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Analytic Treatment of Time Fractional Nonlinear Dynamic System by Modified Decomposition Method

Prabhat Kumar[1,a)], M.M. Dixit [1,b)]

[1]*Department of Applied Sciences & Humanities, NERIST, Nirjuli, Department of Mathematics, A.P. India*

[2] *Department of Applied Sciences & Humanities, NERIST, Nirjuli, Department of Mathematics, A.P. India.*

[a)]***Corresponding author:*** *Prabhat Kuma, Department of Applied Sciences & Humanities, NERIST, Nirjuli, Department of Mathematics, A.P. India*

Abstract: In the present paper analytical and numerical results are reported for analytical approximate solution of nonlinear dynamic system containing fractional derivative by modified decomposition method. Comparison with the exact and numerical solution shows that the present method performs extremely well in terms of accuracy, efficiency and simplicity.

Keywords : Analytic Treatment ,Decomposition Method

1. Introduction

Recently a great deal of interest has been focused on Adomian's Decomposition Method (ADM) and its applications to a wide class of physical problems containing fractional derivatives [6,7].The fractional differential equations appear more and more frequently in different research areas and engineering applications.

The fractional derivatives have been occurring in many physical problems such as frequency dependent damping behavior of materials, motion of a large thin plate in a Newtonian fluid, creep and relaxation functions for viscoelastic materials, the $PI^{\lambda}D^{\mu}$ controller for the control of dynamical systems, etc. [15,20]. Phenomena in electromagnetic, acoustics, viscoelasticity, and electrochemistry and material science are also described by differential equations of fractional order [1,11]. The solution of the differential equation containing fractional derivative is much involved. An effective and easy-to-use method for solving such equations is essentially needed for the tackling of physical situations.

Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. For that reason we needed a reliable and efficient technique for the solution of fractional differential equations. In this connection, it is worthwhile to mention that the recent papers on numerical solutions of fractional differential equations are available from the notable works of Diethelm *et al*[16].

The ADM employed here is adequately discussed in published literature [2,4,21], but it still deserves emphasis to point out the very significant advantages over other methods. The said method can also be an effective procedure for the solution of time fractional nonlinear dynamic system. Nonlinear systems are generally analyzed by approximation methods which involve some sort of linearization. These replace an actual nonlinear system with a so-called equivalent linear system and employ averaging which is not generally valid. While the linearization commonly used is adequate in some cases, they may be grossly inadequate in others since essentially real phenomena (shock waves in gas dynamics, for example) can occur in nonlinear systems which cannot occur in linear systems. Thus, the correct solution of a nonlinear system is a much more significant matter than simply getting more accuracy when we solve the nonlinear system rather than a linearized approximation. Thus, if we want to know how a physical system behaves, it is essential to retain the nonlinearity, not just solve a convenient mathematical model.

In this paper, we use Adomian's modified decomposition method [2,10] to obtain a solution for nonlinear time fractional dynamic systems. The decomposition method provides an effective procedure for analytical solution of a wide class of linear and nonlinear dynamical systems representing real physical problems. This method efficiently works for initial value or boundary value problems and for linear and nonlinear ordinary, partial, differential equations and integral equations. Moreover, we have the advantage of a single global method for solving linear and nonlinear systems. Recently, solution of fractional differential equations and the solution of some complicated physical system containing fractional derivatives has been obtained by decomposition and modified decomposition method [5,7].

2. Mathematical Aspects

2.1 Mathematical Definition

The mathematical definition of fractional calculus has been the subject of several different approaches [2,4]. The most frequently encountered definition of an integral of fractional order is the Reimann-Liouville integral, in which the fractional order integral is defined as

$$\frac{d^{-q} f(x)}{dx^{-q}} = \frac{1}{\Gamma(q)} \int_0^x \frac{1}{(x-t)^{1-q}} f(t)\,dt\,,$$

(2.1)

while the definition of fractional order derivative is

$$\frac{d^q}{dx^q} f(x) = \frac{d^n}{dx^n} \left\{ \frac{d^{-(n-q)}}{dx^{-(n-q)}} f(x) \right\} = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_0^x \frac{1}{(x-t)^{1-n+q}} f(t)\,dt\,,$$

(2.2)

where $q \in R_0^+$,is the order of the operation and n is an integer that satisfies $n-1 \le q < n$.

