Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 45 No. 3 (2024)

Geodetic Decomposition of Zero Divisor Graph

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Abstract:- A graph $\Gamma(R)$ is said to be non-zero zero divisor graph of commutative ring R with identity if $u, v \in V(\Gamma(R))$ and $(u, v) \in E(\Gamma(Z))$ if and only if $uv = 0$. Let $K_{n,n}$ denote a balanced complete bipartite graph with parts of size n and let S_k denote a star with k edges. Let C_k denotes a cycle of length K, i.e., $S_k \equiv K_{1,k}$. Let $D(K_{m,n})$ be the decomposition of complete bipartite graph. Let $L = \{H_1, H_2, \ldots, H_r\}$ be a family of subgraphs of $\Gamma(R)$. An L-decomposition of G is an edge-disjoint decomposition of $\Gamma(R)$ into positive integer α_i copies of H_i where $i \in \{1,2,3, \ldots, r\}$. Furthermore, if each $H_i (i \in \{1,2,3, \ldots, r\})$ is isomorphic to a graph H, then we say that G has an H-decomposition. In this paper, we investigate the concept of geodetic decomposition of zero divisor graph. Let $\Gamma(Z_n) = (V(\Gamma(Z_n)), E(\Gamma(Z_n))))$ be the zero divisor graph. For a non-empty set S of $V(\Gamma(Z_n))$ we define $I[S] = \cup I[x, y]$, for some $x, y \in S$, where $I[x, y]$ is the closed interval consisting of x, y and all vertices lying on some $x - y$ geodesic of $\Gamma(Z_n)$. We have discussed the geodetic number of zero divisor graph $\Gamma(R)$ and determine the geodesic decomposition of zero divisor graph of the ring $R = Z_{2p}, Z_{pq}, Z_{p^2}, Z_p \times Z_p$.

***Keywords**:* Geodetic graph, graph decomposition, zero divisor graph.

1. Introduction

All graphs considered here are finite and undirected, unless otherwise noted. For the standard graph-theoretic terminology the reader is referred to [1]. Geodetic graphs were first defined by Ore in 1962. He also proposed to characterize them. Characterized geodetic graphs. Before that [2] characterized edge geodetic covers graphs. M. Atici [3] revealed that the characteristic of edge geodetic number of graph. Constructed geodetic graphs based on pulling subgraphs homeomorphic to complete graphs in 1984.

The zero divisor graph is very useful to find the algebraic structures and properties of rings. The idea of a zero divisor graph of a commutative ring was introduced by I. Beck's in [4]. Given a ring R, let G(R) denote the graph whose vertex set is R, such that distinct vertices r and s are adjacent provided that rs=0. I.Beck's main interest was the chromatic number χ(G(R)) of the graph G(R). The general terminology, notation everything based on the papers [[5] - [12]].

Definition 1 [10] Let R be a commutative ring (with 1) and let $Z(R)$ be its set of zero-divisors. We associate a (simple) graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisor of R, and for distinct $x, y \in Z(R)^*$ the vertices x and y are adjacent if and only if $xy = 0$. Thus $\Gamma(R)$ is the empty graph if and only if R is an integral domain.

Example 1 A graph G is said to be weakly geodetic if every two vertices u and v of G with distance $d_G(u, v) = 2$ are joined by exactly one shortest path.

Definition 2 Figure 1, the graph G_2 is weakly geodetic. But the graph G_1 is not weakly geodetic, as there are two shortest path between v_1 and v_3 with $d_G(v_1, v_3) = 2$ in G_2.

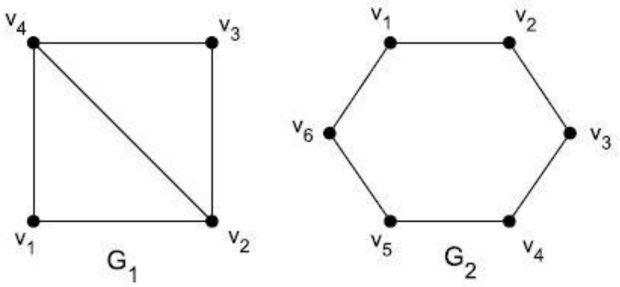

Figure 1. G_2 is weakly geodetic and G_1 is not weakly geodetic

Definition 3 A graph G is geodetic if for any two vertices u and v of G there exist at most one shortest path between them. Alternatively, a graph G is geodetic if given any pair of vertices u and v of G, there exist a unique $u - v$ geodesic in G, where $u - v$ geodesic in a graph is a shortest $u - v$ path P. Thus the distance $d_G(u, v) = |P|$.

Example 2 Figure 2, the graph G_3 is geodetic. But the graph G_2 is not geodetic, as there are two shortest path between v_1 and v_4 in G_1.

