Dynamic Behaviour of an Eco-Epidemiological model with a Fear, Refuge and Harvesting in Fractional order

N.P.Deepak [1*] , S.Muthukumar [1] , M. Siva Pradeep[1] , M. Sivabalan [1] , A. Yasotha [2] , *T. Aasaithambi* [3] .

[1]*Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore, Tamilnadu, India.*

[2]*Department of Science and Humanities, United Institute of Technology, Coimbatore, Tamilnadu, India.*

[3]*Department of Mathematics ,Sri Shanmugha College of Engineering and Technology, Salem, Tamilnadu, India*

Abstract:-This study discusses the fractional order to examine how fear, refuge, and harvesting affect the dynamic behavior of the predator-prey interaction. The model has been used as the functional response of Crowley Martin in a non-delayed model. The eigenvalues of a model are used to test its stability using critical points. Furthermore, the boundlessness, uniqueness, existence, and positivity of the solutions have been studied. The locally asymptotically stable model has been analyzed using the critical points and the globally asymptotically stable model has been examined using the Lyapunov function. The incidence of Hopf bifurcation for fractional order has been examined. Finally, the analytical solutions are confirmed through numerical simulations.

Keywords: Fractional order, Holling type II, Stability, Lyapunov function, Hopf bifurcation.

1. Introduction

The relationships between predators and prey, diseases within a population, and populations of susceptible and diseased prey are all studied using ecoepidemiological models. The study of eco-epidemiology examines how diseases spread among interacting organisms that have a major influence on the environment [5, 15]. Given that functional response is one of the most crucial elements in the prey-predator population, epidemiological models have garnered a lot of attention since Kermack-Mckendrick's groundbreaking work on SIRS [3].Mathematical models are crucial for understanding, studying, and investigating the spread and management of infectious diseases [6,26].The predator-prey models in coupled systems of non-linear differential equations, created by Lotka [1] and Volterra [2], are regarded as early advances in modern mathematical ecology. We examine disease dynamics and predator-prey models in eco-epidemiology. Numerous studies on the dynamic behaviour of Crowley-Martin ecoepidemiological models have been carried out.Because of its influence on prey abundance, the fear effect is one of the factors that regulates the dynamic behaviours of prey-predator systems [7,14]. Fractional calculus is a generalisation of the classical differentiation and integration of arbitrary orders. Many researchers are interested in scientific and engineering fields, including biology, fluid dynamics, and medicine [8,11].Due to its numerous applications, fractional-order calculus has attracted the interest of researchers throughout the last two decades. Fractional-order biological models have recently attracted the interest of many authors. The main reason lies in the fact that memory-based systems, which exist in a large number of biological systems, are easily relatable to fractional-order models. The fractional-order derivative has the benefit that it allows you to remember the concept of numerical derivative calculation as well as important information about derivative values. Javidi studied the biological behaviors of a prey-predator model with fractional order [12]. This article includes an investigation of the stability of a derivative of a fractional-order model of the mutualistic interaction between two species with infection [23, 25]. Alidousti studied how the capture of predators and scavengers was affected by a prey-predator model with fractional order [10].Mukherjee et al. [17] studied the existence, uniqueness, and boundedness of solutions to a fractional-order prey-predator system in restricted space. Recently, fractional calculations have developed rapidly and displayed a wide range of possible applications. However, due to memory effects, fractional-order derivatives in the biological model are more sensible than integer-order derivatives [18, 27]. To change ordinary calculus to fractional calculus, it is important to use the RiemannLioville and Caputo fractional derivatives. One of the most important processes in any natural ecosystem is the predator-prey model [13]. Caputo introduced the Caputo-type derivative at 1967 [22].An investigation was conducted on a fractional order system that exhibited a Holling-type II functional response. The condition for stability of a system in fractional order, which was developed using Routh-Hurwitz criteria, is that any function that depends on both the current and previous states is a fractional-order derivative. A system with non-linear fractional order stability with the use of the Routh-Jurwitz criterion was investigated by Ahmed et al. [16]. Garappa studied how to solve fractional-order nonlinear differential equations [21]. In a prey-predator model with fractional order, Javidi and Nyamoradi investigated the effects of harvesting [19,28]. There are several mathematical models in fractional order that can be used to solve real-world problems.The discussion that follows provides motivation to study the fear effect's dynamic behaviour on the fractional prey-predator model. The unique aspect of this work is to examine the prior history of the prey-predator model with the fear effect, refuge in prey and harvesting. The novelty of this work is to investigate the stability analysis of the prey-predator model through fractional-order derivatives. The analysis demonstrates that fractional calculus is well suited to explain the memory and inherited features of several techniques and materials that are not taken into consideration by classical integer models. The paper is organized as follows: A mathematical model is developed in Section 2. Section 3 has evaluated the uniqueness and boundedness of the suggested model. In Section 4, the stability analysis of the proposed model has been examined. In Section 5, the Hopf bifurcation of the system is examined. The suggested model's numerical simulations are examined in Section 6. Section 7 contains the paper's conclusion as well as the biological implications of our mathematical findings.