3. The Decomposition Method

Let us discuss a brief outline of the Adomian decomposition method, in general. For this, let us consider an equation in the form

$$Lu + Ru + Nu = g\,,$$

(3.1)

where L is an easily or trivially invertible linear operator, R is the remaining linear part and N represents a nonlinear operator.

The general solution of the given equation is decomposed into the sum

$$u = \sum_{n=0}^{\infty} u_n\,,$$

(3.2)

where u_0 is the complete solution of $Lu = g$.

From eq. (3.1), we can write

$$Lu = g - Ru - Nu\,.$$

(3.3)

Because L is invertible, an equivalent expression is

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu\,.$$

(3.4)

For initial-value problems we conveniently define L^{-1} for $L \equiv \frac{d^n}{dt^n}$ as the $n-$ fold definite integration operator from 0 to t. For the operator $L \equiv \frac{d^2}{dt^2}$, for example,

We have

$$L^{-1}Lu = u(0) + tu'(0)$$

(3.5)

and therefore,

$$u = u(0) + tu'(0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu .$$

(3.6)

For boundary value problems (and if desired, for initial value problems as well) indefinite integrations are used and the constants are evaluated from the given conditions. Solving for u yields

$$u = A + Bt + L^{-1}g - L^{-1}Ru - L^{-1}Nu .$$

(3.7)

The first three terms in eq. (3.6) or eq.(3.7) are identified as u_0 in the assumed decomposition $u = \sum_{n=0}^{\infty} u_n$.

Finally, assuming Nu is analytic, we write

$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, ..., u_n)$,where A_n 's are special set of polynomials obtained for the particular nonlinearity $Nu = f(u)$ and were generated by Adomian [11,12].These A_n polynomials depend, of course, on the particular nonlinearity.

The A_n 's are given as follows:

$$A_0 = f(u_0),$$

$$A_1 = u_1 \frac{d}{du_0} f(u_0),$$

$$A_2 = u_2 \frac{d}{du_0} f(u_0) + \frac{u_1^2}{2!} \frac{d^2}{du_0^2} f(u_0),$$

$$A_3 = u_3 \frac{d}{du_0} f(u_0) + u_1 u_2 \frac{d^2}{du_0^2} f(u_0) + \frac{u_1^3}{3!} \frac{d^3}{du_0^3} f(u_0)$$

and can be found from the formula(for $n \geq 1$)

$$A_n = \sum_{\nu=1}^{n} C(\nu, n) f^{\nu}(u_0),$$

(3.8)

where the $C(\nu, n)$ are products or sums of products of ν components of u whose subscripts sum to n, divided by the factorial of the number of repeated subscripts [11]. This formula is easy to set computer code to get as many polynomials as we need in calculation of the numerical as well as explicit solutions.

Recently, the Adomian decomposition method is reviewed and a mathematical model of Adomian polynomials is introduced [7,13].Therefore, the general solution becomes

$$u = u_0 - L^{-1}\left[R\sum_{n=0}^{\infty} u_n\right] - L^{-1}[Nu],$$

(3.9)

$$u = u_0 - L^{-1}\left[R\sum_{n=0}^{\infty} u_n\right] - L^{-1}\left[\sum_{n=0}^{\infty} A_n\right],$$

(3.10)

where $u_0 = \phi + L^{-1}g$ and $L\phi = 0$,

(3.11)

so that $u_{n+1} = -L^{-1}[Ru_n] - L^{-1}\left[\sum_{n=0}^{\infty} A_n\right], n \geq 0$.

(3.12)

Using the known u_0 , all components $u_1, u_2, ..., u_n, ...$ etc. are determinable by using eq. (3.12). Substituting these $u_0, u_1, u_2, ..., u_n, ...$ etc. in eq. (3.2), u is obtained.