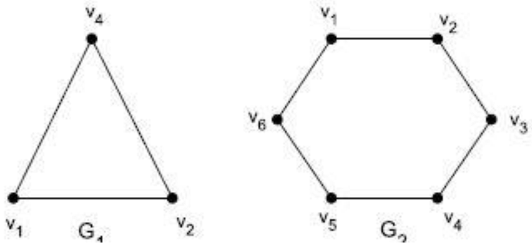

Figure 2. G_3 is geodetic and G_2 is not geodetic

Definition 4 A graph G is called strongly geodetic if every two vertices of G are joined by at most one path of length less than or equal to the diameter of G.

Example 3 Figure 3, the graph G_4 is strongly geodetic. But the graph G_3 is not strongly geodetic, as there is path between v_1 and v_3 of length 2 in G_3, where diameter of G_3 is 1.

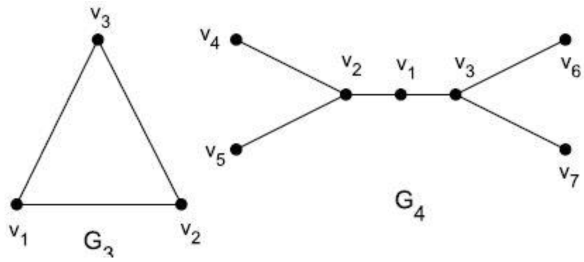

Figure 3. G_4 is strongly geodetic and G_3 is not strongly geodetic

Remark 1 Every strongly geodetic graph is geodetic and every geodetic graph is weakly geodetic.

Note that a graph G is weakly geodetic if and only if G contains no induced subgraph isomorphic to C_4 or K_4-e. It is obvious that a graph G is geodetic if and only if each block of G is geodetic. A forest is unique disconnected strongly geodetic graph. Moreover, it is easy to observe that if a strongly geodetic graph G contains a triangle, then G is a complete graph.

Theorem 1 For any graph $\Gamma(Z_{2p})$, the edge geodetic number of $\Gamma(Z_{2p})$ is equal to the number of end-vertices in $\Gamma(Z_{2p})$.

Theorem 2 For the star $\Gamma(Z_{2p}) \cong K_{1,p-1}, g_e\left(\Gamma(Z_{2p})\right) = p - 1 = q$.

2. Geodetic Decomposition of Zero Divisor Graph $\Gamma(Z_{pq})$

In this section we investigate the problem of geodetic number in zero divisor graph $\Gamma(Z_{pq})$, for each p and q are distinct prime numbers with $q > p$.

Theorem 3 For any graph $\Gamma(Z_{p^2}), g\left(\Gamma(Z_{p^2})\right) = \frac{(p-1)(p-2)}{2}$ if and only if $\Gamma(Z_{p^2})$ is complete, where p is any prime number with $p > 3$.

Proof. If $\Gamma(Z_{p^2})$ is complete, then $E\left(\Gamma(Z_{p^2})\right)$ is the unique geodesic decomposition of $\Gamma(Z_{p^2})$, so that $g\left(\Gamma(Z_{p^2})\right) = \frac{(p-1)(p-2)}{2}$.

Conversely, let us assume that $g\left(\Gamma(Z_{p^2})\right) = \frac{(p-1)(p-2)}{2}$. Let us show that $\Gamma(Z_{p^2})$ is a complete graph. Suppose $\Gamma(Z_{p^2})$ is not complete. Let u and w be any two non-adjacent vertices of $\Gamma(Z_{p^2})$. Let $P = (u = u_1, u_2, \cdots, u_n = w)$ be a shortest u-w path. Then, $\gamma = \{P\} \cup \left\{E\left(\Gamma(Z_{p^2})\right) \backslash E(P)\right\}$ is a geodesic decomposition of $\Gamma(Z_{p^2})$ and $|\phi|$, so that $g\left(\Gamma(Z_{p^2})\right) < \frac{(p-1)(p-2)}{2}$ which is a contradiction. Hence, $\Gamma(Z_{p^2})$ is complete graph.

Theorem 4 For any graph $\Gamma(Z_{2p})$ and p is any prime number with $p \geq 5$ then $g\left(\Gamma(Z_{2p})\right) = p - 2$.

Proof. Let $\Gamma(Z_{2p})$ be a graph with p vertices, where p is any prime number is greater than 2. We know that $\Gamma(Z_{2p})$ is isomorphic to $K_{1,p-1}$. That is $\Gamma(Z_{2p})$ is a complete bipartite graph with p vertices. Let us take $p = 7$.

Then, $\Gamma(Z_{2p}) = \Gamma(Z_{14}) \cong K_{1,6}$

The vertex set of $\Gamma(Z_{2p}) = \Gamma(Z_{14}) = \{2,4,6,7,8,10,12\}$.

Let us take the geodetic decomposition number of $\Gamma(Z_{14}) = g(K_{1,6})$ is, $\{2,7,8\}, \{7,10\}, \{7,12\}, \{4,7\}, \{6,7\}$

(or)

$\{4,7,10\}, \{2,7\}, \{7,12\}, \{7,8\}, \{6,8\}$

(or)

$\{6,7,12\}, \{2,7\}, \{4,7\}, \{7,8\}, \{7,10\}$

and so on.

