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A Mathematical Model for Mucus Transport in Human Lungs Airways under an Oscillatory Pressure Gradient due to Coughing

Vikash Rana[1], Prashant Maurya[2]
[1,2]*Department of Mathematics and Statistics,*
Deen Dayal Upadhyaya Gorakhpur University, Gorakhpur, U.P., India

Abstract:- This paper presents a two layers circular unsteady state mathematical model for mucus transport in the human lung airways by considering the time dependent pressure gradient induced mucus transport in the respiratory system caused by coughing. It is also assumed that the mucus is a visco-elastic fluid. A comprehensive discussion is given to the pressure gradient, which is particular case of a sinusoidal function of time. The effects of various parameters on mucus transport are examined and found that mucus transport rate increase when the viscosities of mucus and serous layer fluid and shear modulus of elasticity decreases. It is also found that mucus transport rate increase when oscillating pressure gradient, air velocity and thickness of mucus increase but mucus transport rate decrease when duration of cough increase.

Keywords: *Mucus visco-elasticity, Cilia Beating, Sinusoidal Pressure.*

1. Introduction

Biologists and physiological scientists have created a great deal of mathematical models to gain understanding of complicated biological and physiological mechanisms for mucus transport in human airways. Computing simulation, Statistical analysis, approximation techniques and other numerical approaches are the foundations of the solution strategies used to solve this problems.

Muco-ciliary clearance is one of the primary defense mechanism of the human lung airways. Mucus transport is made possible by ciliary movement. The whip-like action of cilia on a cell surface of the respiratory airway is known as ciliary movement. A layer of mucus moves towards the pharynx as a result of the cilia's whip-like action, cleaning the passages of any particles that may have been stuckin the mucus. Underscoring the roles of cilia, Barton et al. (1967) conducted an analytical study of cilia-induced mucus ow. Clark (1973) used a two-phase model to analysis the bronchial clearance flow. However, diseases that harm the lungs, such as cystic fibrosis, chronic bronchitis, bronchial asthma, lung cancer, ciliary dyskinesia, etc., are included in human lung pathology. Most of these diseases cause immotile cilia, which may eventually culminate in the loss of cilia. Diseases such ascystic fibrosis are characterized by immotile cilia and a deficiency of periciliary fluid. Due to the high sensitivity of mammalian respiratory tracts, foreign particles (dust, straw, carcinogens, etc.) or other causes of irritation (sneezing) can induce spasms of the airways or the cough re ex, whichis the release of mucus and air into the airways. The cough reflex is associated with a number of illnesses, including asthma and chronic bronchitis.

Not only healthy but also unhealthy individuals experience the natural mechanism of coughing.It is reliant on several factors influencing various conditions such as asthma, chronic bronchitis and cystic fibrosis. Coughing helps secretions and other contaminants travel through various airways, preventing aspiration of the lungs. Because of the way the tracheobronchial system is set up, velocities for a given flow rate are higher in broader airways and lower in smaller airways since each generation of bronchi has an increasingly larger total cross-sectional area as it approaches the alveoli. It follows that cleaning the bigger airways with a cough should be more effective than clearing the smaller ones. Chronic bronchitis and cystic fibrosis are two respiratory disorders that cause excessive mucus production, which is expelled by coughing or forced expiration. When immotile cilia syndrome affects the airways, coughing is the primary defensive mechanism for removing mucus.

Many investigators have been studying the mucus transport in the human lungs in the last few decades. Black (1975) examined the two-layer Newtonian fluid model, which included one serous layer fluid and other mucus layer. He also emphasized the significance of gravity and the impact of air movement on mucus transport.[Scherer and Burtz (1978)] hypothesized from a study carried outin vitro that the effect of coughing can extend to the seventeenth airways generation (respiratory bronchioles) in conditions of excess mucus production and that the complex relationship between the viscosity of mucus, elasticity, and surface tension determines the effectiveness of the cough.[Black and Winet (1980)] proposed in their mathematical study that the mucus transport rate would be significantly increased if cilia could only pierce the upper, considerably more viscous layer. Coughing may not be particularly helpful to remove a very thin layer of mucus, thus the thickness of the mucuslayer is also important.

A planar two-layer fluid model have been presented by [King et al. 1993] to explain muco-ciliary transport in the respiratory tract caused by cilia beating and air motion by treating mucus as aviscoelastic fluid and demonstrated that mucus transport rises with an increase in shear stress fromair motion, pressure drop and mean velocity of cilia tip. It has been demonstrated that the mucus transport rate will peak at a certain amount of serous fluid thickness. Verma et al. (1997) presented a two layer symmetrical unsteady state mathematical model taking an accounts of pressure gradient due to coughing by Dirac delta function which demonstrates that mucus transport rate decreases with increasing coughing time, viscosity of serous fluid and modulus of mucus. Sharma and Singh (2011) considered a mathematical model for mucus transport due to coughing in serous layer under time dependent pressure gradient which is special case for sinusoidal function of time. It has been showsthat on increasing coughing time and serous layer thickness the mucus transport rate become low. A mathematical model of mucus transport in human lung airways has been proposed by Verma andRana (2015) considering mucus as a visco-elastic fluid and accounts for air motion, cilia beatingand porosity parameter under steady-state conditions. It has been demonstrated that the mucus transport rate falls with increasing serous and mucus layer viscosities and rises with air motion, cilia beating, and porosity parameter. By taking into account the impact of constriction on the airways Kumar et al. (2016) examined mucus transport in unhealthy airways. In order to study mucus transport in human lung airways Rana et al. (2021) have proposed a two-layer circular steady-state mathematical model that takes into account the impacts of mucus visco-elasticity, cilia beating, and porosity parameter. By taking into account the porosity parameter, pressure gradient and shear stress brought on by air motion, [Raut et al. (2023)] have produced a two-layer planar unsteady state model.It has been demonstrated that an increase in the porosity parameter, pressure gradient, and shearstress brought on by air motion all contribute to an increase in the mucus transport rate.

