The Connected Detour Edge Semi Toll Number of a Graph

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Abstract

A detour edge semi toll set of a connected graph $G$ is called a connected detour edge semi toll set of $G$ if the induced subgraph $G[S]$ is connected. The minimum cardinality of a connected detour semi toll set of $G$ is the connected detour semi toll number and is denoted by $cdn_{est}(G)$. Any connected detour semi Euler set of cardinality $cdn_{est}(G)$ is called a $cdn_{est}$-set of $G$. Some general properties satisfied by this concept are studied. Some standard graphs are determined. It is shown that for every pair of $a$ and $b$ of integers with $2 \leq a < b$, there exists a connected graph $G$ such that $dn_{est}(G) = a$ and $cdn_{est}(G) = b$.

Keywords: connected detour edge semi toll number, detour edge semi toll number, detour number.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. For basic graph theoretic terminology, we refer to [2]. Two vertices $u$ and $v$ are said to be adjacent if $uv$ is an edge of $G$. Two edges of $G$ are said to be adjacent if they have a common vertex. A walk is defined as a finite length alternating sequence of vertices and edges. The total number of edges covered in a walk is called as length of the walk. It is a trail in which neither vertices nor edges are repeated i.e. if we traverse a graph such that we do not repeat a vertex and nor we repeat an edge. Any connected graph is called as an Euler Graph if and only if all its vertices are of even degree. If there exists a walk in the connected graph that starts and ends at the same vertex and visits every edge of the graph exactly once with or without repeating the vertices, then such a walk is called as an Euler circuit.

The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u$-$v$ path in $G$. An $u$-$v$ path of length $d(u, v)$ is called an $u$-$v$ geodesic. The detour distance $D(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ from $u$ to $v$ is defined as the length of a longest $u$-$v$ path in $G$. An $u$-$v$ path of length $D(u, v)$ is called an $u$-$v$ detour. A vertex $x$ is said to lie on an $u$-$v$ detour $P$ if $x$ is a vertex of $P$ including the vertices $u$ and $v$. A detour set of $G$ is a set $S \subseteq$
\textit{V} (G) such that every vertex of \( G \) is contained in a detour joining some pair of vertices in \( S \). The \textit{detour number} \( dn(G) \) of \( G \) is the minimum order of a detour set and any detour set of order \( dn(G) \) is called \textit{minimum detour set} of \( G \) or a \textit{dn-set} of \( G \). These concepts were studied in [3-7,9].

A tolled walk \( T \) between \( u \) and \( v \) in \( G \) in a sequence at vertices of the form \( T : u, v_1, v_2, \ldots, v \) where \( k \geq 1 \) which enjoys the following three conditions.

- \( w_iw_{i+1} \in E(G), \forall i \)
- \( uw_i \in E(G), \text{ iff } i = 1 \).
- \( vw_i \in E(G), \text{ iff } i = k \).

\( T[u,v] \) = set of vertices lying in the \( uv \) tolled walk including \( u \) and \( v \).

For \( S \subseteq V(G) \), the tolled closure of \( G \) is \( T[S] = \bigcup_{u,v \in S} T[u,v] \). A set \( S \subseteq V(G) \) is called a tolled set if \( T[S] = V[G] \). The minimum cardinality of a tolled set is called the \textit{tolled number} of \( G \) and is denoted by \( T(G) \). This concept was studied in [1,8]. The following theorem are used in sequel.

In this paper, we define a new parameter.

\textbf{Theorem 1.1[3]} Each end vertex of a connected graph \( G \) belongs to every detour set of \( G \) is called the detour semi toll number of a graph.

\textbf{Theorem 1.2[3]} For the star \( G = K_{1,n-1} (n \geq 2), d_n(G) = n - 1 \).

2. The \textbf{Connected detour edge semi toll number of a graph}

\textbf{Definition 2.1.} A detour edge semi toll set of a connected graph \( G \) is called a connected detour edge semi toll set of \( G \) if the induced subgraph \( G[S] \) is connected. The minimum cardinality of a connected detour edge semi toll set of \( G \) is the connected detour edge semi toll number and is denoted by \( cdn_{est}(G) \). Any connected detour edge semi toll set of cardinality \( cdn_{est}(G) \) is called a \( cdn_{est} \)-set of \( G \).