Clearly, $g(\Gamma(Z_{14})) = 5 = 7 - 2 = p - 2$

Let us take $p = 11$. Then, $\Gamma(Z_{2p}) = \Gamma(Z_{22}) \cong K_{1,10}$.

The vertex set of $\Gamma(Z_{2p}) = \Gamma(Z_{22}) = \{2,4,6,8,10,11,12,14,16,18,20\}$.

Clearly, the geodetic decomposition of $\Gamma(Z_{22}) = g(K_{1,10})$ is

$\{2,11,14\}, \{4,11\}, \{6,11\}, \{8,11\}, \{10,11\}, \{11,12\}, \{11,16\}, \{11,18\}, \{11,20\}$

(or)

$\{4,11,14\}, \{2,11\}, \{6,11\}, \{8,11\}, \{10,11\}, \{11,12\}, \{11,16\}, \{11,18\}, \{11,20\}$

and so on.

Clearly, $g(\Gamma(Z_{22})) = 9 = 11 - 2 = p - 2$

Generally, we continue the same process, finally we get for any $\Gamma(Z_{2p})$, the geodetic decomposition number is p-2, where p is any prime number.

Example 4 For example of above Theorem 4, $\Gamma(Z_{14}) = \{2,4,6,7,8,10,12\}$. The Fig. 4 clearly shows that geodetic decomposition number of zero divisor graph of $g(\Gamma(Z_{14})) = 7 - 2 = 5$

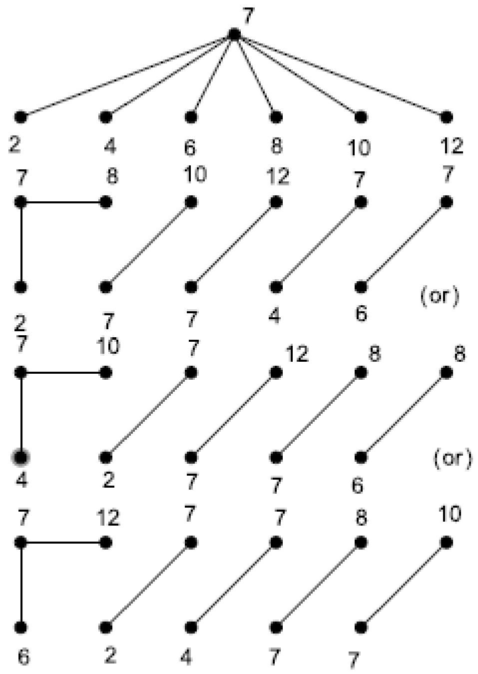

Figure 4. Geodetic number of $g(\Gamma(Z_{14}))$

Theorem 5 For any graph $\Gamma(Z_{3p})$ and p is any prime number with $p \geq 5$ then $g\left(\Gamma(Z_{3p})\right) = p - 1$.

Proof. Let $\Gamma(Z_{3p})$ be a complete bipartite graph with $p + 1$ vertices, where p is any prime number which is greater than 3. Since, $\Gamma(Z_{3p})$ is isomorphic to $K_{2,p-1}$ (see Fig. 5). That is, $\Gamma(Z_{3p}) \cong K_{2,p-1}$.

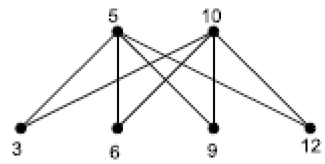

Figure 5: $g(\Gamma(Z_{15}))$

Let (V_1, V_2), where $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, v_2, \ldots, v_{p-1}\}$ be the vertex partition of $\Gamma(Z_{3p})$.

Let $p = 5$. Then, $\Gamma(Z_{3p}) = \Gamma(Z_{15}) \cong K_{2,4}$.

The vertex set of $\Gamma(Z_{15}) = \{3,5,6,9,10,12\}$.

Then, $V_1 = \{5,10\}$ and $V_2 = \{3,6,9,12\}$ be the vertex set of $\Gamma(Z_{15})$.

Let us take the geodetic decomposition of $\Gamma(Z_{15}) = g(K_{2,4})$ is,

$\{5,3,10\}, \{5,6,10\}, \{5,9,10\}, \{5,12,10\}$

(or)

$\{3,5,6\}, \{3,10,6\}, \{9,10,12\}, \{9,5,12\}$

and so on.

Clearly, $g(\Gamma(Z_{15})) = 4 = 5 - 1 = p - 1$.

Now let us take $p = 7$. Then, $\Gamma(Z_{3p}) = \Gamma(Z_{21}) \cong K_{2,6}$ (see Fig. 6)

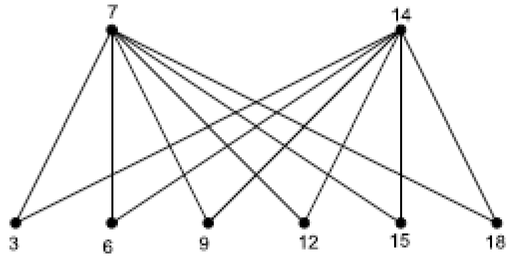

Figure 6: $g(\Gamma(Z_{21})$

The vertex set of $\Gamma(Z_{21}) = \{3,6,7,9,12,14,15,18\}$.

