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Qualitative Results for Nonlocal Fractional Hilfer Boundary Value Problems (BVP) using Fixed Point Theory

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***Abstract:* -** The work in this study focuses upon the qualitative analysis of the nonlinear boundary value problem (BVP) via Hilfer fractional derivative. This study demonstrates the qualitative results for the uniqueness and existence of a solution utilizing the concepts of fixed point theory comprising Banach, Schaefer, and Krasnoselskii's fixed point theorem. Subsequently, the stability of the solution of the amused differential equations is proposed via the theory of Ulam Hyers (UH) stability which adds significance to the quality of the findings. In order to demonstrate the application and validation of the derived results some numerical examples are also provided.

Keywords: *Hilfer fractional derivative; Nonlocal boundary conditions; Existence and Uniqueness; UH stability.*

1 Introduction

Fractional order differential equations have emerged as a more useful tool in the past few decades than integer order differential equations in the model formulation of many problems occurring in numerous fields of engineering and science. Differentials of fractional order are a crucial tool for describing the memory and inherited characteristics of distinct materials and processes (see [13, 17, 24]).

Also, the applications of functional analysis in interpretation of differential equations of fractional order has become very significant during the past few years. The monographs by Deimling [2], Diethlem [3], Kilbas et al. [12] have emphasized the uses of functional analysis for differetial systems of fractional order. For further evolution of fixed point theory in order to study the differential equations involving derivative of fractional order can be seen in [14, 15, 19, 22]. Also, one essential element of the qualitative theory of dynamical systems is the notion of stability. As an outcome of applications, the theory of stability has received considerable interest in a number of different research domains. Particularly, the Ulam-Hyers stability analysis and its relevancy to many kinds of differential equations have drawn the attention of numerous researchers. The Hilfer fractional derivative, which is a generalization of the Riemann-Liouville fractional derivative as well as an interpolation between R-L and Caputo fractional derivative, was introduced by Hilfer [8]. Theoretical simulations of dielectric relaxation in glass-forming materials [9], a thermally sensitive resistor problem [20], etc. are all modelled using Hilfer fractional derivative. The first publication has been offered by Furati et al. [6] involving Hilfer derivative in which the authors proposed the qualitative study about the uniqueness and existence of the solution for the initial value problem in 2012. Following the works of Furati et al. [6] the researchers have been continuously devoting their efforts to study the different phenomenon involving Hilfer derivative. Dhawan et al. [5] proposed anallytical study on the well-posedeness for the implicit fractional BVP, where the existence and uniqueness of the solution is derived using fixed point theorems and the stability is investigated using the approach of UH. For further information on the theoretical advancement of the differential equations involving Hilfer, one can go through [1, 4, 7, 10, 18, 21, 23].

To the authors best knowledge, less research has been done on BVP with the Hilfer fractional derivative. In this study, which is driven by the literature, we will develop existence and uniqueness results for the solution to the

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nonlocal Hilfer fractional BVP provided below as well as examine its UH stability:

$$\begin{cases} {}^{H}D^{\zeta,\alpha}\chi(\mathfrak{z}) + \wedge(\mathfrak{z},\chi) = 0, \mathfrak{z} \in [\mathfrak{p},\mathfrak{q}], \mathfrak{p} \geq 0, \\ \chi(\mathfrak{p}) = \chi'(\mathfrak{p}) = 0, \chi(\mathfrak{q}) = k\chi(\tau), \end{cases} \tag{1}$$

where $\chi \in \mathcal{C}^3([\mathfrak{p},\mathfrak{q}],\mathbb{R})$, $2 < \zeta \leq 3$, $0 \leq \alpha \leq 1$, $\tau \in (\mathfrak{p},\mathfrak{q})$, $k \in \mathbb{R}$, $\wedge: [\mathfrak{p},\mathfrak{q}] \times \mathbb{R} \to \mathbb{R}$ is a function such that $\wedge(\mathfrak{z},0) \neq 0$ and fractional derivative by Hilfer is denoted by ${}^{H}D^{\zeta,\alpha}$ where ζ is the order and α is parameter.

The rest of the article is structured as follows: In section 2, Some basic preliminary facts related to the definitions of fractional calculus and stability of differential equations are explained which would be used in the later sections. In section 3, we derive existence and uniqueness results of solution of nonlocal fractional Hilfer BVP problem (1). In section 4, we establish UH and gUH type stability results. To support our findings, Section 5 provides some examples which is followed by the conclusion of the work done in the manuscript.

2 Preliminaries

The authors have put together a complete set of requirements for the existence of the solutions to the nonlocal Hilfer BVP (1). We give some fundamental ideas in fractional calculus and some of its associated features with suitable justifications in order to move forward analysis.

Definition 1. *[12] "The Riemann-Liouville fractional integral of order $\zeta > 0$ for $\chi: (\mathfrak{p},\infty) \to \mathbb{R}$ is defined as*

$$I^{\zeta}\chi(\mathfrak{z}) = \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}\chi(s)ds, \tag{2}$$

provided the integral converges at the right sides over $(\mathfrak{p},\infty)$, $\mathfrak{p} \geq 0$."

Definition 2. *[12] "The Riemann-Louville fractional derivative of order $\zeta > 0$, for a function $\chi \in \mathcal{C}^n((\mathfrak{p},\infty),\mathbb{R})$, $\mathfrak{p} \geq 0$ is defined as*

$${}^{RL}D^{\zeta}\chi(\mathfrak{z}) = \frac{1}{\Gamma(n-\zeta)}\frac{d^n}{d\mathfrak{z}^n}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{n-\zeta-1}\chi(s)ds, \tag{3}$$

$n-1 < \zeta \leq n$, where $n = [\zeta]+1$, provided that the right hand side is point wise defined on $(\mathfrak{p},\infty)$."