2. Model Formulation

The model has basically two types of population:(i) Prey population and (ii) Predator population.

Melese [8] studied and discussed refuge and harvesting in the prey-predator model with the Holling type II functional response. In the prey-predator model, I have expanded the fear effect, refuge, and harvesting with the Crowley-Martin functional response. The model they provide is as follows:

$$\left.\begin{aligned} \frac{dS}{dT} &= RS\left(1-\frac{S+I}{K}\right) - \lambda IS - \frac{\alpha_1 SP}{a_1+S}, \\ \frac{dI}{dT} &= \lambda IS - d_1 I - \frac{b_1 IP}{a_1+I}, \\ \frac{dP}{dT} &= -d_2 P + \frac{c\alpha_1 SP}{a_1+S} + \frac{cb_1 IP}{a_1+P}. \end{aligned}\right\} \quad (2.1)$$

First, the fear effect will be included in the model (2.1). The fear effect, also known as predator avoidance behaviour, is incorporated into prey-predator models in order to replicate the behaviour of animals in nature.

The condition for the fear effect is

$$\mathcal{F}_1(\mathrm{k},\mathrm{p}) = \frac{1}{1+\mathrm{kp}}$$

This refers to the predator fear level towards susceptible prey. Here, k represents the level of fear. The condition is strongly acceptable, given the epidemiological meaning of k:

$$k(0,p) = \mathcal{F}(k,0) = 1.$$

$$\lim_{k\to\infty} \mathcal{F}_1(k,p) = 0 = \lim_{p\to\infty} \mathcal{F}_1(k,p).$$

$$\frac{\partial \mathcal{F}_1(k,p)}{\partial k} < 0.$$

$$\frac{\partial \mathcal{F}_1(k,p)}{\partial w} < 0.$$

To introduce the fear effect, we have to add the term $\frac{1}{1+\mathrm{kp}}$ in the model (2.1) which represents the fear effect with parameter k as a level of fear.Thus, by adding the fear effect the model (2.1) becomes,

Here the conditions are S(0) ≥0, I(0) ≥ 0 and P(0) ≥ 0.

The following set of equations arises from the model's explanation of the interactions between the diseased prey system and fear, refuge, and harvesting. The suggested model (2.1) was used to examine the dynamic prey and predator mathematical models,

$$\left.\begin{aligned} \frac{dS}{dT} &= \frac{RS}{1+\mathcal{F}P}\left(1-\frac{S+I}{K}\right) - \lambda IS - \frac{\alpha_1 SP}{(1+\zeta S)(1+\eta P)} - H_1E_1S\,, \\ \frac{dI}{dT} &= \lambda IS - d_1 I - \frac{b_1(1-g)IP}{a_1+(1-g)I} - H_2E_2I, \qquad (2.2) \\ \frac{dP}{dT} &= -d_2P + \frac{c\alpha_1 SP}{(1+\zeta S)(1+\eta P)} + \frac{cb_1(1-g)IP}{a_1+(1-g)I}. \end{aligned}\right\}$$

Parameters	Environmental representation
U	Susceptible prey
V	Infected prey
W	Predator
R	Growth rate of prey
K	Carrying capacity of environment
a_1	Constant of half-saturation
α_1	Predation rate of Susceptible prey
b_1	Predation rate of Infected prey
C	Predator-to-prey conversion rate
D_1	Death rate of Infected prey
D_2	Death rate of Predator
λ	Infection rate
$\mathcal{F}$	Fear effect
G	Refuge

Table 1: Biological representation of system (2.2) parameters

To decrease the number of parameters of the system (2.2), it is appropriate to modify the variables to