\textbf{Example 2.2}. For the graph \( G \) given in Figure 2.1, \( S_1 = \{v_3,v_4,v_5,v_6\}, S_2 = \{v_2,v_3,v_5, v_7\}, S_3 = \{v_3,v_4,v_6,v_7\} \) and \( S_4 = \{v_2,v_3,v_5,v_6\} \) are four connected detour edge semi toll sets of \( G \) so that \( cdn_{est}(G) = 4 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.1.png}
\caption{Figure 2.1}
\end{figure}

\textbf{Observation 2.3} (i) Each end vertex of \( G \) belongs to every connected detour edge semi toll set of \( G \).

(ii) Each cut vertex of \( G \) belongs to every connected detour edge semi toll set of \( G \).

(iii) \( 2 \leq cdn_{est}(G) \leq cdn_{est}(G) \leq n \), where \( n \geq 2 \).

\textbf{Theorem 2.4}. For the path \( G = P_n \) \( (n \geq 3) \), \( cdn_{est}(G) = n \).

\textbf{Proof}. From Observation 2.3 (i) and (ii) each end vertices and cut vertices are belongs to every connected detour edge semi toll set of \( G \) so that \( cdn_{est}(G) = n \).

\textbf{Theorem 2.5}. For the complete graph \( G = K_n \) \( (n \geq 3) \), \( cdn_{est}(G) = 2 \).
Proof. Let $u$ and $v$ be any two adjacent vertices of $G$. Then $S = \{u, v\}$ is a connected detour edge semi toll set of $G$ so that $cdn_{est}(G) = 2$. ■

**Theorem 2.6.** For the cycle $G = C_n (n \geq 3)$, $cdn_{est}(G) = 2$.

**Proof.** Let $u, v$ be any two adjacent vertices of $G$. Then $S = \{u, v\}$ is a connected detour edge semi toll set of $G$ so that $cdn_{est}(G) = 2$. ■

**Theorem 2.7.** For the fan graph $G = K_1 + P_{n-1} (n \geq 4)$, $cdn_{est}(G) = 2$.

**Proof.** Let $V(K_1) = \{x\}$ and $V(P_{n-1}) = \{v_1, v_2, ..., v_{n-1}\}$. Then $S = \{x, v_1\}$ is a connected detour edge semi toll set of $G$ so that $cdn_{est}(G) = 2$. ■

**Theorem 2.8.** For the wheel graph $G = K_1 + C_{n-1} (n \geq 4)$, $cdn_{est}(G) = 2$.

**Proof.** Let $V(K_1) = \{x\}$ and $V(C_{n-1}) = \{v_1, v_2, ..., v_{n-1}\}$. Then $S = \{x, v_1\}$ is a connected detour edge semi toll set of $G$ so that $cdn_{est}(G) = 2$. ■

**Theorem 2.9.** For the star graph $G = K_{1, n-1} (n \geq 3)$, $cdn_{est}(G) = n$.

**Proof.** Let $S$ be the end vertices and cut vertex of $G$. Then by Observation 2.3 (i) and (ii), $S$ is a subset of every connected detour edge semi toll set of $G$ and that $cdn_{est}(G) \geq |S|$. Since $S$ is a connected detour edge semi toll set of $G$ so that $cdn_{est}(G) = n$. ■

**Theorem 2.10.** For the Helm graph $G = H_n, cdn_{est}(G) = 2n$, where $n$ is the number of end vertices.

**Proof.** Let $S$ be the set of end vertices and cut vertices of $G$. Then by Observation 2.3 (i) and (ii), $cdn_{est}(G) = 2n$.

**Theorem 2.11.** For the complete bipartite graph $G = K_{m,n}, m \geq 2, n \geq 2$, $cdn_{est}(G) = 2$.

**Proof.** Let $u, v$ be any two vertices of $G$. Then $S = \{u, v\}$ is a connected detour edge semi toll set of $G$ so that $cdn_{est}(G) = 2$. ■

**Definition 2.12.** A vertex $v$ is said to be detour edge toll vertex if $v$ is not an internal vertex of any $x$-$y$ detour edge semi toll path of $G$.

**Example 2.13.** For the graph $G$ given in Figure 2.2, $v_9$ is not an internal edge of any $x$-$y$ detour edge semi toll path of $G$, so that $v_9$ is the detour edge semi toll vertex of $G$.