Definition 3. *[12] "The Caputo fractional derivative of order $\zeta > 0$, for a function $\chi \in \mathcal{C}^n((\mathfrak{p},\infty),\mathbb{R})$, $\mathfrak{p} \geq 0$ is defined as*

$${}^{C}D^{\zeta}\chi(\mathfrak{z}) = \frac{1}{\Gamma(n-\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{n-\zeta-1}\frac{d^n}{ds^n}\chi(s)ds, \tag{4}$$

$n-1 < \zeta \leq n$, where $n = [\zeta]+1$, provided that the right hand side is point wise defined on $(\mathfrak{p},\infty)$."

Definition 4. *[8] "The generalized Riemann-Liouville fractional derivative or Hilfer fractional derivative of order $\zeta > 0$ and parameter α of a function $\chi \in \mathcal{C}^n((\mathfrak{p},\infty),\mathbb{R})$, $\mathfrak{p} \geq 0$ is defined by*

$${}^{H}D^{\zeta,\alpha}\chi(\mathfrak{z}) = I^{\alpha(n-\zeta)}D^{n}I^{(1-\alpha)(n-\zeta)}\chi(\mathfrak{z}), \tag{5}$$

where $n-1 < \zeta \leq n$, $0 \leq \alpha \leq 1$, $D = \frac{d}{d\mathfrak{z}}$."

Remark 1. *If $\alpha = 0$, then Hilfer fractional derivative given by definition 4 is brought down as Riemann-Louville fractional derivative presented by definition 2, also if $\alpha = 1$, then Hilfer fractional derivative is reduced to Caputo derivative given by definition 3.*

Lemma 1. *[12] Let $2 < \zeta \leq 3$, $\mathfrak{z} > \mathfrak{p}$, then*

$$I^{\zeta}({}^{RL}D^{\zeta}\chi(\mathfrak{z})) = \chi(\mathfrak{z}) - c_1(\mathfrak{z}-\mathfrak{p})^{\zeta-1} - c_2(\mathfrak{z}-\mathfrak{p})^{\zeta-2} - c_3(\mathfrak{z}-\mathfrak{p})^{\zeta-3}. \tag{6}$$

Next we give the definitions of UH Stability and gUH stability for the fractional differential equation (1).

Definition 5. *[16] "For every $\epsilon > 0$, the function $z \in \mathcal{C}^3([\mathfrak{p},\mathfrak{q}],\mathbb{R})$ satisfies*

$$|{}^{H}D^{\zeta,\alpha}z(\mathfrak{z}) + \wedge(\mathfrak{z},z(\mathfrak{z}))| \leq \epsilon, \mathfrak{z} \in [\mathfrak{p},\mathfrak{q}], \tag{7}$$

where the function $\wedge$ is defined in (1). Let $x \in \mathcal{C}^3([\mathfrak{p},\mathfrak{q}],\mathbb{R})$ be a solution of the problem (1). If there is a positive

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constant K such that

$$|z(\mathfrak{z}) - x(\mathfrak{z})| \leq K\epsilon, \mathfrak{z} \in [\mathfrak{p}, \mathfrak{q}]. \tag{8}$$

Then the problem (1) is said to be UH stable."

Definition 6. *[16] "Assume that $z \in \mathcal{C}^3([\mathfrak{p},\mathfrak{q}],\mathbb{R})$ satisfies the inequality (7) and $x \in \mathcal{C}^3([\mathfrak{p},\mathfrak{q}],\mathbb{R})$ is a solution of the problem (1). If there is a function $\phi_\wedge(\epsilon) \in \mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$ with $\phi_\wedge(0) = 0$ satisfying*

$$|z(\mathfrak{z}) - x(\mathfrak{z})| \leq \phi_\wedge(\epsilon), \mathfrak{z} \in [\mathfrak{p}, \mathfrak{q}]. \tag{9}$$

Then the problem (1) is said to be gUH stable."

Remark 2. *If there is a function $\psi \in \mathcal{C}([\mathfrak{p},\mathfrak{q}],\mathbb{R})$(independent of z),such that*

- $|\psi(\mathfrak{z})| \leq \epsilon$, for all $\mathfrak{z} \in [\mathfrak{p},\mathfrak{q}]$,
- ${}^{H}D^{\zeta,\alpha}z(\mathfrak{z}) + \wedge(\mathfrak{z}, z(\mathfrak{z})) = \psi(\mathfrak{z})$, $\mathfrak{z} \in [\mathfrak{p},\mathfrak{q}]$.

Then a function $z \in \mathcal{C}^3([\mathfrak{p},\mathfrak{q}],\mathbb{R})$ is a solution of inequality (7).

3 Qualitative Results

In this section, the authors have derived a set of sufficient conditions such that the nonlocal Hilfer BVP possesses a solution. In order to achieve the desired goals we have used the theory of fixed point theorems as a consequence of integral equations. Applications of certain fixed point theorems, like Banach, Schaefer and Krasnoselskii's fixed point theorems (see [12]) are also demonstrated.

To get going the analysis further, let us define the Banach space of the continuous functions, χ from $[\mathfrak{p},\mathfrak{q}] \to \mathbb{R}$ denoted by $\mathcal{C} = \mathcal{C}^3([\mathfrak{p},\mathfrak{q}])$ equipped with

$$\| \chi \| = \sup_{t\in[\mathfrak{p},\mathfrak{q}]} |\chi(\mathfrak{z})|.$$

Lemma 2. *Let*

$$\Delta = (\mathfrak{q} - \mathfrak{p})^{\gamma-1} - k(\tau - \mathfrak{p})^{\gamma-1} \neq 0, \tag{10}$$

then the solution χ of nonlocal fractional Hilfer BVP

$$\begin{cases} {}^{H}D^{\zeta,\alpha}\chi(\mathfrak{z}) + h(\mathfrak{z}) = 0, \mathfrak{z} \in [\mathfrak{p},\mathfrak{q}], \mathfrak{p} \geq 0, \\ \chi(\mathfrak{p}) = \chi'(\mathfrak{p}) = 0, \chi(\mathfrak{q}) = k\chi(\tau), \end{cases} \tag{11}$$