$s=\frac{S}{K}, i=\frac{I}{K}$ and $p=\frac{P}{K}$ and to account for the dimension time $t=\lambda KT$. We now make the following adjustments:

$$r=\frac{R}{K}, \alpha=\frac{\alpha_1}{K}, a=\frac{a_1}{\lambda K}, d_1=\frac{D_1}{\lambda K}, d_2=\frac{D_2}{\lambda K}, b=\frac{\theta}{\lambda K}, \theta_1=\frac{H_1E_1}{K}, \theta_2=\frac{H_2E_2}{K}.$$

With the aforementioned transformations, the equation (2.2) can be rewritten in the following non-dimensional form:

$$\frac{ds}{dt} = \frac{rs}{1+kp}(1-s-i) - is - \frac{\alpha sp}{(1+\zeta s)(1+\eta p)} - \theta_1 s\,,$$

$$\frac{di}{dt} = is - d_1 i - \frac{\theta(1-g)ip}{a+(1-g)i} - \theta_2 i, \qquad (2.3)$$
$$\frac{dp}{dt} = -d_2 p + \frac{c\alpha sp}{(1+\zeta s)(1+\eta p)} + \frac{\theta(1-g)ip}{a+(1-g)i}$$

The model (2.3) in the system is obtained by calculating the Caputo fractional-order derivative β. The model is then taken in the following form:

$$\left.\begin{aligned} \frac{d^\beta s}{dt^\beta} &= \frac{rs}{1+kp}(1-s-i) - is - \frac{\alpha sp}{(1+\zeta s)(1+\eta p)} - \theta_1 s\,, \\ \frac{d^\beta i}{dt^\beta} &= is - d_1 i - \frac{\theta(1-g)ip}{a+(1-g)i} - \theta_2 i, \\ \frac{d^\beta p}{dt^\beta} &= -d_2 p + \frac{c\alpha sp}{(1+\zeta s)(1+\eta p)} + \frac{\theta(1-g)ip}{a+(1-g)i} \end{aligned}\right\} \qquad (2.4)$$

subject to the initial conditions $s(0) \geq 0, i(0) \geq 0$ and $p(0) \geq 0$.

Existence and Uniqueness of solutions:

Theorem :1

For the non-negative initial conditions, there is only one solution to the fractional-order system (2.4).

Proof:

The existence and uniqueness of system (2.4) will be above in the area,

Where $\chi = \{(s,i,p) \in R^3 : \max(|s|, |i|, |p| \leq \delta\}$.

Now, let us define a mapping $V(X) = (V_1(X), V_2(X), V_3(X))$

Where ,

$$V_1(X) = \frac{rs}{1+kp}(1-s-i) - is - \frac{\alpha sp}{(1+\zeta s)(1+\eta p)} - \theta_1 s,$$

$$V_2(X) = is - d_1 i - \frac{\theta(1-g)ip}{a+(1-g)i} - \theta_2 i,$$

$$V_3(X) = -d_2 p + \frac{c\theta(1-g)ip}{a+(1-g)i} + \frac{c\alpha sp}{(1+\zeta s)(1+\eta p)}.$$

$$\parallel V(X) - V(\bar{X}) \parallel = |V_1(X) - V_1(\bar{X})| + |V_2(X) - V_2(\bar{X})| + |V_3(X) - V_3(\bar{X})|$$

$$\begin{aligned} = &\left|\frac{rs}{1+kp}(1-s-i) - is - \frac{\alpha sp}{(1+\zeta s)(1+\eta p)} - \theta_1 s - \frac{r\bar{s}}{1+k\bar{p}}(1-\bar{s}-\bar{\imath}) + \bar{s}\bar{\imath} + \frac{\alpha\bar{s}\bar{p}}{(1+\zeta\bar{s})(1+\eta\bar{p})} + \theta_1\,\bar{u}\right| \\ &+ \left|si - d_1 i - \frac{\theta(1-g)ip}{a+(1-g)i} - \theta_2\bar{v} - \bar{\imath}\bar{s} + d_1\bar{\imath} + \frac{\theta(1-g)\bar{\imath}\bar{p}}{a+(1-g)\bar{\imath}} + \theta_2\bar{\imath}\right| \\ &+ \left|-d_2 p + \frac{c\theta(1-g)ip}{a+(1-g)i} + \frac{c\alpha sp}{(1+\zeta s)(1+\eta p)} + d_2\bar{p} - \frac{c\theta(1-g)\bar{\imath}\bar{p}}{a+(1-g)\bar{\imath}} - \frac{c\alpha\bar{s}\bar{p}}{(1+\zeta\bar{s})(1+\eta\bar{p})}\right| \end{aligned}$$