Thus, considering all of these, a symmetrical two layer unsteady state model for mucus transport resulting from coughing is examined for low Reynolds number and apply the Laplace transform method to solve the governing equations. In this paper, pressure gradient is taken as an oscillatory function of time.

2. Mathematical Model

In human lung airways, the real situation of mucus transport is idealized by the circular tube geometry which represented by symmetrical two layer mathematical model shown in [Fig.1] with ciliated inner surface wall. It is assumed that the central lumen is filled with air and surrounded by mucus (a highly viscous fluid) which is covered by a serous fluid with a lower viscosity than mucus.

Our model that accounts for the following factors in order to explore mucus transport in the human lung airways:

- Mucus layer fluid is assumed to be viscoelastic and Non-Newtonian fluid, while serous layer fluid is supposed to be an incompressible Newtonian fluid.
- There are two sublayers of the serous layer fluid: one that is interacting with the epithelium and the other that is in contact with mucus. In the serous sublayer in interface with the epithelium, no net flow is expected.
- By imposing an air velocity as a boundary condition at the mucus-air contact, the influence of air motion is included.
- The model also takes into account the effects of gravity and Sinusoidal pressure gradients. Sharma and Singh (2011)

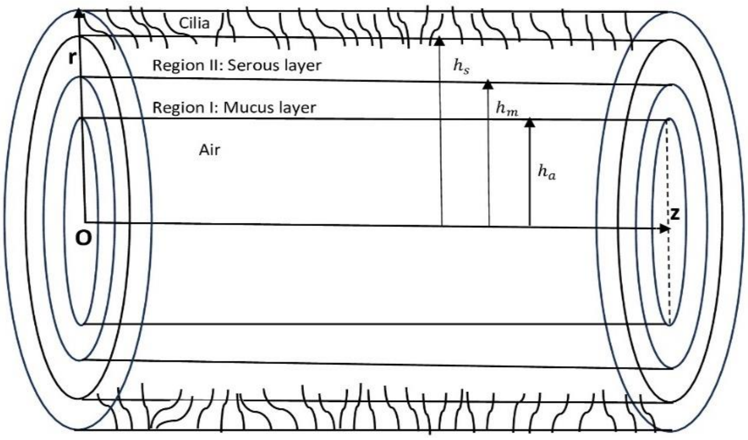

Figure 1: A circular model for mucus transport

Notations used for model are as follows:

u_s (u_m)	Serous(mucus) layer velocity
h_a, h_m and h_s	Air, mucus and serous layer thickness from central axis OZ
ρ_s (ρ_m)	Density of serous(mucus) layer
μ_s (μ_m)	Viscosity of serous(mucus) layer
τ_s (τ_m)	Shear stress of serous(mucus) layer
G	Mucus elastic modulus
U_a	Air velocity
U_o	Cilia beating
U_1	Mucus serous sub layer interface velocity
p	Pressure across the fluid layers
t	Coughing duration
λ	Relaxation time
α	Angle by which the airway under consideration is inclined with vertical

The mucus and serous fluid transport equations under unsteady and low Reynolds number flow approximation are as follows:

Region-I, Mucus Layer ($h_a \le r \le h_m$):

$$\rho_m \frac{\partial u_m}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_m) \quad (1)$$

$$\tau_m + \lambda \frac{\partial \tau_m}{\partial t} = \mu_m \frac{\partial u_m}{\partial r} \quad (2)$$

Region-II, Serous Layer ($h_m \le r \le h_s$):

$$\rho_s \frac{\partial u_s}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\mu_s \frac{\partial u_s}{\partial r}\right) \quad (3)$$

The following conditions are taken for the equations (1)-(3):

Initial condition:

$$u_m = u_s = \tau_m = \tau_s = \frac{\partial \tau_m}{\partial t} = 0, \qquad t = 0 \quad (4)$$

Boundary condition:

$$u_m = U_a, \qquad r = h_a, \qquad t > 0 \quad (5)$$

$$u_s = U_o, \qquad r = h_s, \qquad t > 0 \quad (6)$$

Matching condition:

$$u_m = u_s = U_1(t) \qquad r = h_m \quad (7)$$

$$\tau_m = \tau_s \qquad r = h_m \quad (8)$$

At the mucus-serous sublayer contact, conditions (7) and (8) suggest that the velocities and stresses are continuous.

3. Pressure Gradient due to coughing as Sinusoidal function of time

Sinusoidal pressure refers to a type of pressure gradient that follows a sinusoidal wave that is similar to a sine wave. This type of pressure gradient is common in various fluid flow having characteristic features including periodicity, amplitude, frequency, and phase. They are described mathematically by sinusoidal functions, typically represented as $P(t) = P_0 \sin(\omega t + \psi)$, where P_0 is the amplitude, ω is the angular frequency, t is time and ψ is the phase angle. The effect of sinusoidal pressure on a fluid depends on factors such as the amplitude, frequency, duration, characteristics of the fluid and the surrounding environment.

In this paper, we consider pressure gradient due to coughing be $-\frac{\partial p}{\partial z} = \phi_o f(t)$ where $f(t) = sin\omega t$ and ϕ_o is the amplitude of the oscillating pressure which is constant. Now Laplace transform of $\bar{f}(t) = f(S) = \frac{\omega}{S^2+\omega^2}$.