is presented by

$$\chi(\mathfrak{z}) = \frac{1}{\Delta\Gamma(\zeta)}\left(\int_{\mathfrak{p}}^{\mathfrak{q}} (\mathfrak{q}-s)^{\zeta-1}h(s)ds - k\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}h(s)ds\right)(\mathfrak{z}-\mathfrak{p})^{\gamma-1} - \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}h(s)ds \tag{12}$$

where $\chi \in \mathcal{C}^3([\mathfrak{p},\mathfrak{q}],\mathbb{R})$, $2 < \zeta \leq 3$, $0 \leq \alpha \leq 1$, $\tau \in (\mathfrak{p},\mathfrak{q})$, $k \in \mathbb{R}$, $h: [\mathfrak{p},\mathfrak{q}] \to \mathbb{R}$ is a continuous function, $\gamma = \zeta + 3\alpha - \zeta\alpha$.

Proof. The fractional differential equation in (11) can be written as

$$I^{\alpha(3-\zeta)}D^3I^{(1-\alpha)(3-\zeta)}\chi(\mathfrak{z}) + h(\mathfrak{z}) = 0.$$

Imposing the fractional integral of ζ order on both sides to obtain

$$I^{\zeta}I^{\alpha(3-\zeta)}D^3I^{(1-\alpha)(3-\zeta)}\chi(\mathfrak{z}) + I^{\zeta}h(\mathfrak{z}) = 0.$$

Indeed

$$I^{\zeta}I^{\alpha(3-\zeta)}D^3I^{(1-\alpha)(3-\zeta)}\chi(\mathfrak{z}) = I^{\gamma}D^3I^{(3-\gamma)}\chi(\mathfrak{z}) = I^{\gamma}({}^{RL}D^{\gamma}\chi(\mathfrak{z})),$$

and therefore, we have

$$I^{\gamma}({}^{RL}D^{\gamma}\chi(\mathfrak{z})) + I^{\zeta}h(\mathfrak{z}) = 0.$$

By using Lemma 1, we obtain

$$\chi(\mathfrak{z}) = c_1(\mathfrak{z}-\mathfrak{p})^{\gamma-1} + c_2(\mathfrak{z}-\mathfrak{p})^{\gamma-2} + c_3(\mathfrak{z}-\mathfrak{p})^{\gamma-3} - I^{\zeta}h(\mathfrak{z}).$$

The condition $\chi(\mathfrak{p}) = 0$ implies $c_3 = 0$, thus

$$\chi(\mathfrak{z}) = c_1(\mathfrak{z}-\mathfrak{p})^{\gamma-1} + c_2(\mathfrak{z}-\mathfrak{p})^{\gamma-2} - I^{\zeta}h(\mathfrak{z}). \tag{13}$$

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Now differentiating the equation (13) in order to obtain

$$\chi'(\mathfrak{z}) = (\gamma-1)c_1(\mathfrak{z}-\mathfrak{p})^{\gamma-2} + (\gamma-2)c_2(\mathfrak{z}-\mathfrak{p})^{\gamma-3} - I^{\zeta-1}h(\mathfrak{z}).$$

Again following the same procedure as above the boundary condition $\chi'(\mathfrak{p}) = 0$ gives the value of $c_2 = 0$. Now using the values of constants the solution $\chi(\mathfrak{z})$ becomes

$$\chi(\mathfrak{z}) = c_1(\mathfrak{z}-\mathfrak{p})^{\gamma-1} - I^{\zeta}h(\mathfrak{z}). \tag{14}$$

Now the last boundary condition $\chi(\mathfrak{q}) = k\chi(\tau)$ presents the value of the remaining constant,

$$c_1(\mathfrak{q}-\mathfrak{p})^{\gamma-1} - I^{\zeta}h(\mathfrak{q}) = kc_1(\tau-\mathfrak{p})^{\gamma-1} - kI^{\zeta}h(\tau),$$

from which we get

$$c_1 = \frac{1}{\Delta\Gamma(\zeta)}\left(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}h(s)ds - k\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}h(s)ds\right).$$

Substituting the value of c_1 in (14), the required result is obtained. □

In order to do the necessary analysis of the nonlocal Hilfer BVP (1) using the fixed point theory, construct an operator $A: \mathcal{C} \to \mathcal{C}$ with aiding the help of **Lemma 2** as follows:

$$(A\chi)(\mathfrak{z}) = \frac{(\mathfrak{z}-\mathfrak{p})^{\gamma-1}}{\Delta\Gamma(\zeta)}\left(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}\wedge(s,\chi(s))ds - k\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}\wedge(s,\chi(s))ds\right) - \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}\wedge(s,\chi(s))ds. \tag{15}$$

Note. As we can see that the fixed points of the operator A are nothing but the solutions of the nonlocal Hilfer BVP (1). Thus it is sufficient to analyze the operator A in order to obtain required results. In contemplation of further analysis we assume the following assumptions:

(H1) Assume that there is positive number L which follows

$$|\wedge(\mathfrak{z},\chi_1) - \wedge(\mathfrak{z},\chi_2)| \le L|\chi_1-\chi_2|,$$

for every $\mathfrak{z} \in [\mathfrak{p},\mathfrak{q}]$, $\chi_1, \chi_2 \in \mathbb{R}$.

(H2) $\wedge: [\mathfrak{p},\mathfrak{q}] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H3) There is positive number M which follows $|\wedge(\mathfrak{z},\chi)| \le M$ for each $\mathfrak{z} \in [\mathfrak{p},\mathfrak{q}]$ and $\chi \in \mathbb{R}$.