Then, $V_1 = \{7,14\}$ and $V_2 = \{3,6,9,12,15,18\}$ be the vertex set of $\Gamma(Z_{21})$.

Let us take the geodesic decomposition $g(\Gamma(Z_{21})) = g\left(K_{2,6}\right)$ is,

$$\{7,3,14\}, \{7,6,14\}, \{7,9,14\}, \{7,12,14\}, \{7,15,14\}, \{7,18,14\}$$

(or)

$$\{3,7,6\}, \{3,14,6\}, \{9,7,12\}, \{9,14,12\}, \{15,7,18\}, \{15,14,18\}$$

and so on.

Clearly, $g(\Gamma(Z_{21})) = 6 = 7 - 1 = p - 1$.

Generally, continuing the same process, finally we get for any $\Gamma(Z_{3p})$, the geodesic decomposition number is $p - 1$, where p is any prime number.

Theorem 6 For any graph $\Gamma(Z_{pq})$ where p is any prime > 2, then $g\left(\Gamma\left(Z_{pq}\right)\right) = pq - 2(p+q) + 3$.

Proof. Since $\Gamma(Z_{pq})$ is a complete bipartite graph and $\Gamma(Z_{pq})$ is isomorphic to $K_{p-1,q-1}$, where p and q are distinct prime numbers with $p < q$ (see Fig. 7).

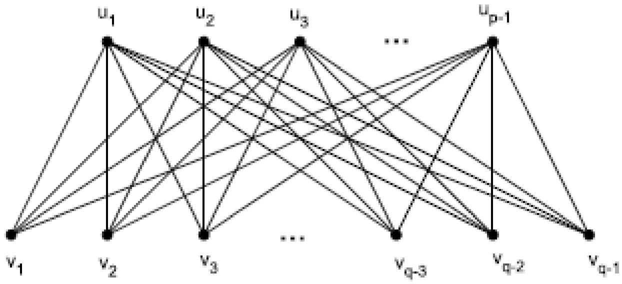

Figure 7: $g(\Gamma(Z_{pq})$

Then, $\{(v_i, u_i, v_{i+1})$ where $1 \le i \le p-2\} \cup \{(v_{p-1}, u_{p-1}, v_1), (u_2, v_1, u_3), (u_3, v_2, u_4), (u_4, v_3, u_1)\} \cup \{(u_1, v_i, u_2)$ where $i \ge 4\}$ is a collection of internally disjoint and edge disjoint shortest paths without exterior points and hence, $g(\Gamma(Z_{pq})) = (p-1)(q-1) - (p+q-2) = (p-1)(q-1) - p - q + 2 = pq - p - q + 1 - p - q + 2 = pq - 2(p+q) + 3$.

Hence proved the theorem.

3. Geodetic decomposition of zero divisor graph of $\Gamma(Z_{p^2})$

Theorem 7 If p and q are distinct prime numbers with $p < q$ then, $g\left(\Gamma(Z_{pq})\right)$ is $\frac{(p-1)(q-1)}{2}$.

Proof. Suppose $\Gamma(Z_{pq})$ is a complete bipartite graph and $\Gamma(Z_{pq})$ is isomorphic to $K_{p-1,q-1}$ where p and q are distinct prime numbers with $p < q$. Let V_1, V_2 be the vertex partition of $\Gamma(Z_{pq})$ where $V_1 = \{u_1, u_2, \dots, u_{p-1}\}$ and $V_2 = \{v_1, v_2, \dots, v_{q-1}\}$. Then, $\{v_i, u_i, v_{i+1}$ where $1 \le i \le p-2\} \cup \{(v_{p-1}, u_{p-1}, v_1), (u_2, v_1, u_3), (u_3, v_2, u_4), (u_4, v_3, u_1)\} \cup \{u_1, v_i, u_2$ where $i \ge 4\}$ is a collection of internally disjoint and edge disjoint shortest paths without exterior points and hence, $g\left(\Gamma(Z_{pq})\right) = \frac{(p-1)(q-1)}{2}$.

Definition 5 A path partition of a graph $\Gamma(Z_n)$ in which every path is a shortest path is called a geodetic partition of $\Gamma(Z_n)$. The minimum cardinality of a geodetic partition of $\Gamma(Z_n)$ is called the geodetic partition number of $\Gamma(Z_n)$ and is denoted by $g(\Gamma(Z_n))$.

Theorem 8 For any complete bipartite graph, $\Gamma(Z_{pq}) \neq K\_1,1$, then $g\left(\Gamma(Z_{pq})\right) = \frac{(p-1)(q-1)}{2}$, where p is prime number.