4. Analytical Solution

After applying Laplace transform, the system of equations (1)-(3) becomes:

$$\frac{d^2\overline{U_m}}{dr^2} + \frac{1}{r}\frac{d\overline{U_m}}{dr} - k_m^2\overline{U_m} = 0 \quad (9)$$

$$\overline{\tau_m} = \frac{\mu_m}{1+\lambda S}\frac{d\overline{u_m}}{dr} \quad (10)$$

$$\frac{d^2\overline{U_s}}{dr^2} + \frac{1}{r}\frac{d\overline{U_s}}{dr} - k_s^2\overline{U_s} = 0 \quad (11)$$

Where,

$$S\rho_m\overline{u_m} - \phi_o f(s) = \overline{U_m}, \qquad S\rho_s\overline{u_s} - \phi_o f(s) = \overline{U_s}, \quad \phi_o f(t) = -\frac{\partial p}{\partial z}, \quad f(s) = \frac{\omega}{s^2+\omega^2}$$

$$k_m^2 = \frac{S\rho_m}{\mu_m}(1+\lambda S) = \frac{S\rho_m}{G}(S+\alpha), \qquad k_s^2 = \frac{S\rho_s}{\mu_s}, \qquad \alpha = \frac{G}{\mu_m}, \qquad \lambda = \frac{1}{\alpha}$$

Using boundary and matching conditions (5)-(8) and solving equations (9)-(11),we have

$$\overline{u_m} = \frac{\phi_o f(S)}{s\rho_m}\left[1 - \frac{\{K_0(k_mh_m) - K_0(k_mh_a)\}I_0(k_mr) - \{I_0(k_mh_m) - I_0(k_mh_a)\}K_0(k_mr)}{I_0(k_mh_a)K_0(k_mh_m) - I_0(k_mh_m)K_0(k_mh_a)}\right]$$
$$+\overline{U_a}\left[\frac{K_0(k_mh_m)I_0(k_mr) - I_0(k_mh_m)K_0(k_mr)}{I_0(k_mh_a)K_0(k_mh_m) - I_0(k_mh_m)K_0(k_mh_a)}\right]$$
$$+\overline{U_1}\left[\frac{K_0(k_mh_a)I_0(k_mr) - I_0(k_mh_a)K_0(k_mr)}{I_0(k_mh_m)K_0(k_mh_a) - I_0(k_mh_a)K_0(k_mh_m)}\right] \tag{12}$$

and

$$\overline{u_s} = \frac{\phi_o f(S)}{s\rho_s}\left[1 - \frac{\{K_0(k_sh_m) - K_o(k_sh_s)\}I_0(k_sr) - \{I_0(k_sh_m) - I_0(k_sh_s)\}K_0(k_sr)}{I_0(k_sh_s)K_0(k_sh_m) - K_o(k_sh_s)I_0(k_sh_m)}\right]$$
$$+\overline{U_1}\left[\frac{K_o(k_sh_s)I_0(k_sr) - I_0(k_sh_s)K_0(k_sr)}{K_o(k_sh_s)I_0(k_sh_m) - I_0(k_sh_s)K_0(k_sh_m)}\right]$$
$$+\overline{U_o}\left[\frac{K_0(k_sh_m)I_o(k_sr) - I_0(k_sh_m)K_o(k_sr)}{I_0(k_sh_s)K_0(k_sh_m) - K_o(k_sh_s)I_0(k_sh_m)}\right] \tag{13}$$

Where $\overline{U_1}$ is given by

$$\overline{U_1}\left[k_s\mu_s\left\{\frac{K_o(k_sh_s)I_1(k_sh_m) + I_0(k_sh_s)K_1(k_sh_m)}{K_o(k_sh_s)I_0(k_sh_m) - I_0(k_sh_s)K_0(k_sh_m)}\right\}\right.$$
$$\left.+\frac{\mu_m k_m}{1+\lambda S}\left\{\frac{K_0(k_mh_a)I_1(k_mh_m) + I_0(k_mh_a)K_1(k_mh_m)}{I_0(k_mh_a)K_0(k_mh_m) - I_0(k_mh_m)K_0(k_mh_a)}\right\}\right]$$
$$= \frac{\phi_o f(S)}{k_s}\left[\frac{\{K_o(k_sh_m) - K_0(k_sh_s)\}I_1(k_sh_m) + \{I_0(k_sh_m) - I_0(k_sh_s)\}K_1(k_sh_m)}{I_0(k_sh_s)K_0(k_sh_m) - K_o(k_sh_s)I_0(k_sh_m)}\right]$$
$$+\frac{\phi_o f(S)}{k_m}\left[\frac{\{K_0(k_mh_m) - K_0(k_mh_a)\}I_1(k_mh_m) + \{I_0(k_mh_m) - I_0(k_mh_a)\}K_1(k_mh_m)}{I_0(k_mh_m)K_0(k_mh_a) - I_0(k_mh_a)K_0(k_mh_m)`}\right]$$
$$+\frac{\overline{U_a}k_m\mu_m}{1+\lambda S}\left[\frac{I_0(k_mh_m)K_1(k_mh_m) + K_0(k_mh_m)I_1(k_mh_m)}{I_0(k_mh_a)K_0(k_mh_m) - I_0(k_mh_m)K_0(k_mh_a)}\right]$$
$$+k_s\mu_s\overline{U_o}\left[\frac{K_0(k_sh_m)I_1(k_sh_m) + I_0(k_sh_m)K_1(k_sh_m)}{K_o(k_sh_s)I_0(k_sh_m) - I_0(k_sh_s)K_0(k_sh_m)}\right] \tag{14}$$

Where I_o, I_1 are modified bessel function of first kind and K_o, K_1 are modified bessel function of second kind with order zero and unity. [Bowmann (1958)]