$$\begin{aligned} \leq |s-\bar{s}|\{3r\psi + 3rk\psi + 2\psi + \alpha a\psi + c\alpha a\psi\} + |i-\bar{\imath}|\{r\psi + rk\psi + 2\psi + d_1\psi + ba\psi + cba\psi\} \\ + |p-\bar{p}|\{3rk\psi + \alpha a\psi + \alpha\psi + ba\psi + b\psi + cba\psi + cb\psi + c\alpha a\psi + c\alpha\psi\} \end{aligned}$$

$$\leq \mathcal{M}|X - \bar{X}|$$

Where

$$\begin{aligned} \mathcal{M} = \max\{&3r\psi + 3rk\psi + 2\psi + \alpha a\psi + c\alpha a\psi, r\psi + rk\psi + 2\psi + d_1\psi + ba\psi + cba\psi, 3rk\psi + \alpha a\psi + \alpha\psi \\ &+ ba\psi + b\psi + cba\psi + cb\psi + c\alpha a\psi + c\alpha\psi\} \end{aligned}$$

Hence, the solution of the system (2.4) exists and unique.

Boundedness of the solutions:

Theorem:2

The system's (2.4) solutions are uniformly bounded and non-negative.

Proof:

Define a function

$$V(t) = S(t) + I(t) + P(t)$$

Then for any positive number λ, we obtain

$$C_D^{\beta}(t) + \lambda V(t) = \left(\frac{rs}{1+kp}(1-s-i) - is - \frac{\alpha sp}{(1+\zeta s)(1+\eta p)} - \theta_1 s\right) + \left(is - d_1 i - \frac{b(1-g)ip}{a+(1-g)i} - \theta_2 i\right) + \left(-d_2 p + \frac{cb(1-g)ip}{a+(1-g)i} + \frac{c\alpha sp}{(1+\zeta s)(1+\eta P)}\right) + \lambda(S+I+P)$$

$$= \frac{rs}{1+kp} - \frac{rs^2}{1+kp} - \frac{rsi}{1+kp} - \frac{\alpha sp}{(1+\zeta s)(1+\eta p)} + \frac{c\alpha sp}{(1+\zeta s)(1+\eta p)} + \lambda s - d_1 i - \frac{b(1-g)i}{a+(1-g)i} + \frac{cb(1-g)i}{a+(1-g)i} - d_2 p + \lambda S + \lambda I + \lambda P$$

$$= \frac{rs(1-s-i)}{1+kp} - \frac{\alpha sp(1-c)}{(1+\zeta s)(1+\eta p)} - d_1 i - \frac{b(1-g)ipc}{a+(1-g)i} + \lambda I - d_2 p + \lambda P$$

$$\leq \frac{rs}{1+kp} - d_2 p - d_1 i$$

$$\leq \frac{\epsilon}{4}$$

Where $\lambda = \min(d_1, d_2)$

Applying the standard comparison theorem for fractional order, we get

$$V(t) \leq V(0) G_{\beta}\left(-\lambda(t)^{\beta}\right) + \left(\frac{\epsilon}{4}\right) t^{\beta} G_{\{\beta,\beta+1\}}(-\lambda(t)^{\beta})$$

Where G_{β} is the Mittag-Leffler function.

$$V(t) \leq \frac{\epsilon}{4\lambda}, t \to \infty$$

Therefore, all solutions of fractional-order system (2.4) which are initiating in R_3^+ , will enter the region

$$\Delta = \left\{S, I, P \in R_3^+ : V \leq \frac{\epsilon}{4\lambda} + \epsilon, \epsilon > 0\right\}$$

Critical Points and Stability Analysis:

The critical point of the model (2.4) are given by the following possible critical points arise from solving the above equations:

(i) The trivial critical point is $E_0(0,0,0)$.

(ii) The axial critical point is $E_1\left(\frac{r-\theta_1}{r}, 0,0\right)$.

(iii) The predator free critical point is $E_2(\hat{s}, \hat{\iota}, 0)$,

where $\hat{s} = d_1$ and $\hat{\iota} = \frac{r(1-d_1)}{r+1}$.