Proof. Suppose $\Gamma(Z_{2p})$ is a star graph, that is, $\Gamma(Z_{2p}) \cong K_{1,p-1}$, then $g\left(\Gamma(Z_{2p})\right) = \left\lceil \frac{p-1}{2} \right\rceil$.

Let $p = 5$, then $\Gamma(Z_{2p}) = \Gamma(Z_{10})$. The vertex set of $\Gamma(Z_{10})$ is {2,4,5,6,8} (see Fig. 8).

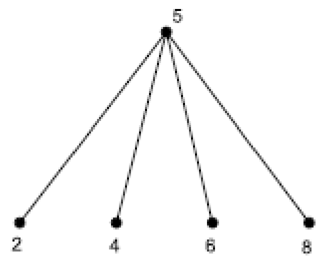

Figure 8: $\Gamma(Z_{10})$

The geodetic partition of $\Gamma(Z_{10})$ is {2,5,4} and {6,5,8}.

That is, $g\left(\Gamma(Z_{2p})\right) = 2 = \left\lceil \frac{5-1}{2} \right\rceil = \left\lceil \frac{p-1}{2} \right\rceil$, where $p = 5$.

This result is true for all $\Gamma(Z_{2p})$.

That is, $g\left(\Gamma(Z_{2p})\right) = \left\lceil \frac{p-1}{2} \right\rceil$ where, $p-1$ is the number of edges in $\Gamma(Z_{2p})$.

Let $\Gamma(Z_{pq})$ where p and q are distinct prime numbers with $p < q$.

Let (V_1, V_2), where $V_1 = \{u_1, u_2, \dots, u_{p-1}\}$ and $V_2 = \{v_1, v_2, \dots, v_{q-1}\}$ be the vertex partition of $\Gamma(Z_{pq})$ where p and q are distinct prime numbers. Since $\Gamma(Z_{pq})$ is a complete bipartite graph and $\Gamma(Z_{pq})$ is isomorphic to $K_{p-1,q-1}$. By the inequality assumption, $g(\Gamma(Z_n)) \ge \left\lceil \frac{(p-1)(q-1)}{2} \right\rceil$.

Since p and q are distinct prime numbers, so $p-1$ and $q-1$ are all even numbers, except the prime number 2. Let $(p-1) = 2k$. Then, $\{v_{2i-1}, u_j, v_{2i} | 1 \le j \le k, 1 \le j \le q-1\}$ geodetic partition of $\Gamma(Z_{pq})$ with the cardinality $\frac{(p-1)(q-1)}{2}$ and hence, $g(\Gamma(Z_n)) = \left\lceil\frac{(p-1)(q-1)}{2}\right\rceil$ where p is prime number, which is equal to $\frac{(p-1)(q-1)}{2}$. Hence proved.

4. Geodetic Decomposition of Zero Divisor Graph $\Gamma(Z_n)$

In this section we investigate the problem of decomposition of zero divisor graph $\Gamma(Z_n)$ of a ring $Z_n = Z_{pq}, Z_{p^2}, Z_p \times Z_p$ into complete, star graph, complete bipartite graph for each p and q are distinct prime numbers with $q > p$.

Definition 6 Let $\Gamma(Z_n)$ be a any connected zero divisor graph and $(\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_n)$ be the decomposition of $\Gamma(Z_n)$. The zero divisor graph $\Gamma(Z_n)$ admits a Geodetic Decomposition, if the following conditions are satisfied.

1. Each Γ_i is connected.
2. Each edge of $\Gamma(Z_n)$ is in exactly on Γ_i .
3. $g(\Gamma_i) = i + 1 (i \leq 1)$, where $g(\Gamma(Z_n))$ is the geodetic number of a zero divisor graph $\Gamma(Z_n)$.

Example 5 The following Fig. 9 illustrates the geodetic decomposition of G.

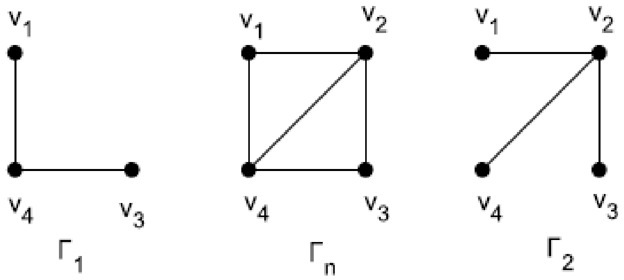

Figure 9: Geodetic decomposition of (Γ_1, Γ_2) in Γ_n

Remark 2 Every path as a geodetic decomposition, since the geodetic number of each path is 2.

Theorem 9 If p is a prime number with $p > 7$, then $g\left(\Gamma(Z_{7p})\right) = \Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_6$ and $K_{6,p-7}$.