Now the volumetric flow rate in the two layer is defined as,

$$\overline{Q_m} = \int_{h_a}^{h_m} 2\pi r\, \overline{u_m} dr \quad \text{and} \quad \overline{Q_m} = \int_{h_m}^{h_s} 2\pi r\, \overline{u_s} dr$$

after using equations (12)-(14) the rate flow becomes,

$$\overline{Q_m} = \frac{2\pi\overline{U_1}}{k_m}\left[\frac{\{h_a I_1(k_m h_a) - h_m I_1(k_m h_m)\}K_0(k_m h_a) - \{h_m K_1(k_m h_m) - h_a K_1(k_m h_a)\}I_0(k_m h_a)}{I_0(k_m h_a)K_0(k_m h_m) - I_0(k_m h_m)K_0(k_m h_a)}\right]$$
$$+\frac{2\pi\overline{U_a}}{k_m}\left[\frac{\{h_m K_1(k_m h_m) - h_a K_1(k_m h_a)\}I_0(k_m h_m) + \{h_m I_1(k_m h_m) - h_a I_1(k_m h_a)\}K_0(k_m h_m)}{I_0(k_m h_a)K_0(k_m h_m) - I_0(k_m h_m)K_0(k_m h_a)}\right]$$
$$+\frac{2\pi\phi_o f(S)}{S\rho_m k_m}\left[\frac{\{h_m I_1(k_m h_m) - h_a I_1(k_m h_a)\}\{K_0(k_m h_a) - K_0(k_m h_m)\}}{I_0(k_m h_a)K_0(k_m h_m) - I_0(k_m h_m)K_0(k_m h_a)}\right]$$
$$+\frac{2\pi\phi_o f(S)}{S\rho_m k_m}\left[\frac{\{h_m K_1(k_m h_m) - h_a K_1(k_m h_a)\}\{I_0(k_m h_a) - I_0(k_m h_m)\}}{I_0(k_m h_a)K_0(k_m h_m) - I_0(k_m h_m)K_0(k_m h_a)}\right] + \frac{\pi\phi_o f(S)}{S\rho_m}(h_m^2 - h_a^2) \tag{15}$$

and

$$\overline{Q_s} = \frac{2\pi\overline{U_1}}{k_s}\left[\frac{\{h_m I_1(k_s h_m) - h_s I_1(k_s h_s)\}K_o(k_s h_s) - \{h_s K_1(k_s h_s) - h_m K_1(k_s h_m)\}I_0(k_s h_s)}{I_0(k_s h_s)K_0(k_s h_m) - K_o(k_s h_s)I_0(k_s h_m)}\right]$$
$$+\frac{2\pi\overline{U_o}}{k_s}\left[\frac{\{h_s I_1(k_s h_s) - h_m I_1(k_s h_m)\}K_o(k_s h_m) + \{h_s K_1(k_s h_s) - h_m K_1(k_s h_m)\}I_o(k_s h_m)}{I_0(k_s h_s)K_0(k_s h_m) - K_o(k_s h_s)I_0(k_s h_m)}\right]$$
$$+\frac{2\pi\phi_o f(S)}{S\rho_s k_s}\left[\frac{\{h_s I_1(k_s h_s) - h_m I_1(k_s h_m)\}\{K_o(k_s h_s) - K_0(k_s h_m)\}}{I_0(k_s h_s)K_0(k_s h_m) - K_o(k_s h_s)I_0(k_s h_m)}\right]$$
$$+\frac{2\pi\phi_o f(S)}{S\rho_s k_s}\left[\frac{\{h_s K_1(k_s h_s) - h_m K_1(k_s h_m)\}\{I_0(k_s h_s) - I_0(k_s h_m)\}}{I_0(k_s h_s)K_0(k_s h_m) - K_o(k_s h_s)I_0(k_s h_m)}\right] + \frac{\pi\phi_o f(S)}{S\rho_s}(h_s^2 - h_m^2) \tag{16}$$

$\overline{Q_m}$ and $\overline{Q_s}$ are approximated for large arguments of modified Bessel function then, we have

$$\overline{Q_m} = \frac{2\pi}{k_m}\left(\overline{U_1} - \frac{\phi_o f(S)}{S\rho_m}\right)\left[h_m coth\{k_m(h_m - h_a)\} - (h_a h_m)^{1/2} cosech\{k_m(h_m - h_a)\}\right]$$
$$+\frac{2\pi}{k_m}\left(\overline{U_a} - \frac{\phi_o f(S)}{S\rho_m}\right)\left[h_a coth\{k_m(h_m - h_a)\} - (h_a h_m)^{1/2} cosech\{k_m(h_m - h_a)\}\right]$$
$$+\frac{\pi\phi_o f(S)}{S\rho_m}(h_m^2 - h_a^2) \tag{17}$$

and

$$\overline{Q_s} = \frac{2\pi}{k_s}\left(\overline{U_1} - \frac{\phi_o f(S)}{S\rho_s}\right)\left[h_m coth\{k_s(h_s - h_m)\} - (h_s h_m)^{1/2} cosech\{k_s(h_s - h_m)\}\right]$$
$$+\frac{2\pi}{k_s}\left(\overline{U_o} - \frac{\phi_o f(S)}{S\rho_s}\right)\left[h_s coth\{k_s(h_s - h_m)\} - (h_s h_m)^{1/2} cosech\{k_s(h_s - h_m)\}\right]$$
$$+\frac{\pi\phi_o f(S)}{S\rho_s}(h_s^2 - h_m^2) \tag{18}$$

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where U_1 is given by

$$\overline{U_1}\left[k_s\mu_s\{coth[k_s(h_s-h_m)]\}+\frac{k_m\mu_m}{(1+\lambda S)}coth[k_m(h_m-h_a)]\right]$$

$$=\frac{\phi_o f(S)}{k_s}\left[coth[k_s(h_s-h_m)]-(h_s/h_m)^{1/2}cosech[k_s(h_s-h_m)]\right]$$

$$+\frac{\phi_o f(S)}{k_m}\left[coth[k_m(h_m-h_a)]-(h_a/h_m)^{1/2}cosech[k_m(h_m-h_a)]\right]$$

$$+\frac{\overline{U_a}k_m\mu_m}{1+\lambda S}\left[(h_a/h_m)^{1/2}cosech[k_m(h_m-h_a)]\right]$$

$$+k_s\mu_s\overline{U_o}\left[(h_s/h_m)^{1/2}cosech[k_s(h_s-h_m)]\right] \quad (19)$$