Theorem 1. *Let $\Delta \neq 0$ and assume that* **(H1)** *holds. If $L\Omega < 1$ where Ω is defined as*

$$\Omega = \frac{1}{\Gamma(\zeta+1)}\left(\frac{(\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1}}{|\Delta|} + \frac{|k|(\tau-\mathfrak{p})^{\zeta}(\mathfrak{q}-\mathfrak{p})^{\gamma-1}}{|\Delta|} + (\mathfrak{q}-\mathfrak{p})^{\zeta}\right). \tag{16}$$

Then the nonlocal Hilfer BVP (1) possesses unique solution on $[\mathfrak{p},\mathfrak{q}]$.

Proof. Let $\chi_1, \chi_2 \in \mathcal{C}$, then for all $\mathfrak{z} \in [\mathfrak{p},\mathfrak{q}]$, we have

$$\begin{aligned}
|A(\chi_1(\mathfrak{z})) - A(\chi_2(\mathfrak{z}))| &\le \frac{1}{|\Delta|\Gamma(\zeta)}\Big(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}|\wedge(s,\chi_1(s)) - \wedge(s,\chi_2(s))|ds \\
&\quad + |k|\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}|\wedge(s,\chi_1(s)) - \wedge(s,\chi_2(s))|ds\Big)(\mathfrak{z}-\mathfrak{p})^{\gamma-1} \\
&\quad + \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}|\wedge(s,\chi_1(s)) - \wedge(s,\chi_2(s))|ds \\
&\le \frac{L\|\chi_1-\chi_2\|}{|\Delta|\Gamma(\zeta)}\left(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}ds + |k|\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}ds\right)(\mathfrak{z}-\mathfrak{p})^{\gamma-1} + \frac{L\|\chi_1-\chi_2\|}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}ds \\
&\le \frac{L\|\chi_1-\chi_2\|}{|\Delta|\Gamma(\zeta+1)}\left((\mathfrak{q}-\mathfrak{p})^{\zeta} + |k|(\tau-\mathfrak{p})^{\zeta}\right)(\mathfrak{z}-\mathfrak{p})^{\gamma-1} + \frac{L\|\chi_1-\chi_2\|}{\Gamma(\zeta+1)}(\mathfrak{z}-\mathfrak{p})^{\zeta} \\
&\le \frac{L\|\chi_1-\chi_2\|}{|\Delta|\Gamma(\zeta+1)}\left((\mathfrak{q}-\mathfrak{p})^{\zeta} + |k|(\tau-\mathfrak{p})^{\zeta}\right)(\mathfrak{q}-\mathfrak{p})^{\gamma-1} + \frac{L\|\chi_1-\chi_2\|}{\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta} \\
&= \frac{L\|\chi_1-\chi_2\|}{|\Delta|\Gamma(\zeta+1)}\left((\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1} + |k|(\tau-\mathfrak{p})^{\zeta}(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\right) + \frac{L\|\chi_1-\chi_2\|}{\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta} \\
&= \frac{L\|\chi_1-\chi_2\|}{\Gamma(\zeta+1)}\left(\frac{(\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1}}{|\Delta|} + |k|\frac{(\tau-\mathfrak{p})^{\zeta}(\mathfrak{q}-\mathfrak{p})^{\gamma-1}}{|\Delta|} + (\mathfrak{q}-\mathfrak{p})^{\zeta}\right) \\
&= L\Omega\|\chi_1-\chi_2\|.
\end{aligned}$$

Which implies that $||A(\chi_1) - A(\chi_2)|| \le L\Omega||\chi_1-\chi_2||$. Since $L\Omega < 1$ from which we can assert that A follows

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contraction. Thus, A has an unique fixed point by Banach Contraction principle and from this we conclude that the nonlocal hilfer fractional BVP (1) possesses a unique solution on $[\mathfrak{p},\mathfrak{q}]$.

Theorem 2. *Let (H2), (H3) hold, then the nonlocal Hilfer BVP (1) has atleast one solution on* $[\mathfrak{p},\mathfrak{q}]$.

Proof. This result is derived as the direct consequence of Schaefer's theorem in which we establish a set of sufficient conditions to prove that operator A defined by (15) has a fixed point. This theorem's proof has been split down into multiple steps.

Step 1. A is continuous:
Consider a sequence $\chi_n \to \chi$ in $\mathcal{C}^3([\mathfrak{p},\mathfrak{q}],\mathbb{R})$, then for each $\mathfrak{z} \in [\mathfrak{p},\mathfrak{q}]$,

$$\begin{aligned}
&|A(\chi_n(\mathfrak{z})) - A(\chi(\mathfrak{z}))| \\
&\leq \frac{1}{|\Delta|\Gamma(\zeta)}\Big(\int_{\mathfrak{p}}^{\mathfrak{q}} (\mathfrak{q}-s)^{\zeta-1}|\wedge(s,\chi_n(s)) - \wedge(s,\chi(s))|ds \\
&\quad + |k|\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}|\wedge(s,\chi_n(s)) - \wedge(s,\chi(s))|ds\Big)(\mathfrak{z}-\mathfrak{p})^{\gamma-1} \\
&\quad + \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}|\wedge(s,\chi_n(s)) - \wedge(s,\chi(s))|ds \\
&\leq \frac{\|\wedge(.,\chi_n(.)) - \wedge(.,\chi(.))\|}{|\Delta|\Gamma(\zeta)}\Big(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}ds + |k|\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}ds\Big)(\mathfrak{z}-\mathfrak{p})^{\gamma-1} \\
&\quad + \frac{\|\wedge(.,\chi_n(.)) - \wedge(.,\chi(.))\|}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}ds \\
&\leq \frac{\|\wedge(.,\chi_n(.)) - \wedge(.,\chi(.))\|}{|\Delta|\Gamma(\zeta+1)}\big((\mathfrak{q}-\mathfrak{p})^{\zeta} + |k|(\tau-\mathfrak{p})^{\zeta}\big)(\mathfrak{z}-\mathfrak{p})^{\gamma-1} \\
&\quad + \frac{\|\wedge(.,\chi_n(.)) - \wedge(.,\chi(.))\|}{\Gamma(\zeta+1)}(\mathfrak{z}-\mathfrak{p})^{\zeta} \\
&\leq \frac{\|\wedge(.,\chi_n(.)) - \wedge(.,\chi(.))\|}{|\Delta|\Gamma(\zeta+1)}\big((\mathfrak{q}-\mathfrak{p})^{\zeta} + |k|(\tau-\mathfrak{p})^{\zeta}\big)(\mathfrak{q}-\mathfrak{p})^{\gamma-1} \\
&\quad + \frac{\|\wedge(.,\chi_n(.)) - \wedge(.,\chi(.))\|}{\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta} \\
&= \frac{\|\wedge(.,\chi_n(.)) - \wedge(.,\chi(.))\|}{|\Delta|\Gamma(\zeta+1)}\big((\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1} + |k|(\tau-\mathfrak{p})^{\zeta}(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\big) \\
&\quad + \frac{\|\wedge(.,\chi_n(.)) - \wedge(.,\chi(.))\|}{\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta} \\
&= \frac{\|\wedge(.,\chi_n(.)) - \wedge(.,\chi(.))\|}{\Gamma(\zeta+1)}\Big(\frac{(\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1}}{|\Delta|} + |k|\frac{(\tau-\mathfrak{p})^{\zeta}(\mathfrak{q}-\mathfrak{p})^{\gamma-1}}{\Delta} + (\mathfrak{q}-\mathfrak{p})^{\zeta}\Big) \\
&= \Omega\,\|\wedge(.,\chi_n(.)) - \wedge(.,\chi(.))\|.
\end{aligned}$$