Recently, Wazwaz [13] proposed that the construction of the zeroth component of the decomposition series can be defined in a slightly different way. In [13],he assumed that, if the zeroth component $u_0 = g$ and the function g is possible to divide into two parts such as g_1 and g_2 , then one can formulate the recursive algorithm for u_0 and general term u_{n+1} in a form f the modified recursive scheme as follows :

$$u_0 = g_1,$$

$$u_2 = g_2 - L^{-1}[Ru_0] - L^{-1}[A_0],.$$

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$$u_{n+1} = -L^{-1}[Ru_n] - L^{-1}[A_n],\ n \geq 0.$$

(3.13)

This type of modification is giving more flexibility to the Adomian decomposition method (ADM) in order to solve complicated nonlinear differential equations. In many cases the modified decomposition scheme avoids the unnecessary computation especially in calculation of the Adomian polynomials. The computation of these polynomials will be reduced very considerably by using the modified decomposition method (MDM).It is worth noting that the zeroth component u_0 is defined, then the remaining components u_n , $n \geq 1$ can be completely determined. As a result, the components $u_0, u_1, ...$ is identified and the series solutions thus entirely determined. However, in many cases the exact solution in a closed form may be obtained. The decomposition series solution (3.2) generally converges very rapidly in real physical problems [11]. The rapidity of this convergence means that only few terms are required for this purpose. Convergence of this method has been rigorously established by Cherruault [7], Abbaoui and Cherruault [8] and Himoun at al [9].

The practical solution will be the $n-$ term approximation ϕ_n .

$$\phi_n = \sum_{i=0}^{n-1} u_i\ , n \geq 1,$$

(3.14)

with $\lim_{n\to\infty} \phi_n = u$.

4. Fractional Dynamic Models and the Solution

Considering the viscoelastic behaviour of an one-degree-of-freedom oscillator consisting of a discrete mass and a viscoelastic spring governed by a fractional calculus law in conjunction with Newton's second law, we have the equation in the form

$$mD_t^2 x(t) + cD_t^{\alpha} x(t) + kg(x(t)) = f(t),$$

(4.1)

subject to the initial condition

$$x(0) = x_0, x'(0) = x_1,$$

(4.2)

where m , c and k represent the mass, damping and stiffness coefficients respectively, $f(t)$ is the externally applied force, $x(t)$ is the displacement and $D_t^{1/2} x(t)$ is the fractional derivative of order $\frac{1}{2}$ of the displacement $x(t)$.

We adopt Modified Decomposition Method (ADM) for solving eq.(4.1) under initial conditions (4.2). In the light of this method, we assume that

$$x(t) = x_0(t) + x_1(t) + x_2(t) + \ldots$$

to be the solution of eq. (4.1).

Now, eq. (4.1) can be written as

$$\frac{d^2}{dt^2} x(t) + \frac{c}{m} \frac{d^{1/2}}{dt^{1/2}} x(t) + \frac{k}{m} g(x(t)) = \frac{1}{m} f(t)$$

(4.3)

Let us suppose that, $L \equiv \frac{d^2}{dt^2}$, which is an easily invertible linear operator. Now comparing eqs. (4.3) and (3.1), we can observe that $\frac{d^{1/2}}{dt^{1/2}}$ in eq. (3) represents the remaining linear operator and the nonlinear part N_x is $\frac{k}{m} g(x(t))$.

Therefore,

$$N_x = f(x) = \sum_{n=0}^{\infty} A_n(x_0, x_1, x_2, \ldots, x_n) = \frac{k}{m} g(x(t)).$$

(4.4)

The Adomian polynomials A_n , as discussed in section 3, becomes in the present case.

