi) + \sqrt{\phi}\; v^2\left(\frac{\phi}{2}\right) - \frac{1}{\phi}\, v^2(2\phi)\right)'' - \frac{c}{\phi^2}\, v\left(\phi\left(1 - \frac{2}{3}\sin\,(\ln \phi\,)\right)\right) = 0, \;\; \phi > 0, \;\; c > 0.$$

(3.1)

This is a special form of (1.1), where $b(\phi) = 1$, $w(\phi) = v(\phi) + \sqrt{\phi}\ v^2\left(\frac{\phi}{2}\right) - \frac{1}{\phi}\ v^2(2\phi)$, $p(\phi) = \frac{c}{\phi^2}$ with c is a constant, $\xi(\phi) = \phi\left(1 - \frac{2}{3}\sin(\ln\phi)\right)$, $\alpha = 1, k = 1$, $f\left(v(\xi(\phi))\right) = v(\xi(\phi))$, $\phi_0 = 0$, $\theta = 2 > 1$, $q_1(\phi) = \sqrt{\phi}$, $q_2(\phi) = \frac{1}{\phi}$, $\chi(\phi) = \frac{\phi}{2}$, $\mu(\phi) = 2\,\phi$. Clearly, the deviating argument $\xi(\phi)$ is of mixed type.

If we choose $\phi_k = e^{\left(\frac{\pi}{2}\right)+2k\pi}$, k = 1, 2, 3, …, then $\phi_k \in D_\xi$ and moreover $\xi(\phi_k) = \frac{1}{3}\ e^{\left(\frac{\pi}{2}\right)+2k\pi}$.

Condition (2.5) takes the form

$$\limsup_{k\to\infty} n_1 \int_{\xi(\phi_k)}^{\phi_k} \frac{c}{s^2}\left[\frac{1}{3}\ e^{\left(\frac{\pi}{2}\right)+2k\pi} - s\left(1 - \frac{2}{3}\sin(\ln s)\right)\right] ds$$

$$= \limsup_{k\to\infty} c\, n_1 \int_{\xi(\phi_k)}^{\phi_k} \left[\frac{1}{3}\ e^{\left(\frac{\pi}{2}\right)+2k\pi}\ \frac{1}{s^2} - \frac{1}{s} + \left(\frac{2}{3}\right)\left(\frac{1}{s}\right)\sin(\ln s)\right] ds$$

$$= \limsup_{k\to\infty} c\, n_1 \left[-\frac{1}{3}\ e^{\left(\frac{\pi}{2}\right)+2k\pi}\left(\frac{1}{s}\right)_{\xi(\phi_k)}^{\phi_k} - (\ln s)_{\xi(\phi_k)}^{\phi_k} - \frac{2}{3}\left(\cos(\ln s)\right)_{\xi(\phi_k)}^{\phi_k}\right]$$

$$= c\, n_1\left[\frac{2}{3} + \ln\frac{1}{3} - \frac{2}{3}\sin\left(\ln\frac{1}{3}\right)\right]$$

$$> 1,$$

which (by Theorem 2.1) guarantees that $W_1 = \emptyset\ \left(i.e.\ for\ c > \frac{1}{n_1}\ 6.181523945\right)$.

On the other hand, if we choose $s_k = e^{\left(\frac{3\pi}{2}\right)+2k\pi}$, k = 1, 2, 3, …, then $s_k \in A_\xi$ and moreover $\xi(s_k) = \frac{5}{3}\ e^{\left(\frac{3\pi}{2}\right)+2k\pi}$.