Which implies that $\| A(\chi_n) - A(\chi) \| \leq \Omega \|\wedge(.,\chi_n(.)) - \wedge(.,\chi(.))\|$. Now the continuity of $\wedge$ implies that $\| A(\chi_n) - A(\chi) \| \to 0$ as $n \to \infty$.

Step 2. A maps bounded sets into bounded sets.
For any arbitrary $r > 0$, we define $B_r = \{\chi \in \mathcal{C}: \|\chi\| \leq r\}$, for each $\chi \in B_r$, by (H3) we have for each $\mathfrak{z} \in [\mathfrak{p},\mathfrak{q}]$

$$\begin{aligned}
&|(A\chi)(\mathfrak{z})| \leq \frac{(\mathfrak{z}-\mathfrak{p})^{\gamma-1}}{|\Delta|\Gamma(\zeta)}\Big(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}|\wedge(s,\chi(s))|ds \\
&\quad + |k|\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}|\wedge(s,\chi(s))|ds\Big) + \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}|\wedge(s,\chi(s))|ds \\
&\leq \frac{M(\mathfrak{z}-\mathfrak{p})^{\gamma-1}}{|\Delta|\Gamma(\zeta)}\Big(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}ds + |k|\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}ds\Big) + \frac{M}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}ds \\
&\leq \frac{M}{|\Delta|\Gamma(\zeta+1)}\big((\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1} + |k|(\tau-\mathfrak{p})^{\zeta}(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\big) + \frac{M}{\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta} \\
&= M\Omega \\
&= l.
\end{aligned}$$

Therefore, $AB_r \subseteq B_l$ from which we can infer that bounded sets are mapped into bounded sets by A.

Step 3. A maps bounded sets into equicontinuous sets of $\mathcal{C}$.
Let $t_1, t_2 \in [\mathfrak{p},\mathfrak{q}], t_1 < t_2$ and B_r be a bounded set of $\mathcal{C}$ as assumed in Step 2, and consider $\chi \in B_r$, then

$$|(A\chi)(t_2) - (A\chi)(t_1)|$$

$$\leq \frac{((\mathfrak{t}_2-\mathfrak{p})^{\gamma-1}-(\mathfrak{t}_1-\mathfrak{p})^{\gamma-1})}{|\Delta|\Gamma(\zeta)}(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}|\wedge(s,\chi(s))|ds$$
$$+|k|\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}|\wedge(s,\chi(s))|ds)$$
$$+\frac{1}{\Gamma(\zeta)}(\int_{\mathfrak{p}}^{\mathfrak{t}_2}(\mathfrak{t}_2-s)^{\zeta-1}|\wedge(s,\chi(s))|ds-\int_{\mathfrak{p}}^{\mathfrak{t}_1}(\mathfrak{t}_1-s)^{\zeta-1}|\wedge(s,\chi(s))|ds)$$
$$\leq \frac{M((\mathfrak{t}_2-\mathfrak{p})^{\gamma-1}-(\mathfrak{t}_1-\mathfrak{p})^{\gamma-1})}{|\Delta|\Gamma(\zeta)}(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}ds+|k|\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}ds)$$
$$+\frac{M}{\Gamma(\zeta)}(\int_{\mathfrak{p}}^{\mathfrak{t}_2}(\mathfrak{t}_2-s)^{\zeta-1}ds-\int_{c}^{\mathfrak{t}_1}(\mathfrak{t}_1-s)^{\zeta-1}ds)$$
$$=\frac{M}{|\Delta|\Gamma(\zeta+1)}((\mathfrak{q}-\mathfrak{p})^{\zeta}+|k|(\tau-\mathfrak{p})^{\zeta})((\mathfrak{t}_2-\mathfrak{p})^{\gamma-1}-(\mathfrak{t}_1-\mathfrak{p})^{\gamma-1})$$
$$+\frac{M}{\Gamma(\zeta)}(\int_{\mathfrak{p}}^{\mathfrak{t}_1}(t_2-s)^{\zeta-1}ds+\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-s)^{\zeta-1}ds-\int_{\mathfrak{p}}^{\mathfrak{t}_1}(\mathfrak{t}_2-s)^{\zeta-1}ds)$$
$$=\frac{M}{\Gamma(\zeta+1)}(\left[\frac{(\mathfrak{q}-\mathfrak{p})^{\zeta}}{|\Delta|}+|k|\frac{(\tau-\mathfrak{p})^{\zeta}}{|\Delta|}\right]((\mathfrak{t}_2-\mathfrak{p})^{\gamma-1}-(\mathfrak{t}_1-\mathfrak{p})^{\gamma-1})$$
$$-(\mathfrak{t}_2-\mathfrak{t}_1)^{\zeta}+(\mathfrak{t}_2-\mathfrak{p})^{\zeta}-(\mathfrak{t}_1-\mathfrak{p})^{\zeta}+(\mathfrak{t}_2-\mathfrak{t}_1)^{\zeta}).$$

Now, as $\mathfrak{t}_1\to\mathfrak{t}_2$, then $|(A\chi)(\mathfrak{t}_2)-(A\chi)(\mathfrak{t}_1)|\to 0$. Therefore combining the steps 1,2 and 3 with the aid of Arzela-Ascoli theorem, A is compact operator.