![Figure 2.2](image-url)

**Remark 2.14.** Every end vertex of $G$ is the detour edge semi toll vertex of $G$ but the converse need not be true.
Observation 2.15. Every detour edge semi toll vertex set belong to any connected detour edge semi toll set of $G$.

Theorem 2.16. Let $G$ be the connected graph of order $n \geq 2$. If $cdn_{est}(G) = 2$, then every vertex of $G$ lies on a $u$-$v$ detour edge semi toll diametral walk of $G$.

Proof. Let $cdn_{est}(G) = 2$. Let $S = \{u, v\}$ be a $cdn_{est}$-set of $G$. Then every vertex of $G$ lies on $u$-$v$ detour edge semi toll diametral walk of $G$. On the contrary, suppose that $P$ is not a $u$-$v$ detour edge semi toll diametral walk of $G$. Then there exists at least one vertex, say $x \in V(G) - V(P)$ such that $x$ is not an internal vertex of $u$-$v$ detour edge semi toll walk of $G$, which is a contradiction. Therefore every vertex of $G$ lies on a $u$-$v$ detour edge semi toll diametral walk of $G$.

Remark 2.17. The converse of Theorem 2.16 need not be true. For $G = P_n (n \geq 3)$ with $V(G) = \{v_1, v_2, ..., v_n\}$, every vertex of $G$ lies on the $v_1$-$v_n$ detour edge semi toll walk of $G$. However by Theorem 2.3, $cdn_{est}(G) = n (n \geq 3)$.

Theorem 2.18. Let $G$ be the connected graph of order $n \geq 3$. Then $cdn_{est}(G) = n$ if and only if every vertex of $G$ is either a cut vertex or a detour edge semi toll vertex of $G$.

Proof. If every vertex of $G$ is either a cut vertex or a detour edge semi toll vertex of $G$, then the result follows from Observation 2.3 (ii) and 2.15.

Conversely, let $cdn_{est}(G) = n$. We prove that every vertex of $G$ is either a cut vertex or a detour edge semi toll vertex of $G$. On the contrary, suppose that there exists a vertex $x$ such that $x$ is neither a cut vertex nor an detour edge semi toll vertex of $G$. Let $S = V(G) - \{x\}$. Since $x$ is not a pendant vertex of $G$, $x \in dn_{est}(G)$. Since $x$ is not a cut vertex of $G$, $G[S]$ is connected. Then $S$ is a connected detour edge semi toll set of $G$ and so $cdn_{est}(G) \leq n - 1$, which is a contradiction. Therefore every vertex of $G$ is either a cut vertex of $G$ or a detour edge semi toll vertex of $G$.

Theorem 2.19. For every pair of $a$ and $b$ of integers with $2 \leq a < b$, there exists a connected graph $G$ such that $dn_{est}(G) = a$ and $cdn_{est}(G) = b$.

Proof. For $a = b$ let $G = K_{1,a}$. Then by Theorems 1.2 and 2.9, $dn_{est}(G) = cdn_{est}(G) = a$. So let $a < b$. Let $P_{b-a+1} : u_0, u_1, u_2, ..., u_{b-a}$ be a path of order $b - a + 1$. Let $P_i : x_i, y_i (1 \leq i \leq b - a)$ be a copy of pathon two vertices. Let $G$ be the graph obtained from $P_{b-a+1}$ and $P_i (1 \leq i \leq b - a - 1)$ by joining $x_i$ and $y_i (1 \leq i \leq b - a - 1)$ with $u_i (1 \leq i \leq b - a - 1)$. The graph $G$ is shown in Figure 2.3.

First we prove that $dn_{est}(G) = a$. Let $Z = \{u_0, z_1, z_2, ..., z_{a-1}\}$ be the set of all end vertices of $G$. Then by Observation 2.3(ii), $Z$ is a subset of every detour edge semi toll set of $G$ and so $dn_{est}(G) \geq a$. Since $Z$ is a detour edge semi toll set of $G$, we have $dn_{est}(G) = a$.

Next we prove that $cdn_{est}(G) = b$. Let $Z_1 = Z \cup \{u_1, u_2, ..., u_{b-a}\}$ be the set of all end vertices and cut vertices of $G$. By Observation 2.3 (i) and (ii), $Z_1$ is a subset of every connected detour edge semi toll set of $G$ and so $cdn_{est}(G) \geq b$. Since $Z_1$ is a connected detour edge semi toll set of $G$, $cdn_{est}(G) = b$.
References