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# Tunable Bands in Photonic Lieb Lattice by Graph Laplacian Approach

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***Abstract***: We theoretically investigate the dispersion relation of a waveguide array made of metamaterials. The unit cell contains three waveguides with adjustable optical properties. We model the propagation of high-intensity electromagnetic waves in the array by a generalized nonlinear Schrodinger equation replacing the Laplacian operator with the graph Laplacian. We found that when the nonlinear coefficient matrix is a null matrix, the dispersion curve supports three branches, each corresponding to three energy bands and one perfectly flat. On the other hand when it is a matrix of ones the degeneracy of the dispersion curve is decreased and the flat band is located on the top of the dispersive bands. Hence, this study revealed the impact of high-intensity optical light to manipulate the band structure of a photonic Lieb lattice made of metamaterial.

***Keywords*:** Negative index materials, Lieb lattice, Graph Laplacian, Photonic lattices.

## 1. Introduction

The graph theory approach has gained popularity in the field of complex optical networks in recent years. The propagation of electromagnetic waves in a complex optical network in the nonlinear regime has been discussed adopting stationary scattering theory on quantum graphs [1]. Utilizing the diophantine equations and the mapping procedure onto a Cayley graph, an iteration algorithm has been proposed to solve the nonlinear Schrodinger lattices [2]. The continuous-time quantum walks spatial search on a planar triangular optical lattice has been analyzed through numerical simulations and has been realized experimentally [3]. The dynamics of light through coupled arrays of non-identical waveguides with nonuniform coupling strengths has been studied using the graph Laplacian approach [4]. The completely integrable nonlinear Schrodinger model with an infinite number of constants of motion on the networks with vertices and bonds has been studied [5]. The analysis of soliton solutions of the cubic nonlinear Schrodinger equation on the network with the tadpole graph has been investigated [6]. The optical bistability and Ikeda instability during the dynamics of light within InGaAsP-InP-modified add-drop resonator has been investigated using a graphical approach [7].

On the other hand, there is a great interest in the investigations of the photonic spectrum of optical lattices with flat bands because of their potential application. The dispersion relation with flat bands admitted by photonic lattices of optical waveguides fabricated by Bessel beam multiplexing optical induction in the photorefractive media has been studied [8]. The localization of electromagnetic wave in photonic flat-band lattices in the quantum regime has been studied [9]. The circumstances to exhibit the diffractionless modes in a Kerr nonlinear Lieb lattice have been investigated [10]. The stability of the flat band modes in a rhombic nonlinear optical waveguide array depends on the intensity per waveguide [11]. Moreover, the experimental realization of the studies on flat bands has been reported [12, 13]. The emergence of nontrivial topological nearly flat bands can be modeled by the short-ranged tight-binding Hamiltonian [14]. The two-dimensional lattices formed by Sierpinski gasket fractal geometries as the basic unit cells can exhibit dispersion relations with multiple flat bands [15]. The coincidence of the flat-band eigenstate in both real and momentum spaces is necessary for the real-energy flat band in a two-dimensional non-Hermitian Lieb lattice [16].

Moreover, the nonlinear evolution of electromagnetic waves in waveguide arrays made of metamaterials has attracted widespread interest in recent years. The condensate solutions for electromagnetic pulse propagating in metamaterial waveguide array have been discussed [17]. The generation of staggered and unstaggered discrete solitons in metamaterial waveguide array near the zero-diffraction points has been reported [18]. The modulation instability and the generation of ultrashort pulses are highly influenced by the interaction of nonlinear effects of individual waveguides [19]. The diffractionless wave solutions in the non-Kerr photonic Lieb lattice with metamaterials have been derived and the stability of the solution has been analyzed [20].

Here we investigate the photonic spectrum pertaining to a waveguide array made of metamaterials. The unit cell of the waveguide array consists of three waveguides of different optical properties as depicted in Fig. 1. We model the nonlinear evolution of electromagnetic waves in the array by a generalized nonlinear Schrodinger equation replacing the Laplacian operator with the graph Laplacian. Taking the engineering freedom of metamaterial to tune material parameters into account, we analyze the dispersion relation for all possible combinations of nonlinear coefficient matrix including the defocussing nonlinearity. Here we report the impact of high-intensity optical light to manipulate the band structure of a photonic Lieb lattice made of metamaterial.