Step 4. A priori bounds:
Let $\epsilon=\{\chi\in\mathcal{C}^3([\mathfrak{p},\mathfrak{q}],\mathbb{R}):\chi=\lambda A(\chi)$ for some $0<\lambda<1\}$ and we show that ϵ is bounded. To prove our claim let us consider $\chi\in\epsilon$, then $\chi=\lambda A(\chi)$ for some $0<\lambda<1$. Therefore for each $\mathfrak{z}\in[\mathfrak{p},\mathfrak{q}]$, we have

$$\chi(\mathfrak{z})=\frac{\lambda}{\Delta\Gamma(\zeta)}(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}\wedge(s,\chi(s))ds-k\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}\wedge(s,\chi(s))ds)(\mathfrak{z}-\mathfrak{p})^{\gamma-1}$$
$$-\frac{\lambda}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}\wedge(s,\chi(s))ds.$$

Now using step 2, we have $|A\chi(\mathfrak{z})|\leq l$ which implies $|\chi(\mathfrak{z})|=|A\chi(\mathfrak{z})|\leq\lambda l$ and so ϵ is bounded. Thus, according to Schaefer's Fixed Point Theorem, A has a fixed point which provides the solution to the nonlocal Hilfer BVP equation (1).

Theorem 3. *(Krasnoselskii's fixed point theorem [11]) "Let M be a closed,convex and non empty subset of a Banach space X and let A,B be the operators such that*

- $Ax+By\in M$ whenever $x,y\in M$.
- A is compact and continuous.
- B is contraction.

Then there exists $z\in M$ such that $z=Az+Bz$."

Theorem 4. *Consider* $\wedge:[\mathfrak{p},\mathfrak{q}]\times\mathbb{R}\to\mathbb{R}$ *satisfies (H1) and (H2) along with*

$$|\wedge(\mathfrak{z},\chi)|\leq\phi(\mathfrak{z})\ \ for\ all\ \ (\mathfrak{z},\chi)\in[\mathfrak{p},\mathfrak{q}]\times\mathbb{R}\ \ and\ \ \phi\in\mathcal{C}([\mathfrak{p},\mathfrak{q}],\mathbb{R}^+).$$

Then, the nonlocal Hilfer BVP (1) has at least one solution on $[\mathfrak{p},\mathfrak{q}]$, provided

$$L\mu<1, \tag{17}$$

where $\mu=\frac{(\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1}+|k|(\tau-\mathfrak{p})^{\zeta}(\mathfrak{q}-\mathfrak{p})^{\gamma-1}}{|\Delta|\Gamma(\zeta+1)}$.

Proof. The proof of this theorem includes the application of both compact and contraction operators on separating the operator A into two parts say A_1 and A_2. Setting $\sup_{\mathfrak{z}\in[\mathfrak{p},\mathfrak{q}]}\phi(\mathfrak{z})=\|\phi\|$ and choosing ρ such that $\rho\geq\|\phi\|\,\Omega$ and consider $B_\rho=\{\chi\in\mathcal{C}:\|\chi\|\leq\rho\}$. Now, we define two operators A_1 and A_2 on B_ρ as follows:

$$(A_1\chi)(\mathfrak{z})=\ -\frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}\wedge(s,\chi(s))ds,$$
$$(A_2\chi)(\mathfrak{z})=\ \frac{(\mathfrak{z}-\mathfrak{p})^{\gamma-1}}{\Delta\Gamma(\zeta)}(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}\wedge(s,\chi(s))ds-k\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}\wedge(s,\chi(s))ds).$$

Now for any $\chi_1,\chi_2\in B_\rho$ and for all $\mathfrak{z}\in[\mathfrak{p},\mathfrak{q}]$, we have

$$|(A_1\chi_1)(\mathfrak{z})+(A_2\chi_2)(\mathfrak{z})|\leq\frac{\|\phi\|}{|\Delta|\Gamma(\zeta)}(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}ds+|k|\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}ds)(\mathfrak{z}-\mathfrak{p})^{\gamma-1}$$
$$+\frac{\|\phi\|}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}ds$$
$$\leq\frac{\|\phi\|\left((\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1}+|k|(\tau-\mathfrak{p})^{\zeta}(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\right)}{|\Delta|\Gamma(\zeta+1)}+\frac{\|\phi\|(\mathfrak{q}-\mathfrak{p})^{\zeta}}{\Gamma(\zeta+1)}$$

$$=\| \phi \| \Omega$$
$$\leq \rho.$$

This shows that $A_1\chi_1 + A_2\chi_2 \in B_\rho$. Also, the continuity of the nonlinear function $\wedge$ can be followed from (H2) and from this we can ascertain that that A_1 is continuous. Now we claim that A_1 is uniformly bounded. For this purpose, consider $m = \frac{\|\phi\|(\mathfrak{q}-\mathfrak{p})^\zeta}{\Gamma(\zeta+1)}$ and let $\chi_1 \in B_\rho$, then

$$|(A_1\chi_1)(\mathfrak{z})| \leq \frac{\|\phi\|}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}ds$$
$$= \frac{\|\phi\|(\mathfrak{z}-\mathfrak{p})^\zeta}{\Gamma(\zeta+1)}$$
$$\leq \frac{\|\phi\|(\mathfrak{q}-\mathfrak{p})^\zeta}{\Gamma(\zeta+1)} = m.$$