$$A_0 = f(x_0) = \frac{k}{m} g(x_0),$$

$$A_1 = x_1 \frac{d}{dx_0} f(x_0) = \frac{k}{m} x_1 \frac{d}{dx_0} f(x_0),$$

$$A_2 = x_2 \frac{d}{dx_0} f(x_0) + \frac{1}{2!} x_1^2 \frac{d^2}{dx_0^2} f(x_0)$$

$$= \frac{k}{m} x_2 \frac{d}{dx_0} g(x_0) + \frac{1}{2!} \frac{k}{m} x_1^2 \frac{d^2}{dx_0^2} g(x_0),$$

$$A_3 = x_3 \frac{d}{dx_0} f(x_0) + x_1 x_2 \frac{d^2}{dx_0^2} f(x_0) + \frac{1}{3!} x_1^3 \frac{d^3}{dx_0^3} f(x_0)$$

$$= \frac{k}{m} x_3 \frac{d}{dx_0} g(x_0) + \frac{k}{m} x_1 x_2 \frac{d^2}{dx_0^2} g(x_0) + \frac{1}{3!} \frac{k}{m} x_1^3 \frac{d^3}{dx_0^3} g(x_0)$$

and so on.
(4.5)
Therefore, by decomposition method, we can write,

$$x(t) = x(0) + tD_t x(t)\big|_{t=0} + \frac{1}{m} L^{-1}[f(t)] - L^{-1}\left[\frac{c}{m} D_t^{1/2}\left(\sum_{n=0}^{\infty} x_n(t)\right)\right] - L^{-1}\left[\sum_{n=0}^{\infty} A_n\right]$$

(4.6)

$$= x_0 + tx_1 + \frac{1}{m} L^{-1}[f(t)] - \frac{c}{m} L^{-1}\left[D_t^{1/2}\left(\sum_{n=0}^{\infty} x_n(t)\right)\right] - L^{-1}\left[\sum_{n=0}^{\infty} A_n\right].$$

(4.7)
We employ the modified recursive scheme as follows :

$$x_0(t) = x_0 + tx_1,$$

$$x_1(t) = \frac{1}{m} L^{-1}[f(t)] - \frac{c}{m} L^{-1}\left[D_t^{1/2} x_0(t)\right] - L^{-1}[A_0],$$

$$x_2(t) = -\frac{c}{m} L^{-1}\left[D_t^{1/2} x_1(t)\right] - L^{-1}[A_1],$$

$$x_3(t) = -\frac{c}{m} L^{-1}\left[D_t^{1/2} x_2(t)\right] - L^{-1}[A_2],$$

and so on.
(4.8)
Therefore, the general solution of eq.(4.1) is

$$x(t) = x_0 + tx_1 + \frac{1}{m} L^{-1}[f(t)] - \frac{c}{m}\Big\{L^{-1}\left[D_t^{1/2} x_0(t)\right] + L^{-1}\left[D_t^{1/2} x_1(t)\right] +$$

$$L^{-1}\left[D_t^{1/2} x_2(t)\right] + \ldots\Big] - \left[L^{-1}(A_0) + L^{-1}(A_1) + L^{-1}(A_2) + \ldots\right].$$

(4.9)

5. Implementation of the Method

Let us consider the following motion equation of a one-degree-of-freedom oscillator

$$mD_t^2 x(t) + cD_t^{1/2} x(t) + kx^2(t) = f(t)$$

(5.1)
subject to the initial condition

$$x(0) = 0,\ x'(0) = 0,$$

(5.2)
where

$$f(t)=2\left(t-\frac{3}{10}\right)\left(t-\frac{8}{10}\right)+4t\left(t-\frac{3}{10}\right)+4t\left(t-\frac{8}{10}\right)+2t^2$$
$$+\frac{8}{10\sqrt{\pi}}\left(\frac{128}{35}t^{7/2}-\frac{88}{25}t^{5/2}+\frac{16}{25}t^{3/2}\right)+\left(t^2\left(t-\frac{3}{10}\right)\left(t-\frac{8}{10}\right)\right)^2.$$

(5.3)

For the sake of convenience of the readers, we can give the first few Adomian polynomials for $N(x)=f(x)=kx^2$ of the nonlinearity as

$$A_0=f(x_0)=kx_0^2,$$
$$A_1=x_1\frac{d}{dx_0}f(x_0)=2kx_0x_1,$$
$$A_2=x_2\frac{d}{dx_0}f(x_0)+\frac{1}{2!}x_1^2\frac{d^2}{dx_0^2}f(x_0)=2kx_0x_2+kx_1^2,$$

and so on.