Condition (2.6) takes the form

$$\limsup_{k\to\infty} n_2 \int_{s_k}^{\xi(s_k)} \frac{c}{s^2}\left[s\left(1 - \frac{2}{3}\sin(\ln s)\right) - \frac{5}{3}\ e^{\left(\frac{3\pi}{2}\right)+2k\pi}\right] ds$$

$$= \limsup_{k\to\infty} c\, n_2 \int_{s_k}^{\xi(s_k)} \left[\frac{1}{s} - \left(\frac{2}{3}\right)\left(\frac{1}{s}\right)\sin(\ln s) - \frac{5}{3}\ e^{\left(\frac{3\pi}{2}\right)+2k\pi}\ \frac{1}{s^2}\right] ds$$

$$= \limsup_{k\to\infty} c\, n_2 \left[(\ln s)_{s_k}^{\xi(s_k)} + \frac{2}{3}\left(\cos(\ln s)\right)_{s_k}^{\xi(s_k)} + \frac{5}{3}\ e^{\left(\frac{3\pi}{2}\right)+2k\pi}\left(\frac{1}{s}\right)_{s_k}^{\xi(s_k)}\right]$$

$$= c\, n_2\left[\ln\frac{5}{3} + \frac{2}{3}\sin\left(\ln\frac{5}{3}\right) - \frac{2}{3}\right]$$

$$= c\, n_2\left[-\frac{2}{3} + \ln\frac{5}{3} + \frac{2}{3}\sin\left(\ln\frac{5}{3}\right)\right]$$

$$> 1,$$

which ensures that $W_2 = \emptyset\ \left(i.e.\ for\ c > \frac{1}{n_2}\ 5.879215638\right)$.

Moreover, we can verify that

$$c\int_0^\infty \int_u^\infty \frac{1}{s^2}\, ds du = \infty.$$

that means, (2.4) is also satisfied. Based on the two criteria, we can observe that the condition $c > \frac{1}{n_2}\ 5.879215638$ suggests that (3.1) oscillates.

Example 3.2. The differential equation (3.1) is once again considered.

At first, by theorem 2.2, we shall show that $W_1 = \emptyset$ for c $\geq \frac{1}{n_1}$ 5.80674902. So, we set c $= \frac{1}{n_1}$ 5.80674902. Again taking $\phi_k = e^{\left(\frac{\pi}{2}\right)+2k\pi}$, k = 1, 2, 3, …, then $\xi(\phi_k) = \frac{1}{3}\, e^{\left(\frac{\pi}{2}\right)+2k\pi}$ and $\xi(\xi(\phi_k)) = \left(\frac{1}{3} - \frac{2}{9} cos\left(ln\ \frac{1}{3}\right)\right) e^{\left(\frac{\pi}{2}\right)+2k\pi}$. In view of Lemma 2.2, the condition (2.24) reduces to

$$\frac{2}{3}\, n_1\, c\, sin\,(ln\,\phi)\ \geq \gamma,\quad on\quad (\,\xi(\xi(\phi_k)), \xi(\phi_k)\,),\ \ k\ =\ 1, 2, 3, ...$$

Since $\frac{2}{3}\, n_1\, c\, sin\,(ln\,\phi)$ is increasing function on $(\,\xi(\xi(\phi_k)), \xi(\phi_k)\,)$, we have

$$\gamma = \frac{2}{3}\, n_1\, c\, sin\left(ln\left(\,\xi(\xi(\phi_k))\right)\right) = \frac{2}{3}\, n_1\, c\, cos\left(ln\left(\frac{1}{3} - \frac{2}{9}\, cos\left(ln\left(\frac{1}{3}\right)\right)\right)\right) = 0.4284180863$$

so that γ is the same on each interval $(\,\xi(\xi(\phi_k)), \xi(\phi_k)\,)$.

Now, we verify the condition (2.26).

$$\limsup_{k\to\infty} n_1\, \xi^{\gamma}(\phi_k) \int_{\xi(\phi_k)}^{\phi_k} p(s) \left[\frac{\xi^{1-\gamma}(\phi_k) - \xi^{1-\gamma}(s)}{1-\gamma}\right] ds$$

$$= \limsup_{k\to\infty} \frac{n_1\, c}{1-\gamma} \left(\frac{1}{3}\, e^{\left(\frac{\pi}{2}\right)+2k\pi}\right)^{\gamma} \int_{\xi(\phi_k)}^{\phi_k} \frac{1}{s^2} \left[\left(\frac{1}{3}\, e^{\left(\frac{\pi}{2}\right)+2k\pi}\right)^{1-\gamma} - \left(s\left(1 - \frac{2}{3} sin\,(ln\ s\,)\right)\right)^{1-\gamma}\right] ds$$