Proof. Let $\Gamma(Z_{7p})$ be a zero divisor graph. The vertex set of $\Gamma(Z_{7p})$ is $\{7,14,21,\ldots,7(p-1), p, 2p, 3p, \ldots, 6p\} = \{u_1, u_2, u_3, \ldots, u_{p-1}, v_1, v_2, v_3, v_4, v_5, v_6\}$ and the vertex set can be partition into two subsets are $V_1, V_2 \in V\left(\Gamma(Z_{7p})\right)$ where $V_1 = \{u_1, u_2, u_3, \ldots, u_{p-1}\} V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Let the graph $\Gamma(Z_{7p})$ can be decomposing into two parts namely $K_{6,6}$ and $K_{6,p-7}$.

Case(i) Now, consider the vertex set of $V(K_{6,6})$ are $X = \{u_1, u_2, u_3, \ldots, u_6\}$ and $Y = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Let $U_3 = \{u_4, v_3, v_4, v_5, v_6\}$ be the subset of $V\left(\Gamma(Z_{7p})\right)$. Clearly, $< U_3 >= S_4$. Also $g(S_4) = 4$. Then $g(\Gamma_3) = 4$.

Let $U_4 = \{u_5, v_1, v_3, v_4, v_5, v_6\}$ be the subset of $V\left(\Gamma(Z_{7p})\right)$. Clearly, $< U_4 >= S_5$. Also $g(S_5) = 5$. Then $g(\Gamma_4) = 5$.

Let $U_5 = \{u_6, v_1, v_2, v_3, v_4, v_5, v_6\}$ be the subset of $V\left(\Gamma(Z_{7p})\right)$. Clearly, $< U_5 >= S_6$. Also $g(S_6) = 6$. Then $g(\Gamma_5) = 6$. Now, we construct Γ_1 and Γ_2 as follows:

After constructing the graphs $\Gamma_3, \Gamma_4, \Gamma_5, \ldots, \Gamma_{\frac{p-1}{2}}$. We construct the graphs Γ_1 and Γ_2 with the rest of the edges. Take Γ_2 as $K_{3,6} - K_{1,1}$ with $V(\Gamma(Z_{7p}) = \{u_1, u_2, u_3, v_1, v_2, \ldots, v_6\}$ and $E\left(\Gamma(Z_{7p})\right) = K_{3,6} - \{u_1v_1, u_1v_2, \ldots, u_1v_6\}$.

Let $U_2 = \{u_1, u_2, u_3\}$ is the minimum geodetic set and $g(\Gamma_2) = 3$.

The remaining edges are, $u_1v_1, u_4v_1, u_4v_2, , u_5v_2$. We construct the graph Γ_1 as $K_{1,1}$ + Remaining edges. Then $\Gamma_1 = \{u_1, u_4, u_5, v_1, v_2\}$. In this case $\{u_1, u_5\}$ is the minimum geodetic set and $g(\Gamma_1) = 2$. Hence each Γ_i satisfies the condition $g(\Gamma_i) = i + 1\ \forall i$. Hence $\Gamma(Z_{7p})$ admits geodetic decomposition $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_{\frac{p-1}{2}}$.

Case(ii) Consider, the second part of the graph as $K_{6,p-7}$ this graph decomposition into $\frac{6(p-7)}{4}$ copies of C_4 graph. Clearly, $g(K_{6,p-7})$ geodetic number is 2.

Theorem 10 If p and q are distinct prime numbers with $q > p$, then $g\left(\Gamma(Z_{pq})\right) = \Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_{p-1}$ and $K_{p-1,q-p}$.

Proof Let $\Gamma(Z_{pq})$ be a zero divisor graph. The vertex set of $\Gamma(Z_{pq})$ is $\{p, 2p, 3p, \ldots, p(q-1), q, 2q, 3q, \ldots, q(p-1)\} = \{u_1, u_2, u_3, \ldots, u_{q-1}, v_1, v_2, v_3, \ldots, v_{p-1}\}$ and the vertex set can be partition into two subsets are $V_1, V_2 \in V\left(\Gamma(Z_{pq})\right)$ where $V_1 = \{u_1, u_2, u_3, \ldots, u_{q-1}\} V_2 = \{v_1, v_2, v_3, \ldots, v_{p-1}\}$. Let the graph $\Gamma(Z_{pq})$ can be decomposing into two parts namely $K_{p-1,p-1}$ and $K_{q-p,p-1}$.

Case(i) Now, consider the vertex set of $V(K_{p-1,p-1})$ are $X = \{u_1, u_2, u_3, \ldots, u_{p-1}\}$ and $Y = \{v_1, v_2, v_3, \ldots, v_{p-1}\}$. Let $U\_3 = \{u_4, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}\}$ be the subset of $V\left(\Gamma(Z_{pq})\right)$. Clearly, $< U\_3 >= S_4$. Also $g(S_4) = 4$. Let it be Γ_3. Then $g(\Gamma_3) = 4$.

Let $U_4 = \{u_5, v_1, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}\}$ be the subset of $V\left(\Gamma(Z_{pq})\right)$. Clearly, $< U_4 >= S_5$. Also $g(S_5) = 5$. Then $g(\Gamma_3) = 4$.