Therefore A_1 is uniformly bonded on B_ρ. For compactness of A_1, let $t_1, t_2 \in [\mathfrak{p},\mathfrak{q}], t_1 < t_2$

$$|(A_1\chi)(t_2) - (A_1\chi)(t_1)|$$
$$= \left|\frac{1}{\Gamma(\zeta)}(-\int_{\mathfrak{p}}^{t_2}(t_2-s)^{\zeta-1}\wedge(s,\chi(s))ds + \int_{\mathfrak{p}}^{t_1}(t_1-s)^{\zeta-1}\wedge(s,\chi(s))ds)\right|$$
$$= \left|\frac{1}{\Gamma(\zeta)}(\int_{\mathfrak{p}}^{t_2}(t_2-s)^{\zeta-1}\wedge(s,\chi(s))ds - \int_{\mathfrak{p}}^{t_1}(t_1-s)^{\zeta-1}\wedge(s,\chi(s))ds)\right|$$
$$= \left|\frac{1}{\Gamma(\zeta)}(\int_{\mathfrak{p}}^{t_2}((t_2-s)^{\zeta-1} - (t_1-s)^{\zeta-1})\wedge(s,\chi(s))ds + \int_{t_1}^{t_2}(t_2-s)^{\zeta-1}\wedge(s,\chi(s))ds)\right|$$
$$\leq \frac{\|\phi\|}{\Gamma(\zeta+1)}(-(t_2-t_1)^\zeta + (t_2-\mathfrak{p})^\zeta - (t_1-\mathfrak{p})^\zeta + \int_{t_1}^{t_2}(t_2-s)^{\zeta-1}ds),$$

as if $t_2 \to t_1$, then $|(A_1\chi)(t_2) - (A_1\chi)(t_1)| \to 0$. Thus A_1 is equicontinuous. So A_1 is relatively compact on B_ρ. Hence by Arzela-Ascoli theorem, A_1 is compact on B_ρ.

Now using (H1) and (17), we prove that A_2 is a contraction.

$$|(A_2\chi_1)(\mathfrak{z}) - (A_2\chi_2)(\mathfrak{z})| \leq \frac{L\|\chi_1-\chi_2\|}{|\Delta|\Gamma(\zeta)}(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}ds + |k|\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}ds)(\mathfrak{z}-\mathfrak{p})^{\gamma-1}$$
$$\leq \frac{L\|\chi_1-\chi_2\|}{|\Delta|\Gamma(\zeta+1)}((\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1} + |k|(\tau-c)^\zeta(\mathfrak{q}-\mathfrak{p})^{\gamma-1})$$
$$= L\mu \| \chi_1 - \chi_2 \|.$$

So A_2 is a contraction, and hence concluding by Krasnoselskii's fixed theorem, the nonlocal Hilfer BVP (1) has atleast one solution on $[\mathfrak{p},\mathfrak{q}]$.

4 Stability results

Theorem 5. *If $z \in \mathcal{C}^3([\mathfrak{p},\mathfrak{q}],\mathbb{R})$ satisfies the inequality (7), then for arbitrary $\epsilon \in (0,1]$, z is a solution of the inequality*

$$|z(\mathfrak{z}) - A(z(\mathfrak{z}))| \leq \Omega\epsilon.$$

Proof. From Lemma 2 and Remark 2, we can write

$$z(\mathfrak{z}) = \frac{(\mathfrak{z}-\mathfrak{p})^{\gamma-1}}{\Delta\Gamma(\zeta)}(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}(\wedge(s,z(s)) + \psi(s))ds - k\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}(\wedge(s,z(s)) + \psi(s))ds)$$
$$-\frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}(\wedge(s,z(s)) + \psi(s))ds,$$
$$A(z(\mathfrak{z})) = \frac{(\mathfrak{z}-\mathfrak{p})^{\gamma-1}}{\Delta\Gamma(\zeta)}(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{p}-s)^{\zeta-1}\wedge(s,z(s))ds - k\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}\wedge(s,z(s))ds)$$
$$-\frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}\wedge(s,z(s))ds.$$

Further

$$|z(\mathfrak{z}) - A(z(\mathfrak{z}))| = \left|\frac{1}{\Delta\Gamma(\zeta)}(\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}\psi(s)ds - k\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta-1}\psi(s)ds)(\mathfrak{z}-\mathfrak{p})^{\gamma-1}\right.$$
$$\left.-\frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}\psi(s)ds\right|$$
$$\leq \frac{\|\psi\|}{|\Delta|\Gamma(\zeta+1)}((\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1} + |k|(\tau-\mathfrak{p})^\zeta(\mathfrak{q}-\mathfrak{p})^{\gamma-1}) + \frac{\|\psi\|(\mathfrak{q}-\mathfrak{p})^\zeta}{\Gamma(\zeta+1)}$$

$\leq \epsilon\Omega.$

Theorem 6. *If (H1) and (H2) are fulfilled and* $1 - L\Omega \neq 0$ *holds, then the nonlocal fractional Hilfer problem (1) is UH stable.*

Proof. Suppose $z \in \mathcal{C}^3([\mathfrak{p},\mathfrak{q}],\mathbb{R})$ is solution of inequality (7) and due to Therorem 3, let x be the unique solution of nonlocal fractional Hilfer BVP (1). Let $\mathfrak{z} \in [\mathfrak{p},\mathfrak{q}]$, then

$$|z(\mathfrak{z}) - x(\mathfrak{z})| = |z(\mathfrak{z}) - A(z(\mathfrak{z})) + A(z(\mathfrak{z})) - x(\mathfrak{z})|$$
$$\leq |z(\mathfrak{z}) - A(z(\mathfrak{z}))| + |A(z(\mathfrak{z})) - x(\mathfrak{z})|$$
$$\leq \Omega\epsilon + \Omega L|z(\mathfrak{z}) - x(\mathfrak{z})|$$

As $(1 - L\Omega)|z(\mathfrak{z}) - x(\mathfrak{z})| \leq \Omega\epsilon$, i.e., $\| z - x \| \leq \frac{\Omega\epsilon}{1-L\Omega}$. Now, by setting $K = \frac{\Omega}{1-L\Omega}$, we obtain $\| z - x \| \leq K\epsilon$, $K > 0$. Therefore, the nonlocal fractional Hilfer BVP (1) is UH stable.