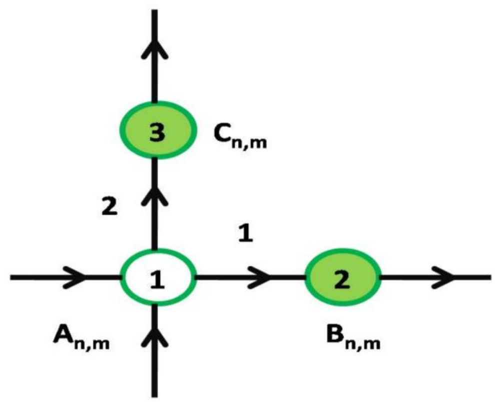


**FIG. 1:** The graphical representation of the unit cell of the photonic Lieb lattice with metamaterials. The vertices in the graph stand for waveguides and bonds represent the coupling between them. The arrows are oriented arbitrarily.

## 2. Theoretical model

The unit cell of the lattice consists of three waveguides, 1, 2 and 3. $A_{n,m}, B_{n,m}$ and $C_{n,m}$ stand for the amplitude of the waves propagating in waveguide 1, 2 and 3, respectively. The waveguides are nonidentical, they have different electromagnetic wave propagation characteristics such as refractive index, dielectric polarization propagation constant, etc. We will replace the waveguide array with its graphical representation. We consider the array as a graph $G(\rho, \sigma)$, which consists of a node-set $\rho$ and a branch set $\sigma$. Here the branch set $(\sigma)$ is an unordered pair of distinct nodes. Let us assume the elements in node set and branch set as $\rho = \{1, 2, 3 \dots . j\}$ and $\sigma = \{1, 2, 3 \dots . k\}$, respectively such that $j > 1$ and $k > 0$. The graphical representation of a cross-section of $n^{th}$ unit cell of the photonic Lieb lattice made of metamaterials is shown in Fig. 2, which has $j$ = 3 nodes and $k$ = 2 branches. The vertices in the graph stand for waveguides and bonds represent the coupling between them. The propagation of high-intensity electromagnetic waves in the array can be modeled by a generalized nonlinear Schrodinger equation replacing the Laplacian operator by the graph Laplacian as follows,

$$i\left(\sigma_1 \frac{\partial A_{n,m}}{\partial \xi} + \frac{\partial A_{n,m}}{\partial t}\right) =$$
$$\alpha\left(A_{n,m} - B_{n-1,m}\right) + \alpha\left(B_{n,m} - A_{n,m}\right) + \beta\left(C_{n,m} - A_{n,m}\right) +$$
$$\beta\left(A_{n,m} - C_{n,m-1}\right) - \gamma_1 \left|A_{n,m}\right|^2 A_{n,m}, \quad (1a)$$
$$i\left(\sigma_2 \frac{\partial B_{n,m}}{\partial \xi} + \frac{\partial B_{n,m}}{\partial t}\right) = \alpha\left(B_{n,m} - A_{n,m}\right) + \alpha\left(A_{n+1,m} - B_{n,m}\right) - \gamma_2 \left|B_{n,m}\right|^2 B_{n,m}, \quad (1b)$$
$$i\left(\sigma_3 \frac{\partial C_{n,m}}{\partial \xi} + \frac{\partial C_{n,m}}{\partial t}\right) = \beta\left(C_{n,m} - A_{n,m}\right) + \beta\left(A_{n,m+1} - C_{n,m}\right) - \gamma_3 \left|C_{n,m}\right|^2 C_{n,m}. \quad (1c)$$

where the pair (n, m) of integers locates the unit cell in the two-dimensional Lieb lattice. $\sigma_j$ with $j$ = 1, 2 and 3 stands for the sign of refractive index in respective waveguides. $\gamma_j$ with $j$ = 1, 2 and 3 are the selfphase modulation coefficients in waveguides 1, 2, and 3 respectively.