Let $U_5 = \{u_6, v_1, v_2, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}\}$ be the subset of $V\left(\Gamma(Z_{pq})\right)$. Clearly, $< U_5 >= S_6$. Also $g(S_6) = 6$. Then $g(\Gamma_5) = 6$.

Continuing in this way we get, Γ_{p-2} as S_{p-1} with $U_{p-2}(\Gamma) = \{u_{p-1}, v_1, v_2, v_3, \ldots, v_{p-1}\}$

Clearly,

$$g(\Gamma_{p-2}) = p - 1$$

Now, we construct Γ_1 and Γ_2 as follows:

After constructing the graphs $\Gamma_3, \Gamma_4, \Gamma_5, \ldots, \Gamma_{\frac{p-1}{2}}$. We construct the graphs Γ_1 and Γ_2 with the rest of the edges. Take Γ_2 as $K_{3,p-1} - K_{1,p-6}$ with $V(\Gamma_2) = \{u_1, u_2, u_3, v_1, v_2, \ldots, v_{p-1}\}$ and $E(\Gamma_2) = K_{3,p-1} - \{u_1v_1, u_1v_2, \ldots, u_1v_{p-6}\}$.

Let $U_2 = \{u_1, u_2, u_3\}$ is the minimum geodetic set and $g(\Gamma_2) = 3$.

The remaining edges are, $u_1v_1, u_1v_2, \ldots, u_1v_{p-6}, u_4v_1, u_4v_2, \ldots, u_4v_{p-5}, u_5v_2, u_5v_3, \ldots, u_5v_{p-5}, \ldots, u_{p-2}v_{p-5}$. We construct the graph Γ_1 as $K_{1,p-6}$ + Remaining edges.

Then $\Gamma\_1 = \{u_1, u_4, \ldots, u_{p-2}, v_1, v_2, \ldots, v_{p-5}\}$. In this case $\{u_1, u_{p-2}\}$ is the minimum geodetic set and $g(\Gamma_1) = 2$. Hence each Γ_i satisfies the condition $g(\Gamma_i) = i + 1\ \forall i$. Hence $\Gamma(Z_{pq})$ admits geodetic decomposition $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_{\frac{p-1}{2}}$.

Case(ii) Consider, the second part of the graph as $K_{p-7,q-p}$. This graph decomposition into $\frac{(p-1)(q-p)}{4}$ copies of C_4 graph. Clearly, $g(K_{p-1,q-p})$ geodetic number is 2.

Theorem 11 If p is any prime $p > 2$ then the graph $\Gamma(Z_p \times Z_p)$ is admits a geodetic decomposition into $\Gamma_1, \Gamma_2, \ldots, \Gamma_{p-2}$.

Proof. Trivially $\Gamma(Z_2 \times Z_2), \Gamma(Z_3 \times Z_3), \Gamma(Z_5 \times Z_5)$ admits geodetic decomposition, since $g(\Gamma(Z_3 \times Z_3)) = 2, \Gamma(Z_5 \times Z_5) = K_{1,1} \cup (K_{3,3} - K_{1,1}) \cup K_{1,4}$ and satisfies the condition $g(\Gamma_i) = i + 1$.

Let $\Gamma(Z_p \times Z_p) = K_{p-1,p-1}$ with p is prime > 5. Let (U, V) be the partition of $\Gamma(Z_p \times Z_p)$, where $U = \{u_1, u_2, \ldots, u_{p-1}\}$ and $V = \{v_1, v_2, \ldots, v_{p-1}\}$.

Let $V_3 = \{u_4, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}\}$ be the subset of $V\left(\Gamma(Z_p \times Z_p)\right)$. Clearly, $< V_3 >= S_4$.

Also $g(S_4) = 4$.

Let it be Γ_3. Then, $g(\Gamma_3) = 4$.

Let $V_4 = \{u_5, v_1, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}\}$. Clearly $< V_4 >= S_5$.

Also $g(S_5) = 5$.

Let it be Γ_4. Then, $g(\Gamma_4) = 5$.

Let $V_5 = \{u_6, v_1, v_2, v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}\}$. Clearly $< V_5 >= S_6$. Also $g(S_6) = 6$.

Let it be Γ_5. Then, $g(\Gamma_5) = 6$.

Continuing in this way we get, Γ_{p-2} as S_{p-1} with $V(\Gamma_{p-2}) = \{u_{p-1}, v_1, v_2, \cdots, v_{p-1}\}$.

Clearly $g(\Gamma_{p-2}) = p - 1$.

Now, we construct Γ_1 and Γ_2 as follows:

After construction the graphs $\Gamma_3, \Gamma_4, \ldots, \Gamma_{p-2}$.

We construct the graphs Γ_1 and Γ_2 with the rest of the edges.