(5.4)

The rest of the polynomials can be constructed in a similar manner.

We will then obtain

$$x_0(t)=0,$$
$$x_1(t)=\frac{1}{m}\frac{d^{-2}}{dt^{-2}}f(t)-\frac{c}{m}\frac{d^{-3/2}}{dt^{-3/2}}x_0(t)-\frac{d^{-2}}{dt^{-2}}(A_0)$$
$$=\frac{1}{m}\left\{\frac{1}{90}t^{10}-\frac{11}{360}t^9+\frac{169}{5600}t^8-\frac{11}{875}t^7+\frac{6}{3125}t^6+t^4-\frac{11}{10}t^3+\frac{6}{25}t^2\right.$$
$$\left.+\frac{1}{\sqrt{\pi}}\left(\frac{2048}{17325}t^{11/2}-\frac{1408}{7875}t^{9/2}+\frac{256}{4375}t^{7/2}\right)\right\},$$
$$x_2(t)=-\frac{c}{m}\frac{d^{-3/2}}{dt^{-3/2}}x_1(t)-\frac{d^{-2}}{dt^{-2}}A_1$$
$$=-\frac{c}{m^2}\left\{\frac{32}{105\sqrt{\pi}}t^{7/2}\left(\frac{6}{25}-\frac{2}{3}t\left(\frac{11}{10}-\frac{8}{11}t\left(1+\frac{8}{13}t^2\left(\frac{6}{3125}-\frac{14}{17}t\left(\frac{11}{875}-\frac{16}{19}t\left(\frac{169}{5600}\right.\right.\right.\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\left.\left.\left.-\frac{6}{7}t\left(\frac{11}{360}-\frac{2}{207}t\right)\right)\right)\right)\right)\right)\right)+\frac{7}{128}t^5\left(\frac{256}{4375}-\frac{3}{4}t\left(\frac{1408}{7875}-\frac{1024}{11025}t\right)\right)\right\}+\ldots$$

and so on.

(5.5)

Therefore, the series solution (4.9) becomes

$$x(t)=\frac{1}{m}\left\{\frac{1}{90}t^{10}-\frac{11}{360}t^9+\frac{169}{5600}t^8-\frac{11}{875}t^7+\frac{6}{3125}t^6+t^4-\frac{11}{10}t^3+\frac{6}{25}t^2\right.$$
$$+\frac{1}{\sqrt{\pi}}\left(\frac{2048}{17325}t^{11/2}-\frac{1408}{7875}t^{9/2}+\frac{256}{4375}t^{7/2}\right]-\frac{c}{m^2}\left[\frac{32}{105\sqrt{\pi}}t^{7/2}\left(\frac{6}{25}-\frac{2}{3}t\right.\right.$$

$$\left(\frac{11}{10}-\frac{8}{11}t\left(1+\frac{8}{13}t^2\left(\frac{6}{3125}-\frac{14}{17}t\left(\frac{11}{875}-\frac{16}{19}t\left(\frac{169}{5600}\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.\left.-\frac{6}{7}t\left(\frac{11}{360}-\frac{2}{207}t\right)\right)\right)\right)\right)\right)\right)+\frac{7}{128}t^5\left(\frac{256}{4375}-\frac{3}{4}t\left(\frac{1408}{7875}-\frac{1024}{11025}t\right)\right)\Bigg]+\ldots$$

(5.6)

6. Numerical Results and Discussion

In the present numerical computation we have assumed $m=1$, $c=0.8$ and $k=1$, as is taken in [219]. It is interesting to note that the graph obtained in our case coincide with the exact solution

$x(t)=t^2\left(t-\frac{3}{10}\right)\left(t-\frac{8}{10}\right)$ in fig. 1 eq. (5.6) has been used to draw the graph as shown in fig. 1.