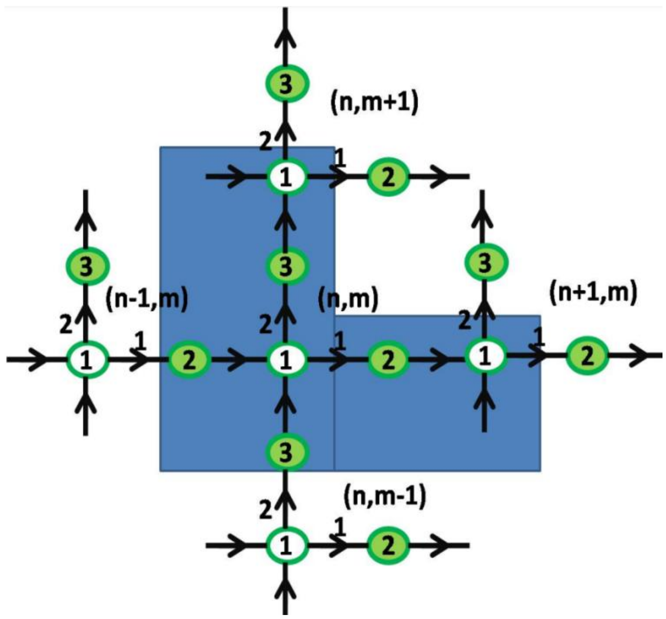


**FIG. 2:** Graphical representation showing the arraignment of unit cells in the waveguide array. Shaded portion in the graph shows the $(n,m)^{th}$ unit cell and adjacent waveguides directly coupled to it in the nearest neighbor unit cells.

Moreover, $\alpha$ and $\beta$ are the coupling coefficients along branches 1 and 2, respectively. To express the above set of Eq. 1 in a generalized form system of equation let us consider Fig. 2. Fig. 2 shows the coupling of $(n,m)^{th}$ unit cell with the nearest neighboring unit cells of the array. The shaded portion in the graph shows the $(n,m)^{th}$ unit cell and adjacent waveguides directly coupled to it in the nearest neighbor unit cells. The generalized form of system of Eq. (1) can be expressed as,

$$iS\Psi_\xi + i\Psi_t = G_1\Psi + G_2\Phi - \Gamma P(\Psi), \quad (2)$$

where $S$ is the refractive index matrix, $\Psi$ amplitude matrix, which corresponds to the $(n,m)^{th}$ unit cell of the lattice and $\Phi$ is the amplitude matrix corresponds to the waveguide connected to $(n,m)^{th}$ unit cell from nearest neighboring unit cells. The $3 \times 3$ matrices $G_1$ and $G_2$ are graph Laplacian. $\Gamma$ and $P(\Psi)$ are the nonlinear coefficient matrix and the matrix corresponds to the self-phase modulation, respectively. Which are given by,

$$S = (\sigma_1 \quad \sigma_2 \quad \sigma_3),$$
$$\Psi = \begin{pmatrix} A_{n,m} \\ B_{n,m} \\ C_{n,m} \end{pmatrix},$$

$$\Phi = \begin{pmatrix} A_{n,m+1} \\ B_{n-1,m} \\ C_{n,m-1} \end{pmatrix},$$

$$G_1 = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} 0 & -\alpha & -\beta \\ \alpha & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix},$$

$$\Gamma = (\gamma_1 \quad \gamma_2 \quad \gamma_3)$$

and

$$P(\Psi) = \begin{pmatrix} |A_{n,m}|^2 A_{n,m} \\ |B_{n,m}|^2 B_{n,m} \\ |B_{n,m}|^2 B_{n,m} \end{pmatrix}.$$

Considering the symmetry of the structure, here we have assumed the amplitudes of the wave propagating in all 1-nodes connected to are $(n,m)^{th}$ unit cell are equal.

## 3. Dispersion Relation

Now we derive the dispersion relations satisfied by the waves in the Lieb lattice made of metamaterials. Let us assume the quasiharmonic waves with the following form as the solution of Eq. (2),

$$\Psi = \psi e^{-i\omega\tau + ik_z\zeta + ik_1 n + ik_2 m}, \tag{3}$$

where

$$\psi = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix},$$

$k_z$ is a small correction to the wave vector along the propagation direction, $k_1 = k_x h$ and $k_2 = k_y h$ are normalized transverse wave-numbers. $h$ is the lattice parameter of the waveguide array. Also $k_x$ and $k_y$ are the quasi (Bloch) momenta of the two dimensional photonic Lieb lattice. After substituting Eq. (3) in Eq. (2) one can arrive in the following equation,

$$D_m \psi = 0 \tag{4}$$

where

$$D_m = \begin{pmatrix} \omega - k_z + f_1 & \kappa_1^* & \kappa_2^* \\ \kappa_1 & \omega + k_z + f_2 & 0 \\ \kappa_2 & 0 & \omega + k_z + f_3 \end{pmatrix},$$

also, where

$$F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \Gamma |\psi|^2 ,$$

which provides the nonlinearity contribution of dispersion relation originated as a result of Kerr nonlinearity. and the refractive index matrix is given by,

$$S = (1 \quad -1 \quad -1).$$

The normalized wave numbers in Eq. (4) are given by ,

$$\kappa_1 = \alpha(1 + e^{ik_1}), \tag{5a}$$

$$\kappa_2 = \beta(1 + e^{ik_2}). \tag{5b}$$

Non-zero solutions of Eq. (4) exist only when the determinant associated the matrix $D_m$ vanishes. This vanishing condition leads to the dispersion satisfied by the array.