Remark 3. *Further, if we take* $\phi_\wedge(\epsilon) = K\epsilon$, $\phi_\wedge(0) = 0$, *which implies the nonlocal fractional Hilfer BVP (1) is generalized UH stable.*

5 Illustrative example

Example 5.1. *Consider the nonlocal fractional Hilfer BVP*

$$\begin{cases} {}^H D^{\frac{27}{10},\frac{1}{3}} \chi(\mathfrak{z}) = \dfrac{1}{2(3+2\mathfrak{z})^2}\left(\dfrac{\chi^2(\mathfrak{z}) + 2|\chi(\mathfrak{z})|}{1+|\chi(\mathfrak{z})|}\right) + \dfrac{3}{2}, \mathfrak{z} \in [0,1], \\ \chi(0) = \chi'(0) = 0, \chi(1) = \dfrac{3}{4}\chi\left(\dfrac{1}{2}\right), \end{cases} \tag{18}$$

On comparing the Hilfer fractional BVP (18) with (1), we can obtain the values of various parameters gievn as, $\zeta = \frac{27}{10}, \alpha = \frac{1}{3}, \gamma = \frac{84}{30}, \mathfrak{p} = 0, \mathfrak{q} = 1, \tau = \frac{1}{2}, k = \frac{3}{4}$ and $\Omega = 1.8011007261$

The assumption (H1) is satisfied for $L = \frac{1}{9}$, as

$$|\wedge(\mathfrak{z},\chi_1) - \wedge(\mathfrak{z},\chi_2)| \leq \frac{1}{9}|\chi_1 - \chi_2|,$$

for all $\mathfrak{z} \in [0,1]$ and $\chi_1, \chi_2 \in \mathbb{R}$. Thus $L\Omega \approx 0.2001223029 < 1$. Here all the postulates of Therorem 1 are satisfied, which gives us the conclusion that nonlocal Hilfer BVP (18) has unique solution on $[0,1]$.

Moreover, all the conditions of Therorem 6 are satisfied, thus from Therorem 6 we can also conclude that the nonlocal Hilfer BVP (18) is also UH stable.

Example 5.2. *If the non linear function* $\wedge(\mathfrak{z},\chi)$ *in (18) is considered as*

$$\wedge(\mathfrak{z},\chi) = \frac{21}{2(3+2\mathfrak{z})^2}\sin\left(\frac{|\chi(\mathfrak{z})|}{1+|\chi(\mathfrak{z})|}\right) + \frac{3}{2}. \tag{19}$$

For $\wedge$ in (19) the assumption (H1) is satisfied with $L = \frac{7}{6}$ as

$$|\wedge(\mathfrak{z},\chi_1) - \wedge(\mathfrak{z},\chi_2)| \leq \frac{7}{6}|\chi_1 - \chi_2|,$$

for all $\mathfrak{z} \in [0,1]$ and $\chi_1, \chi_2 \in \mathbb{R}$. Since, $L\Omega \approx 2.1012841805 > 1$, therefore Therorem 1 can is not applicable. On the other hand $|\wedge(\mathfrak{z},\chi)| \leq \frac{8}{3}$ where $M = \frac{8}{3} > 0$, thus all the postulates for Therorem 2 which asserts us that nonlocal Hilfer BVP (18) with $\wedge$ given by (19) has atleast one solution on $[0,1]$.

Example 5.3. *If the non linear function* $\wedge(\mathfrak{z},\chi)$ *in (18) is considered as*

$$\wedge(\mathfrak{z},\chi) = \frac{5}{(3+2\mathfrak{z})^2}\left(\frac{|\chi(\mathfrak{z})|}{1+|\chi(\mathfrak{z})|}\right) + \frac{3}{2}. \tag{20}$$

For $\wedge$ in (20) the assumption (H1) is satisfied with $L = \frac{5}{9}$ Since

$$|\wedge(\mathfrak{z},\chi_1) - \wedge(\mathfrak{z},\chi_2)| \leq \frac{5}{9}|\chi_1 - \chi_2|,$$

for all $\mathfrak{z} \in [0,1]$ and $\chi_1, \chi_2 \in \mathbb{R}$. Since, $L\Omega \approx 1.0006115145 > 1$ which contradicts to the conditions of Theorem 1 and hence Theorem 1 is not applicable.

The non linear function $\wedge$ is bounded by a function of $\mathfrak{z}$ as

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$$|\wedge(\mathfrak{z},\chi)| \leq \frac{5}{(3+2\mathfrak{z})^2}+\frac{3}{2}=\phi(\mathfrak{z}),$$

and $\mu = 1.5613297687$ and $L\mu \approx 0.867405427 < 1.$ Hence, by Therorem 4 nonlocal Hilfer BVP (18) with $\wedge$ given by (20) has atleast one solution on [0,1].

6 Conclusion

We have demonstrated existence and uniqueness results for the nonlocal fractional Hilfer BVP solution in the present investigation. Our findings were established using the fixed point theorems of Banach, Schafer, and Krasnoselskii. The stability of UH and Generalised UH has been established. We have also provided examples to support the validity of our findings. Keeping in the view of the present analysis, it can be said that fixed point theory have played a key instrument in this study for establishing a variety of fractional BVP conclusions.

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