Table 1 analyzes the two-term approximate numerical solution ϕ_2 , exact solution $x(t)$, and the absolute error and relative error between them.

Table 2 analyzes the three-term approximate numerical solution ϕ_3, exact solution $x(t)$, and the absolute error and relative error between them.

From above two tables we observe that our approximate solution is in good agreement with the exact solution. Of course the accuracy can be improved by computing more terms in the decomposition method.

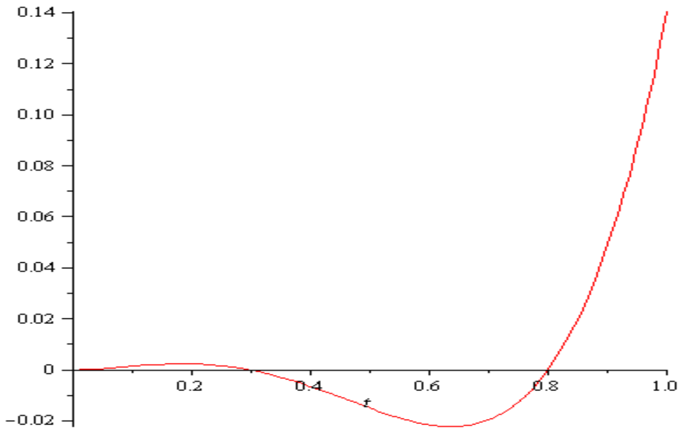

Fig.1.Displacement of the one-degree-of-freedom oscillator (Time vs displacement).

Table I: The exact solution, numerical solution ϕ_2 , absolute error and relative error.

Time t	Exact solution $x(t)$	Numerical solution ϕ_2	Absolute error $\lvert x(t)-\phi_2\rvert$	Relative error $\frac{\lvert x(t)-\phi_2\rvert}{x(t)}$
0.25	0.00171875	0.001812268	9.35177E-05	0.054410287
0.50	-0.015	-0.015065475	6.54751E-05	0.004365005

0.75	-0.01265625	-0.014513086	0.001856836	0.146712982
1.00	0.14	0.138915566	0.001084434	0.007745954

Table II:The exact solution, numerical solution ϕ_3, absolute error and relative error.

Time t	Exact solution $x(t)$	Numerical solution ϕ_3	Absolute error $\lvert x(t)-\phi_3 \rvert$	Relative error $\frac{\lvert x(t)-\phi_3 \rvert}{x(t)}$
0.25	0.00171875	0.001717552	1.19822E-06	0.000697147
0.50	-0.015	-0.015011315	1.13148E-05	0.000754321
0.75	-0.01265625	-0.012610457	4.57934E-05	0.003618248
1.00	0.14	0.140338625	0.000338625	0.00241875

The above results may be compared obtained in [25].

7. Conclusion

Nonlinear phenomena play a crucial role in applied mathematics and physics. The nonlinear problems are solved easily and elegantly without linearizing the problem, by using the Adomian decomposition method (ADM) [2,4,5].The present problem deals with the nonlinear dynamic system containing time fractional derivative and it has been solved analytically by ADM.

The advantage of this global methodology lies in the fact that it not only leads to an analytical continuous approximation which is very rapidly convergent [4,18,20] but also shows the dependence giving insight into the character and behaviour of the solution just as in a closed form solution[4,7,12,20]. The ADM is straight forward and rapid stabilization to an acceptable accuracy is evident when numerical computation of the analytic approximation is carried out [4,7,12].

The present analysis exhibits the applicability of ADM to solve time fractional nonlinear differential equation. In our previous paper we have already as well as successfully exhibit the applicability of ADM to obtain a solution for nonlinear physical systems containing fractional order derivative. Moreover, this method does not require any transformation techniques, linearization, discretization of the variables and it does not make closer approximation or smallness assumptions. If, therefore, provides more realistic series solution that generally converge very rapidly in real physical problems. When solutions are computed numerically, the rapid convergence is obvious. Finally we point out that, although other methods are available, the present method can yield very satisfactory solution without suffering traditional difficulty.

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