### 3.1 Linear case

Let us consider the case where the nonlinear coefficient matrix ($\Gamma$) is a null matrix, which leads to linear polarization in all the nodes of the graph given in Fig. 1. Here we plot the frequency of electromagnetic waves passing through the lattice against the two-dimensional Bloch wave vectors denoted by $k_1$ and $k_2$ (dispersion relation) for linear polarization case in Fig. 3. It should be noted that the dispersion curve supports three branches, each of which corresponds to three energy bands. One of them is a perfectly flat, similar dispersion curve that can be observed in the case of photonic Lieb lattices and kagome lattices [13] and is known as a flat band. The other two are dispersive curved bands. The flat bands are static, and do not support the propagation of the localized state, and hence the localized states of flat bands are diffractionless. From Fig. 3, it is clear that at first Brillouin zone boundaries, the two curved dispersive bands intersect with the degenerate flat band located in the middle of them.

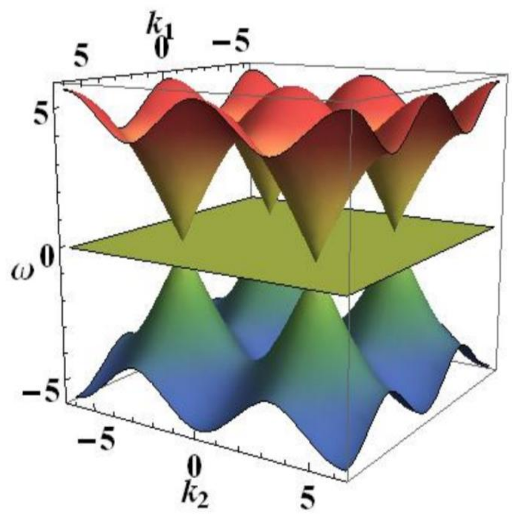


**FIG. 3:** Linear dispersion relation of the Lieb lattice with metamaterials.

### 3.2 Nonlinear case

Now, we will discuss the nature of the dispersion curve when the nonlinear coefficient matrix ($\Gamma$) is not a null matrix. Taking the advantage of engineering freedom of a metamaterial waveguide array, we here consider combinations of linear and nonlinear nodes. As a result, the nonlinear coefficient matrix of the waveguide array is a tunable matrix and hence it can support different dispersion curves depending upon the nature of nonlinearity.

Fig. 4 represents the dispersion curves supported by the waveguide array for different nonlinear coefficient matrices. Fig. 4(a) corresponds to the case where the nonlinear coefficient matrix, $\Gamma = (1 \quad 0 \quad 0)$. Comparing above $\Gamma$ matrix with refractive index matrix $S = (1 \quad -1 \quad -1)$ . one can see that in this case all positive index nodes are nonlinear with focussing nonlinearity, whereas all negative index nodes are linear. Compared with linear case here the flatband is shifted towards high and positive frequency from zero.