$$= \limsup_{k\to\infty} \frac{n_1\, c}{1-\gamma} \left[\frac{2}{3} - \left(\frac{1}{3}\, e^{\left(\frac{\pi}{2}\right)+2k\pi}\right)^{\gamma} \int_{\xi(\phi_k)}^{\phi_k} s^{-1-\gamma} \left(1 - \frac{2}{3} sin\,(ln\ s\,)\right)^{1-\gamma} ds\right].$$

Substituting s $= e^{\left(\frac{\pi}{2}\right)+2k\pi}\phi$, the above equation, we get

$$\limsup_{k\to\infty} n_1\, \xi^{\gamma}(\phi_k) \int_{\xi(\phi_k)}^{\phi_k} p(s) \left[\frac{\xi^{1-\gamma}(\phi_k) - \xi^{1-\gamma}(s)}{1-\gamma}\right] ds$$

$$= \frac{n_1\, c}{1-\gamma}\left[\frac{2}{3} - \left(\frac{1}{3}\right)^{\gamma} \int_{\frac{1}{3}}^{1} \phi^{-1-\gamma} \left(1 - \frac{2}{3} sin\left(ln\left(e^{\left(\frac{\pi}{2}\right)+2k\pi}\,\phi\right)\right)\right)^{1-\gamma} d\phi\right]$$

$$= \frac{n_1\, c}{1-\gamma}\left[\frac{2}{3} - \left(\frac{1}{3}\right)^{\gamma} \int_{\frac{1}{3}}^{1} \phi^{-1-\gamma} \left(1 - \frac{2}{3} cos\,(ln\,\phi\,)\right)^{1-\gamma} d\phi\right].$$

Using Matlab for calculation, we get

$$\int_{\frac{1}{3}}^{1} \phi^{-1-\gamma} \left(1 - \frac{2}{3} cos\,(ln\,\phi\,)\right)^{1-\gamma} d\phi = 0.909774 \quad \text{with} \quad \gamma = 0.4284180863$$

and finally, we get

$$\limsup_{k\to\infty} n_1\, \xi^{\gamma}(\phi_k) \int_{\xi(\phi_k)}^{\phi_k} p(s) \left[\frac{\xi^{1-\gamma}(\phi_k) - \xi^{1-\gamma}(s)}{1-\gamma}\right] ds = 1.000000007 > 1$$

which by Theorem 2.2 guarantees that $W_1 = \emptyset$.

At second, by theorem 2.2, we shall show that $W_2 = \emptyset$ for c $\geq \frac{1}{n_2}$ 4.4183627. So, we set c $= \frac{1}{n_2}$ 4.4183627. Again taking $s_k = e^{\left(\frac{3\pi}{2}\right)+2k\pi}$, k = 1, 2, 3, …, then $\xi(s_k) = \frac{5}{3}\, e^{\left(\frac{3\pi}{2}\right)+2k\pi}$ and $\xi(\xi(s_k)) = \left(\frac{5}{3} + \frac{10}{9} cos\left(ln\ \frac{5}{3}\right)\right) e^{\left(\frac{3\pi}{2}\right)+2k\pi}$. In view of Lemma 2.3, the condition (2.25) reduces to

$$-\frac{2}{3}\, n_2\, c\, sin\,(ln\,\phi)\ \geq \delta,\quad on\quad \left(\xi(s_k), \xi(\xi(s_k))\right),\ \ k\ =\ 1, 2, 3, ...$$

Since $-\frac{2}{3}\, n_2\, c\, sin\,(ln\,\phi)$ is decreasing function on $\left(\xi(s_k), \xi(\xi(s_k))\right)$, we have

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$$\delta = -\frac{2}{3} n_2 c \sin\left(\ln\left(\xi\left(\xi(s_k)\right)\right)\right) = \frac{2}{3} n_2 c \cos\left(\ln\left(\frac{5}{3} + \frac{10}{9}\cos\left(\ln\left(\frac{5}{3}\right)\right)\right)\right) = 1.666983689$$

so that δ is the same on each interval $\left(\xi(s_k), \xi\left(\xi(s_k)\right)\right)$.