Take Γ_2 as $K_{3,p-1} - K_{1,p-6}$ with $V_{\Gamma_2} = \{u_1, u_2, u_3, v_1, v_2, \cdots, v_{p-1}\}$ and $E(\Gamma_2) = K_{3,p-1} - \{u_1v_1, u_1v_2, \cdots, u_1v_{p-6}\}$. Then $\{u_1, u_2, u_3\}$ is the minimum geodetic set and $g(\Gamma_2) = 3$. The remaining edges are,

$u_1v_1, u_1v_2, \cdots, u_1v_{p-6}, u_4v_1, u_4v_2, \cdots, u_4v_{p-5}, u_5v_2, u_5v_3, \cdots, u_5v_{p-5}, \cdots, u_{p-2}v_{p-5}$.

We construct the graph Γ_1 as $K_{1,p-5}$ + Remaining edges.

Then $V(\Gamma_1) = \{u_1, u_4, u_5, \cdots, u_{p-2}, v_1, v_2, \cdots, v_{p-5}\}$.

In this case $\{u_1, u_{p-2}\}$ is the minimum geodetic set and $g(\Gamma_1) = 2$.

Hence each Γ_i satisfies the condition $g(\Gamma_i) = i + 1$ for all i.

Thus $\Gamma(Z_p \times Z_p)$ admits geodetic decomposition $(\Gamma_1, \Gamma_2, \Gamma_3, \cdots, \Gamma_{p-2})$.

Theorem 12 Complete graph $\Gamma(Z_{p^2})$ with any prime $p > 5$ admits a geodetic decomposition $\Gamma\_1, \Gamma\_2, \ldots, \Gamma_{p-3}$.

Proof. Let $v_1, v_2, \ldots, v_{p-1}$ be the vertices of $\Gamma(Z_{p^2})$.

Consider $\Gamma_1 = S_2$ with $V\left(\Gamma(Z_{p^2})\right) = \{v_1, v_2, v_{p-2}\}, E(\Gamma_1) = \{v_1v_2, v_2v_{p-2}\}$ and $\{v_1, v_{p-2}\}$ is the geodetic set of Γ_1 and hence $g(\Gamma_1) = 2$.

Let as construct $\Gamma_2 = S_3$ with $V(\Gamma_2) = \{v_1, v_2, v_{p-2}\}, E(\Gamma_2) = \{v_3v_1, v_3v_2, v_3v_{p-2}\}$ and $\{v_1, v_2, v_{p-2}\}$ is geodetic set of Γ_2 and $g(\Gamma_2) = 3$.

Similarly, we can construct $\Gamma_3 = S_4$ with $V(\Gamma_3) = \{v_1, v_2, v_3, v_4, v_{p-2}\}$,

$E(\Gamma_3) = \{v_4v_1, v_4v_2, v_4v_3, v_4v_{p-2}\}$ and $\{v_1, v_2, v_3, v_4, v_{p-2}\}$ is the geodetic set of Γ_3 and $g(\Gamma_3) = 4$.

Continuing this process we get, $\Gamma_{p-4} = S_{p-3}$ with $V(\Gamma_{p-4}) = \{v_1, v_2, \ldots, v_{p-4}, v_{p-3}, v_{p-2}\}$,

$E(\Gamma_{p-4}) = \{v_{p-3}v_1, v_{p-3}v_2, \ldots, v_{p-3}v_{p-4}, v_{p-3}v_{p-2}\}$ and $\{v_1, v_2, \ldots, v_{p-4}, v_{p-2}\}$ is the geodetic set of Γ_{p-4} and $g(\Gamma_{p-4}) = p - 3$.

Finally, we get, Γ_{p-3} with $V(\Gamma_{p-3}) = \{v_1, v_2, \ldots, v_{p-2}, v_{p-1}\}$,

$E(\Gamma_{p-3}) = \{v_{p-1}v_1, v_{p-1}v_{p-2}, v_1v_{p-2}, v_{p-1}v_2, v_{p-1}v_3, \ldots, v_{p-1}v_{p-3}\}$ and $\{v_1, v_2, \ldots, v_{p-3}, v_{p-2}\}$ is the geodetic set of Γ_{p-3} and $g(\Gamma_{p-3}) = p - 2$.

Hence $\Gamma(Z_{p^2})$ satisfies the condition $g(\Gamma_i) = i + 1$, for all i and hence the proof.

5. Conclusion

In this paper, we discussed the new concept of geodetic decomposition of graphs. Let $\Gamma(Z_n) = (V(\Gamma(Z_n)), E(\Gamma(Z_n)))$ be the graph. For a non empty set S of $V(\Gamma(Z_n))$ we define $I[S] = \cup I[a, b]$, for some $a, b \in S$, where $I[a, b]$ is the closed interval consisting of a, b and all vertices lying on some $a - b$ geodesic of zero divisor graph $\Gamma(Z_n)$. If $\Gamma(Z_n)$ is a graph, then a set S of vertices is a geodetic set if $I[S] = V(G)$. The cardinality of a geodetic set is called the geodetic number of zero divisor graph and is denoted as $g(G)$. We have discussed the geodetic number of zero divisor graph $\Gamma(R)$ and determine the geodesic decomposition of zero divisor graph of the ring $R = Z_{2p}, Z_{pq}, Z_{p^2}, Z_p \times Z_p$.

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