Although determining the accurate inverse transform of $\overline{Q_m}$ and $\overline{Q_s}$ is difficult, its expressions can be estimated by making appropriate assumptions about coughing, such as thick mucus and thin serous layer fluid i.e. $k_m(h_m-h_a)\gg 1$ and $k_s(h_s-h_m)\ll 1$ respectively, then the expression for $\overline{Q_m}$ and $\overline{Q_s}$ becomes:

$$\overline{Q_m}=2\pi h_a\left[\frac{1}{S^{3/2}(S+\alpha)^{1/2}}\right]\left(\frac{G}{\rho_m}\right)^{\frac{1}{2}}U_a+2\pi h_m^{\frac{1}{2}}h_s^{\frac{1}{2}}\left[\frac{1}{S^{\frac{3}{2}}(S+\alpha)^{\frac{1}{2}}}\left(\frac{G}{\rho_m}\right)^{\frac{1}{2}}-(h_s-h_m)\frac{1}{S(S+\alpha)}\left(\frac{G}{\mu_s}\right)\right]U_o$$

$$+\frac{\pi}{\rho_m\omega}\left[(h_m^2-h_a^2)\left\{\frac{1}{S}-\frac{S}{S^2+\omega^2}\right\}-2(h_m+h_a)\left(\frac{G}{\rho_m}\right)^{\frac{1}{2}}\left\{\frac{1}{S^{\frac{3}{2}}(S+\alpha)^{\frac{1}{2}}}-\frac{1}{\sqrt{2}(S^2+\omega^2)}\right\}\right]\phi_o$$

$$+\frac{2\pi}{\mu_s\omega}h_m(h_s-h_m)\left(\frac{G}{\rho_m}\right)\left[\frac{1}{(S+\alpha)S}-\frac{1}{2(S^2+\omega^2)}\right]\phi_o$$

$$+\frac{2\pi}{\rho_s\omega}h_m^{\frac{1}{2}}(h_s^{\frac{1}{2}}-h_m^{\frac{1}{2}})\left(\frac{G}{\rho_m}\right)^{\frac{1}{2}}\left[\frac{1}{\sqrt{2}(s^2+\omega^2)}-\frac{1}{S^{\frac{3}{2}}(S+\alpha)^{\frac{1}{2}}}\right]\phi_o$$

$$+\frac{2\pi}{\rho_s\omega}h_m^{\frac{1}{2}}\left(h_s^{\frac{1}{2}}-h_m^{\frac{1}{2}}\right)(h_s-h_m)\left(\frac{G}{\mu_s}\right)\left[\frac{1}{(S+\alpha)S}-\frac{1}{2(S^2+\omega^2)}\right]\phi_o$$

$$+\frac{2\pi}{\mu_s\omega}h_m(h_s-h_m)^2\left(\frac{G}{\mu_s}\right)\left(\frac{G}{\rho_m}\right)^{\frac{1}{2}}\left[\frac{1}{\sqrt{2}(s^2+\omega^2)}-\frac{1}{S^{\frac{1}{2}}(S+\alpha)^{\frac{3}{2}}}\right]\phi_o \quad (20)$$

and

$$\overline{Q_s}=\frac{2\pi h_s^{\frac{1}{2}}(h_s^{\frac{1}{2}}-h_m^{\frac{1}{2}})}{(h_s-h_m)}\left(\frac{\mu_s}{\rho_s}\right)\frac{1}{S^2}U_o-\frac{2\pi}{\rho_s}\left(h_s^{\frac{1}{2}}-h_m^{\frac{1}{2}}\right)^2\left(\frac{\mu_s}{\rho_s}\right)\frac{1}{S^2}\left(\frac{\omega}{S^2+\omega^2}\right)\phi_o$$

$$+\frac{2\pi h_m^{\frac{1}{2}}(h_m^{\frac{1}{2}}-h_s^{\frac{1}{2}})}{(h_s-h_m)}\left(\frac{\mu_s}{\rho_s}\right)\frac{1}{S}\overline{U_1}+\frac{\pi}{\rho_s}(h_s^2-h_m^2)\frac{1}{S}\left(\frac{\omega}{S^2+\omega^2}\right)\phi_o \quad (21)$$

Now, after taking inverse Laplace transform of equation (20), the value of $\overline{Q_m}$ becomes:

$$Q_m = 2\pi\left(\frac{G}{\rho_m}\right)^{\frac{1}{2}} h_a t e^{-\left(\frac{1}{2}\alpha t\right)}\left[I_o\left(\frac{\alpha t}{2}\right)+I_1\left(\frac{\alpha t}{2}\right)\right]U_a$$
$$+2\pi h_s^{\frac{1}{2}} h_m^{\frac{1}{2}}\left[t e^{-\left(\frac{1}{2}\alpha t\right)}\left\{I_o\left(\frac{\alpha t}{2}\right)+I_1\left(\frac{\alpha t}{2}\right)\right\}\left(\frac{G}{\rho_m}\right)^{\frac{1}{2}}-(h_s-h_m)(1-e^{-\alpha t})\left(\frac{\mu_m}{\mu_s}\right)\right]U_o$$
$$+\frac{\pi}{\rho_m\omega}\left[(h_m^2-h_a^2)\{1-cos(\omega t)\}-2(h_m+h_a)\left(\frac{G}{\rho_m}\right)^{\frac{1}{2}}\left\{t e^{-\left(\frac{1}{2}\alpha t\right)}\left[I_o\left(\frac{\alpha t}{2}\right)+I_1\left(\frac{\alpha t}{2}\right)\right]-\frac{1}{\sqrt{2}\omega}sin(\omega t)\right\}\right]\phi_o$$
$$+\frac{2\pi}{\rho_s\omega}h_m^{\frac{1}{2}}(h_s^{\frac{1}{2}}-h_m^{\frac{1}{2}})\left(\frac{G}{\rho_m}\right)^{\frac{1}{2}}\left[\frac{1}{\sqrt{2}\omega}sin(\omega t)-t e^{-\left(\frac{1}{2}\alpha t\right)}\left[I_o\left(\frac{\alpha t}{2}\right)+I_1\left(\frac{\alpha t}{2}\right)\right]\right]\phi_o$$
$$+\frac{2\pi}{\mu_s\omega}h_m(h_s-h_m)\left(\frac{G}{\rho_m}\right)\left[\frac{\left(1-e^{(-\alpha t)}\right)}{\alpha}-\frac{sin(\omega t)}{2\omega}\right]\phi_o$$
$$+\frac{2\pi}{\mu_s\omega}h_m(h_s-h_m)^2\left(\frac{G}{\mu_s}\right)\left(\frac{G}{\rho_m}\right)^{\frac{1}{2}}\left[\frac{sin(\omega t)}{\sqrt{2}\omega}-t e^{-\left(\frac{1}{2}\alpha t\right)}\left[I_o\left(\frac{\alpha t}{2}\right)-I_1\left(\frac{\alpha t}{2}\right)\right]\right]\phi_o$$
$$+\frac{2\pi}{\rho_s\omega}h_m^{\frac{1}{2}}(h_s^{\frac{1}{2}}-h_m^{\frac{1}{2}})(h_s-h_m)\left(\frac{G}{\mu_s}\right)\left[\frac{\left(1-e^{(-\alpha t)}\right)}{\alpha}-\frac{sin(\omega t)}{2\omega}\right]\phi_o \tag{22}$$

5. Result and Discussion

To study the effect of various parameters on mucus transport rate quantitatively, expression for it given by equation (22), can be written in non-dimensional form as:

$$Q_m^* = 2\pi\left(\frac{G^*}{\rho_m^*}\right)^{\frac{1}{2}} h_a^* t^* e^{-\left(\frac{1}{2}\alpha^* t^*\right)}\left[I_o\left(\frac{\alpha^* t^*}{2}\right)+I_1\left(\frac{\alpha^* t^*}{2}\right)\right]U_a^* + \frac{2\pi}{\mu_s^*\omega^*}h_m^*(1-h_m^*)\left(\frac{G^*}{\rho_m^*}\right)\left[\frac{\left(1-e^{(-\alpha^* t^*)}\right)}{\alpha^*}-\frac{sin(\omega^* t^*)}{2\omega^*}\right]\phi_o^*$$
$$+2\pi h_m^{*\frac{1}{2}}\left[t^* e^{-\left(\frac{1}{2}\alpha^* t^*\right)}\{I_o\left(\frac{\alpha^* t^*}{2}\right)+I_1\left(\frac{\alpha^* t^*}{2}\right)\}\left(\frac{G^*}{\rho_m^*}\right)^{\frac{1}{2}}-(1-h_m^*)(1-e^{-\alpha^* t^*})\left(\frac{\mu_m^*}{\mu_s^*}\right)\right]$$
$$+\frac{\pi}{\rho_m^*\omega^*}\left[(h_m^{*2}-h_a^{*2})\{1-cos(\omega^* t^*)\}-2(h_m^*+h_a^*)\left(\frac{G^*}{\rho_m^*}\right)^{\frac{1}{2}}\times\left\{t^* e^{-\left(\frac{1}{2}\alpha^* t^*\right)}\left[I_o\left(\frac{\alpha^* t^*}{2}\right)+I_1\left(\frac{\alpha^* t^*}{2}\right)\right]-\frac{1}{\sqrt{2}\omega^*}sin(\omega^* t^*)\right\}\right]\phi_o^*$$
$$+\frac{2\pi}{\rho_s^*\omega^*}h_m^{*\frac{1}{2}}(1-h_m^{*\frac{1}{2}})\left(\frac{G^*}{\rho_m^*}\right)^{\frac{1}{2}}\left[\frac{1}{\sqrt{2}\omega^*}sin(\omega^* t^*)-t^* e^{-\left(\frac{1}{2}\alpha^* t^*\right)}\left[I_o\left(\frac{\alpha^* t^*}{2}\right)+I_1\left(\frac{\alpha^* t^*}{2}\right)\right]\right]\phi_o^*$$
$$+\frac{2\pi}{\mu_s^*\omega^*}h_m^*(1-h_m^*)^2\left(\frac{G^*}{\mu_s^*}\right)\left(\frac{G^*}{\rho_m^*}\right)^{\frac{1}{2}}\left[\frac{sin(\omega^* t^*)}{\sqrt{2}\omega^*}-t^* e^{-\left(\frac{1}{2}\alpha^* t^*\right)}\left[I_o\left(\frac{\alpha^* t^*}{2}\right)-I_1\left(\frac{\alpha^* t^*}{2}\right)\right]\right]\phi_o^*$$
$$+\frac{2\pi}{\rho_s^*\omega^*}h_m^{*\frac{1}{2}}(1-h_m^{*\frac{1}{2}})(1-h_m^*)\left(\frac{G^*}{\mu_s^*}\right)\left[\frac{\left(1-e^{(-\alpha^* t^*)}\right)}{\alpha^*}-\frac{sin(\omega^* t^*)}{2\omega^*}\right]\phi_o^* \tag{23}$$

by using the following non-dimensional set of parameter- (King.et.al 1993 and Verma.et.al 1997)

$$\mu_s^* = \frac{\mu_s}{\mu_o}, \quad \mu_m^* = \frac{\mu_m}{\mu_o}, \quad \rho_s^* = \frac{h_s U_o}{\mu_o}\rho_s, \quad \rho_m^* = \frac{h_s U_o}{\mu_o}\rho_m, \quad h_a^* = \frac{h_a}{h_s}, \quad h_m^* = \frac{h_m}{h_s}, \quad \alpha^* = \frac{G^*}{\mu_m^*},$$

$$G^* = \frac{h_s}{\mu_o U_o}G, \quad t^* = \frac{U_o}{h_s}t, \quad U_a^* = \frac{U_a}{U_o}, \quad \omega^* = \frac{h_s}{U_o}\omega, \quad \phi_o^* = \frac{h_s^2}{\mu_o U_o}\phi_o, \quad Q_m^* = \frac{Q_m}{U_o h_s^2}$$

where μ_o and ρ_o are the viscosity and density of the serous sublayer fluid in contact with epithelium and U_o is cilia tip velocity.

