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Solution of Delay Ordinary Differential Equations by Using Emad-Sara Transform

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Abstract:- In this paper, we consider the initial value problem (I.V.P) of a second order linear delay differential equation. We apply Emad – Sara integral transform technique to find the solution of this problem. Furthermore, we introduce examples provided the theoretical results.

***Keywords**:* Emad-Sara transform, inverse of Emad- Sara delay differential equation (D.D.E.), initial value problem (I.V.P).

1. Introduction

The delay differential problems have found applications in a wide variety of science in addition to engineering, physics, biotechnology and other scientific fields [2, 3].

These delay problems have for instance, been applied in networks [2], population [2], and bistable devices [2].

There are many researchers who have investigated oscillation hopf bifurcation, numerical aspect and asymptically stability analysis for delay differential problems [2, 3].

On the other hand integral transforms are used to solve application problems in mathematics and other fields [4-15] therefore, it is useful tool for physicists and engineers.

2. Basic Concepts

Definition (2.1): let $f(t)$ be an integrable function defined for $t \geq 0, v \neq 0$ is a positive real parameter, the Emad–Sara integral transform $T(v) of\ f(t)$by the form:

$$T(v) = \frac{1}{v^2} \int_{t=0}^{\infty} e^{-vt} f(t)dt = ES\{f(t)\}$$

Provided the integral exists for some parameter v, [1] proposition (2.1) Let $f(t)$ be a real function with these features:

1. $f(t)$ is a piecewise continuous in every finite interval $0 < t < t_1\ (t_1 > 0\ )$
2. $f(t)$ is of exponential order, that is, $\exists \alpha, N > 0$ and $t_1 > 0\ e^{\propto t}|f(t)| < N$ for $t > t_0$

Then the Emad-Sara integral transform exist for $v > \propto$ [1].

3. The Emad-Sara Transform for Some Basic Functions [1]

In the following values of some basic important functions in Emad - Sara transform

1. $ES\{p\} = \frac{p}{v^3}, v > 0, p\ is\ a\ constant$
2. $ES\{t^n\} = \frac{n!}{v^{n+3}}, v > 0, n \in Z^+$
3. $ES\{e^{pt}\} = \frac{1}{v^2(v-p)}, v > p, p\ is\ a\ constant$

4. $ES\{\sin(pt)\} = \frac{p}{v^2(v^2+p^2)}, v \neq 0, p\ isa\ constant$
5. $ES\{\cos(pt)\} = \frac{p}{v^2(v^2+p^2)}, v \neq 0, p\ isa\ constant$
6. $ES\{\sinh(pt)\} = \frac{p}{v^2(v^2-p^2)}, v > |p|, p\ isa\ constant$
7. $ES\{\cosh(pt)\} = \frac{p}{v(v^2-p^2)}, v > |p|, p\ isa\ constant$

4. The Inverse of Emad-Sara Integral Transform for Some Importart Functions [1]

In the following the inverse of Emad-Sara transform for some basic functions.

1. $(ES)^{-1}\left\{\frac{1}{v^3}\right\} = 1$
2. $(ES)^{-1}\left\{\frac{1}{v^{n+3}}\right\} = \frac{t^n}{n!}, n \in Z^+$
3. $(ES)^{-1}\left\{\frac{1}{v^2(v+1)}\right\} = e^{-t}$
4. $(ES)^{-1}\left\{\frac{1}{v^2(v^2+1)}\right\} = \sin(t)$
5. $(ES)^{-1}\left\{\frac{1}{v^2(v^2+1)}\right\} = \cos(t)$
6. $(ES)^{-1}\left\{\frac{1}{v^2(v^2-1)}\right\} = \sinh(t)$
7. $(ES)^{-1}\left\{\frac{1}{v(v^2-1)}\right\} = \cosh(t)$

5. The Emad-Sara Integral Transform of Derivatives [1]

Function $f(t)$ defined as the Emad – Sara integral transform of $T(v) = ES\{f(t)\}$ then:

1. $ES\{f'(t)\} = \frac{-f(0)}{v^2} + vT(v)$
2. $ES\{f''(t)\}c = \frac{-f'(0)}{v^2} - \frac{-f(0)}{v^2} + v^2T(v)$
3. In general, $n \in Z^+$

$$ES\{f^n(t)\} = \frac{-f^{(n-1)}(0)}{v^2} + vES\{f^{(n-1)}(0)\}$$

Now consider the delay differential problem:

$$u''(t) + a\,u'(t) + bu'(t-\tau) + c\,u(t) + d\,u(t-\tau) = f(t)$$

Where $t > 0$.

And $u(t) = \vartheta(t)\ \ -\tau \leq t \leq 0, u'(0) = \gamma$

Where $a, b, c\ and\ d$ are real constants, $f(t)$ and $\vartheta(t)$ are given real functions and sufficient smooth functions. γ is a real number and τ is a positive constant large delay.