Now, we verify (2.27).

$$\limsup_{k \to \infty} n_2 \xi^{-\delta}(s_k) \int_{s_k}^{\xi(s_k)} p(s) \left[\frac{\xi^{1+\delta}(s) - \xi^{1+\delta}(s_k)}{1+\delta}\right] ds$$

$$= \limsup_{k \to \infty} \frac{n_2 c}{1+\delta} \left(\frac{5}{3} e^{\left(\frac{3\pi}{2}\right)+2k\pi}\right)^{-\delta} \int_{s_k}^{\xi(s_k)} \frac{1}{s^2} \left[\left(s\left(1-\frac{2}{3}\sin(\ln s)\right)\right)^{1+\delta} - \left(\frac{5}{3} e^{\left(\frac{3\pi}{2}\right)+2k\pi}\right)^{1+\delta}\right] ds$$

$$= \limsup_{k \to \infty} \frac{n_2 c}{1+\delta} \left[\left(\frac{5}{3} e^{\left(\frac{3\pi}{2}\right)+2k\pi}\right)^{-\delta} \int_{s_k}^{\xi(s_k)} s^{\delta-1} \left(1-\frac{2}{3}\sin(\ln s)\right)^{1+\delta} ds - \frac{2}{3}\right].$$

Using $s = e^{\left(\frac{3\pi}{2}\right)+2k\pi}\phi$ in the last integral, we get

$$\limsup_{k \to \infty} n_2 \xi^{-\delta}(s_k) \int_{s_k}^{\xi(s_k)} p(s) \left[\frac{\xi^{1+\delta}(s) - \xi^{1+\delta}(s_k)}{1+\delta}\right] ds$$

$$= \frac{n_2 c}{1+\delta} \left[\left(\frac{5}{3}\right)^{-\delta} \int_1^{\frac{5}{3}} \phi^{\delta-1} \left(1-\frac{2}{3}\sin\left(\ln\left(e^{\left(\frac{3\pi}{2}\right)+2k\pi}\phi\right)\right)\right)^{1+\delta} d\phi - \frac{2}{3}\right]$$

$$= \frac{n_2 c}{1+\delta} \left[-\frac{2}{3} + \left(\frac{5}{3}\right)^{-\delta} \int_1^{\frac{5}{3}} \phi^{\delta-1} \left(1+\frac{2}{3}\cos(\ln\phi)\right)^{1+\delta} d\phi\right].$$

Using Matlab for calculation, we get

$$\int_1^{\frac{5}{3}} \phi^{\delta-1} \left(1+\frac{2}{3}\cos(\ln\phi)\right)^{1+\delta} d\phi = 2.97659 \quad \text{with} \quad \delta = 1.666983689$$

and finally,

$$\limsup_{k \to \infty} n_2 \xi^{-\delta}(s_k) \int_{s_k}^{\xi(s_k)} p(s) \left[\frac{\xi^{1+\delta}(s) - \xi^{1+\delta}(s_k)}{1+\delta}\right] ds = 1.000005958 > 1$$

which by Theorem 2.2 guarantees that $W_2 = \emptyset$. Hence, based on the two criteria, we can observe that the condition $c > \frac{1}{n_2} 4.4183627$ suggests that (3.1) oscillates, while Theorem 2.1 requires $c > \frac{1}{n_2} 5.879215638$.

4. Conclusion

This paper studies a class of second order mixed functional nonlinear differential equations with superlinear neutral terms and establishes some criteria for oscillation. Also, we obtained stronger conditions for equation 3.1 to be oscillatory, and hence, a further development of Theorem 2.1 is Theorem 2.2.

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