Various graphs are plotted for Q_m^* given by equation (23) in Fig. (2) to (7) using the following set of parameters which have been calculated by using typical values of various characteristics related to airways [King et.al. (1993), Agarwal and Verma (1997) and Kongnuan and Pholuang (2012)]:

$\mu_s^* = 5 - 20, \quad \mu_m^* = 10 - 100, \quad \rho_s^* = 1, \ \rho_m^* = 5, \quad U_a^* = 1 - 8, \ h_a^* = 0.776, \quad \omega^* = \pi/2$

$h_m^* = 0.820 - 0.830, \ G^* = 15 - 30, \quad t^* = 0.004 - 0.016, \quad \phi_o^* = 100 - 110$

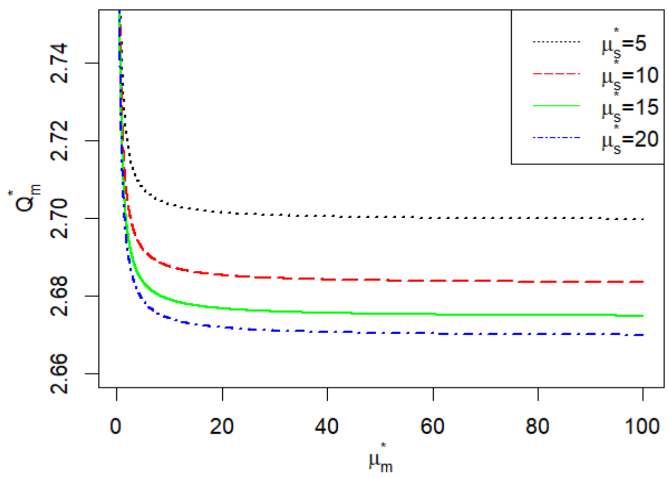

Figure 2: Variation of Q_m^* with μ_s^* for different values of μ_m^*

Figure 2 shows that for $\rho_s^* = 1, \rho_m^* = 5, h_a^* = 0.776, h_m^* = 0.826, G^* = 20, t^* = 0.006, \phi_o^* = 100, U_a^* = 2$, the mucus transport rate decreases with increases the viscosities of serous and mucus layer fluid but it is constant for higher values of mucus viscosity and behave as elastic slab(Corrsin and Ross 1974). This phenomenon is observed in cystic fibrosis patients, who have impaired mucus clearance and airway mucus obstruction. [David and others, 2018]. Furthermore, mucus from bronchitis patients has been shown to be more viscous during flare-ups of the condition [Samet and Cheng, 1994]. This result inlined with the findings of King et al. (1985, 1989) and Agarwal and Verma (1997), Verma (2010), Saxena and Tyagi (2015), Verma and Rana (2015), and Saxena et al. (2020).

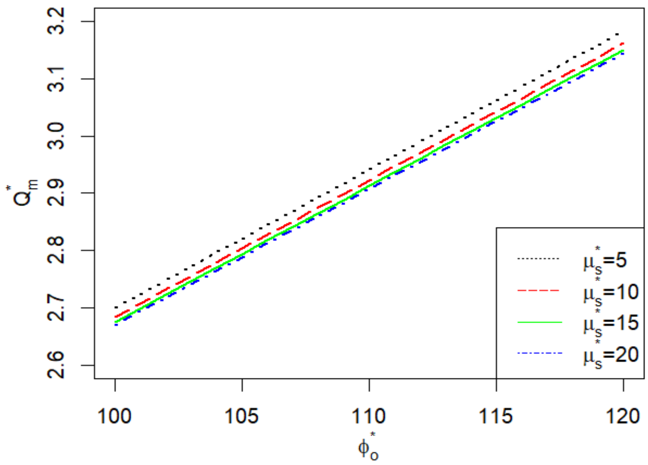

Figure 3: Variation of Q_m^* with ϕ_o^* for different values of μ_s^*

Figure 3 illustrates that for $\rho_s^* = 1, \rho_m^* = 5, h_a^* = 0.776, h_m^* = 0.826, \mu_m^* = 100, \mu_s^* = 5, t^* = 0.006, U_a^* = 2, G^* = 20$, the mucus transport rate rises with increasing oscillatory pressure gradient but falls with increasing serous layer viscosity. This outcome according to the conclusions made in their mathematical models by King et al. (1993) and Agarwal and Verma (1997).

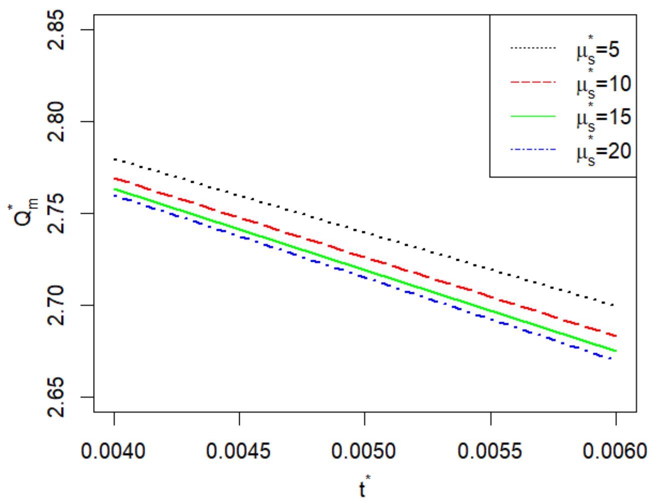

Figure 4: Variation of Q_m^* with t^* for different values of μ_s^*

Figure 4 illustrates that for $\rho_s^* = 1, \rho_m^* = 5, h_a^* = 0.776, h_m^* = 0.826, \mu_m^* = 1000, U^* = 2, \phi_o^* = 100, G^* = 20$,mucus transport rate decreases with an increase in coughing duration and viscosity of serous fluid.The former result is consistent with a study by King et al. (1985).