Therem (5.1): Let $\vartheta(t), \vartheta'(t)$ are continuous function on the closed interval $[-\tau, 0], \tau > 0$ and $T(v)$ is the Emad –Sara transform of $f(t)$ in equation (1). then the exact solution of equations (1) –(2) is

$u(t) = (ES)^{-1}\{\frac{T(v)+F(v)}{k(v)}\}$ where:

$K(v) = v^2 + av + c + (bv + d)e^{-v\tau}$

$$F(v)=\frac{\gamma}{v^2}+\big(v\vartheta(0)+a\vartheta(0)\big)\frac{1}{v^2}-b\,\vartheta''(v)+\frac{be^{-v\tau}\vartheta(0)}{v^2}-d\vartheta'(v)$$

And

$\vartheta'(V)=\frac{1}{v^2}\int_{t=-\tau}^{0}e^{-v(t+\tau)}\vartheta(t)dt\ ;\vartheta''(v)=\frac{1}{v^2}\int_{t=-\tau}^{0}e^{-v(t+\tau)}\vartheta(t)dt$

Proof:

To proof the problem (1)-(2) applying the Emad –Sara transform it is known the Emad- Sara transform of the derivatives of u(t) is:

$ES\{u'(t)\}=\frac{-1}{v^2}u(0)+v\,ES\{u(t)\}$

And

$ES\{u'(t)\}=\frac{-\vartheta(0)}{v^2}+v\,ES\{u(t)\}$

$ES\{u''(t)\}=\frac{-u'(t)}{v^2}-\frac{u(0)}{v}+v^2\,ES\{u(t)\}$

$ES\{u''(t)\}=\frac{-\gamma}{v^2}-\frac{\vartheta(0)}{v}+v^2\,ES\{u(t)\}$

The Emad – Sara integral transform for $(t-\tau)$, from the definition gets:

$ES\{u(t-\tau)\}=\frac{1}{v^2}\int_{t=0}^{\infty}e^{-vt}\,u(t-\tau)dt$

After replacing integral variable by: $t=x+\tau, x=t-\tau$ we find that:

$ES\{u(t-\tau)\}=\frac{1}{v^2}\int_{x=-\tau}^{\infty}e^{-v(x+\tau)}\,u(x)dx$

$$=\frac{1}{v^2}\left[\int_{-\tau}^{\infty}e^{-v(x+\tau)}\,p(x)dt+\int_{-\tau}^{\infty}e^{-v(x+\tau)}\,u(x)dx\right]$$

$$=\frac{1}{v^2}\int_{-\tau}^{\infty}e^{-v(x+\tau)}\,p(x)dt$$

$$+\frac{1}{v^2}e^{-v\tau}\int_{-\tau}^{\infty}e^{-vx}\,u(x)dx$$

Thus we get:

$ES\{u(t-\tau)\}=\vartheta'(v)+e^{-v\tau}ES\{u(t)\}$

Similary, the Emad – Sara transform for $u'(t-\tau)$ we can write as

$ES\{u'(t-\tau)\}=\frac{1}{v^2}\int_{t=0}^{\infty}e^{-vt}\,u'(t-\tau)dt$

$$=\frac{1}{v^2}\int_{x=-\tau}^{\infty}e^{-v(x+\tau)}\,u'(x)dt$$

$$=\frac{1}{v^2}\int_{-\tau}^{\infty}e^{-v(x+\tau)}\,p(x)dt+\frac{1}{v^2}e^{-v\tau}\int_{-\tau}^{\infty}e^{-vx}\,u'(x)dt$$

We have:

$$ES\{u'(t-\tau)\}=p''(v)+e^{-v\tau}Es\{u'(t)\}$$

Using the Emad Sara transform to eq.(1) gets:

$ES\{u''(t)\}+a\,ES\{u'(t)\}+bES\{u'(t)\}+cES\{u(t)\}+d\,ES\{u\,(t-\tau)\}=Es\{f(t)\}.$

And applying above equations we obtain:

$$\frac{-\gamma}{v^2} - \frac{\vartheta(0)}{v} + v^2\,ES\{u(t)\} + a\,v\,ES\{u(t)\} - a\frac{\vartheta(0)}{v^2} + b\,\vartheta''(v) + b\,e^{-v\tau}\,v\,Es\,\{u(t)\} - b\,e^{-v\tau}\frac{\vartheta(0)}{v^2} + c\,ES\,\{u(t)\} + d\,\vartheta'(v) + \,d\,e^{-v\tau}ES\{u(t)\} = ES\{f(t)\}$$

$$(v^2 + av + b\,e^{-v\tau} + c + d\,e^{-v\tau})ES\{u(t)\} = T(v) + \frac{-\gamma}{v^2} + \frac{\vartheta(0)}{v} + a\frac{\vartheta(0)}{v^2} + b\,\vartheta''(v) + b\,e^{-v\tau}\frac{\vartheta(0)}{v^2} - d\,\vartheta'(v)$$

It can be reduced to:

$$ES\{u(t)\} = \frac{T(v) + F(v)}{K(v)}$$

6. Application

In this section we introduce delay differential problem that demonstrate the validity of the obtained result (exact solutions):

Example (6-1): we consider the following delay differential problem:

$u''(t) - 3u'(t) + u'(t-1) + 2u(t) - u(t-1) = 0, where\ t \leq 0$

Subject to $u(t) = e^t, t \in [-1,0], u'(0) = 1$

If we take into consideration

$$ES\{0\} = T(v) = 0, F(v) = \frac{1}{v^2}(v - 2e^{-v}), K(v) = v^2 + 3v + 2 + (v-1)e^{-v}$$

So,

$$ES\{u(t)\} = \frac{1}{v^2}\left(\frac{v - 2 + e^{-v}}{(v-1)(v - 2 + e^{-v})}\right)$$

And

$$ES\{u(t)\} = \frac{1}{v^2(v-1)}\ take\ inverse$$

$u(t) = e^t$ is the exact solution of the above delay differential problem.

7. Conclusion

In this paper, the Emad_Sara integral transform is applied th evaluate the exact solution of a second order linear delay differential equations, demonstrating its efficacy as a technique for finding the exact solutions to a broad class of differential equations. This means that the technique given here can be used to issues of the neutral delay kind as well as the Volterra delay integro-differential kind.

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