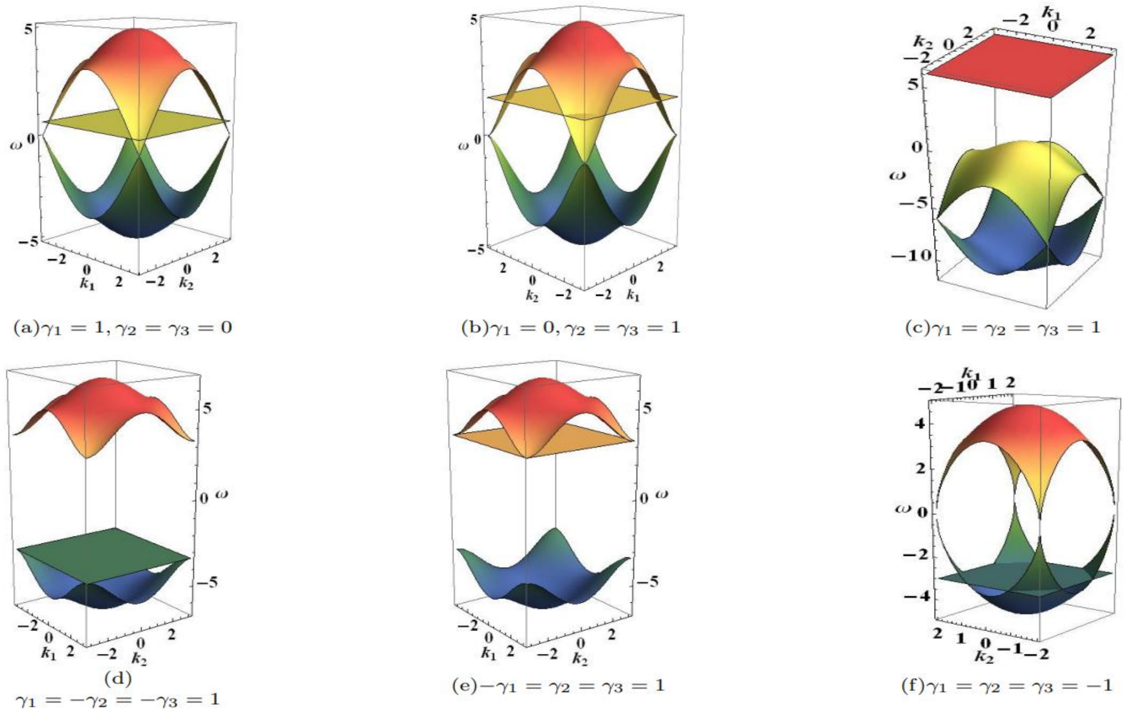


**FIG. 4:** (Color online.) Nonlinear dispersion relation of the Lieb lattice with metamaterials.

Now, consider the case $\Gamma = (0 \quad 1 \quad 1)$., where all positive index nodes are linear and all negative index nodes are nonlinear with focussing nonlinearity. Here the flat band is shifted further towards high and positive frequency, however, the dispersive band remains unchanged. When $\Gamma$ is a matrix of ones, the dispersion relation is depicted in Fig. 4(c). When all the nodes are nonlinear the degeneracy of the dispersion curve is decreased and the flat band is located on the top of the dispersive bands. The dispersive bands are shifted towards low and negative frequency regimes. Now we discuss the impact of defocussing nonlinearity associated with the waveguides. In this situation at least one of the elements in the Γ is a negative quantity. Fig. 4(d) depicts the case with $\Gamma = (1 \quad -1 \quad -1)$. That is all the positive index nodes of the array are with focussing nonlinearity and all the negative index nodes of the array are with defocussing nonlinearity. In this case, the upper conical band is shifted upwards, whereas the flat band and lower dispersive bands are shifted downwards. Now the situation is different when the positive index nodes of the array are with defocussing nonlinearity and all the negative index nodes of the array are with focussing nonlinearity as shown in Fig. 4(e). The lower dispersive band is shifted downwards, on the other hand, the upper dispersive band and flat band are shifted upwards to a positive frequency regime. Now, when all the nodes are with defocusing nonlinear type polarization, the dispersion curve is depicted in Fig. 4(f). That is when $\Gamma = (-1 \quad -1 \quad -1)$, the positions of dispersive bands are similar to the linear case and the flat band shifted downwards to negative frequency. Thus we can conclude that the nonzero nonlinear coefficient matrix alters the position of the flat band and conical band. In other words, the high-intensity optical light can be used to shift degenerate as well as dispersive bands and to manipulate the band structure of a photonic Lieb lattice made of metamaterial.

## 4. Conclusion

In this paper, we have theoretically studied the nature dispersion relation pertaining to a waveguide array made of metamaterials adopting the graph Laplacian approach. The waveguide array consists of unit cells with three waveguides of different optical properties. We have modeled the propagation of the high-intensity electromagnetic wave in the array by a generalized nonlinear Schrodinger equation replacing the Laplacian operator with the graph Laplacian. Considering the engineering freedom of metamaterial, we have studied the dispersion relation for all possible combinations of nonlinear coefficient matrix including the defocussing nonlinearity. We have found that when the nonlinear coefficient matrix is a null matrix the dispersion curve supports three branches, each of them corresponds to three energy bands and one of them is perfectly flat. On the other hand when it is a matrix of ones the degeneracy of the dispersion curve is decreased and the flat band is located on the top of the dispersive bands. Thus this study reports the impact of high-intensity optical light to manipulate the band structure of a photonic Lieb lattice made of metamaterial.

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