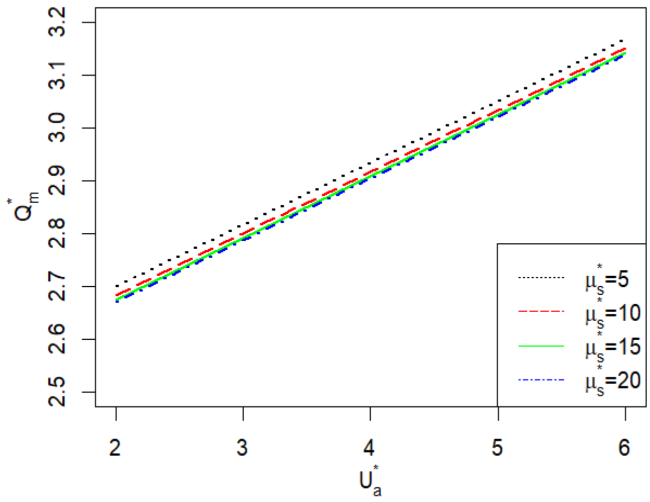

Figure 5: Variation of Q_m^* with U_a^* for different values of μ_s^*

Figure 5 shows that for $\rho_s^* = 1, \rho_m^* = 5, h_a^* = 0.776, h_m^* = 0.826, t^* = 0.006, \mu_m^* = 100, G^* = 20, \phi_o^* = 100, U_a^* = 2,$ mucus flow rate increases with increasing the air mucus interface velocity and falls with viscosity of serous layer.This result is in line with Verma and Tripathi (2013).

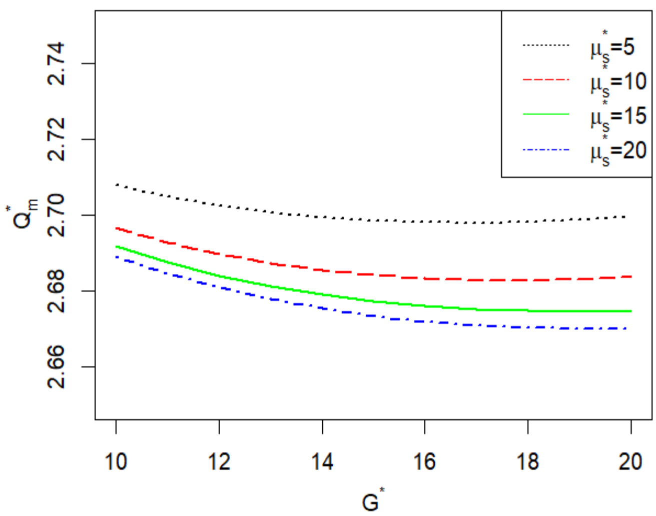

Figure 6: Variation of Q_m^* with G^* for different values of μ_s^*

Figure 6 shows that for $\rho_s^* = 1, \rho_m^* = 5, h_a^* = 0.776, h_m = 0.826, t^* = 0.006, \mu_m^* = 100, \phi_o^* = 100, U_a^* = 2,$ mucus transport rate falls with increasing elastic modulus. This outcome is consistent with the results of Clark et al. (1970) and Verma 1997.

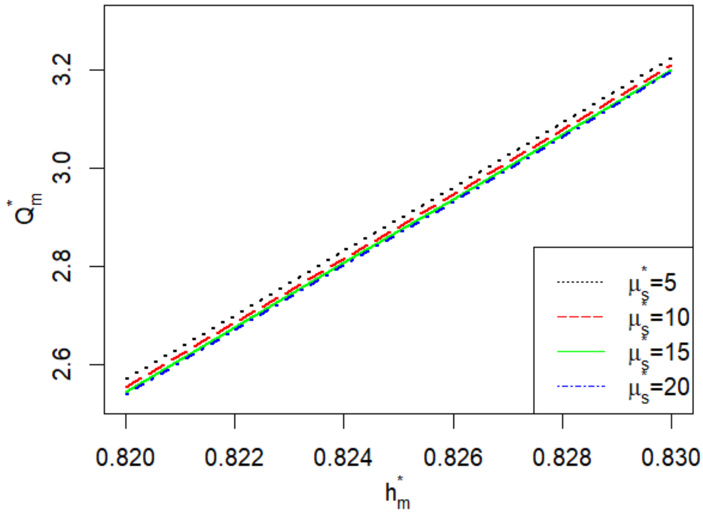

Figure 7: Variation of Q_m^* with h_m^* for different values of μ_s^*

In figure 7, it is seen that for $\rho_s^* = 1, \rho_m^* = 5, h_a^* = 0.776, t^* = 0.006, \mu_m^* = 100, G^* = 20, \phi_o^* = 100, U_a^* = 2$, The mucus transport rate increase when the thickness of mucus increase but it is observed that when serous layer viscosity rises its transport rate falls. This results are in line with Verma and Rana (2015), Rana et al.(2021) and Raut et al.(2023).

6. Conclusion

The results of this study indicate that the mucus transport rate increases when air velocity rises because of air flow, mucus thickness and oscillatory pressure gradient in the fluid layers. It also found that an increase in mucus viscosity, serous fluid viscosity, coughing duration and mucus elastic modulus decrease the mucus transport rate.

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