(iv) The interior critical point $E^*(s^*, i^*, p^*)$

Where,

$$i^* = \frac{a(ad_2 + (d_2 - c\alpha)s^*)}{(c\alpha s^* + (cb - d_2)(a + s^*))}$$

$$p^* = \frac{ac(s^* - d_1)(a+s^*)}{(c\alpha s^* + (cb - d_1)(a+s^*)}$$

and s^* is the only positive root of the equation for a quadratic function.

$$As^2 + Bs + C = 0,$$

Where

$$A = r(c\alpha + cb - d_2),$$

$$B = (cb - d_2)(ar - r) + \alpha c\big((1 + kp - r)\big) + a(d_2(1 + kp) + (d_2 - c\alpha)r),$$

$$C = -a\big(r(1 + kp)\big)(cb - d_2) + (c\alpha(1 + kp)d_1) - ad_2((1 + kp)r))).$$

Stability Analysis:

For local stability analysis, we wish to compute the variational matrix around several critical points. Given an arbitrary point (s, i, p), the variational matrix is given by

$$J(u, v, w) = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}$$

Where,

$$n_{11} = \frac{r}{1 + kp}(1 - 2s - i) - i - \theta_1 - \frac{\alpha p}{(1 + \zeta s)(1 + \eta p)} + \frac{\alpha \zeta sp}{(1 + \zeta s)^2(1 + \eta p)},$$

$$n_{12} = -s\left(\frac{r}{1 + kp} + 1\right),$$

$$n_{13} = (1 - u - v) - \frac{\alpha s}{(1 + \zeta s)(1 + \eta p)} + \frac{\alpha \eta sp}{(1 + \zeta s)(1 + \eta p)^2}$$

$$n_{21} = i, n_{22} = s - d_1 - \frac{ab(1 - g)p}{(a + (1 - g)i)^2}, n_{23} = -\frac{b(1 - g)i}{(a + (1 - g)i)^2},$$

$$n_{31} = \frac{ac\alpha p}{(a + i)^2}, n_{32} = \frac{acb(1 - g)p}{(a + (1 - g)i)^2}, n_{33} = -d_2 + \frac{cb(1 - g)i}{(a + (1 - g)i)}.$$

Theorem:3

The trivial equilibrium point $E_0(0,0,0)$ which is saddle.

Proof: At an critical point E_0, the variational matrix is given by

$$J(E_0) = \begin{pmatrix} r & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix}$$

The eigenvalues at E_0 are $\lambda_1 = r, \lambda_2 = -d_1, \lambda_3 = -d_2$.

Thus, $|\arg(\lambda_1)| = 0 < \frac{\beta\pi}{2}, |\arg(\lambda_2)| = \pi > \frac{\beta\pi}{2}, |\arg(\lambda_3)| = \pi > \frac{\beta\pi}{2}$.

Therefore, E_0 is saddle.

Theorem:4

The axial critical point $E_1(1,0,0)$ is unstable.

Proof: At an critical point E_1, the variational matrix is given by

$$J(E_1) = \begin{pmatrix} -r & -(r+1) & \frac{-\alpha}{a+s} \\ 0 & 1-d_1 & 0 \\ 0 & 0 & -d_2 + \frac{c\alpha}{a+s} \end{pmatrix}$$

The eigenvalues at E_1 are $\lambda_1 = -r, \lambda_2 = 1 - d_1, \lambda_3 = -d_2 - \frac{c\alpha}{a+s}$.

Thus, $|\arg(\lambda_1)| = 0 < \frac{\beta\pi}{2}, |\arg(\lambda_2)| = \pi > \frac{\beta\pi}{2}, |\arg(\lambda_3)| = \pi > \frac{\beta\pi}{2}$.

Due to numerical simulation table values, $1 - d_1$ is positive.

The critical point $E_1(1,0,0)$ is unstable.

Theorem:5

The predator-free critical point $E_2(\hat{s}, \hat{\imath}, 0)$ is locally asymptotically stable if $d_2 > c(\alpha + b)$.

Proof: At an critical point E_2, the variational matrix is given by

$$J(E_1) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Where,

$a_{11} = r(1 - 2\hat{s}) + i(r+1),\ a_{12} = (-1 - r)\hat{s}, a_{13} = \frac{-\alpha\hat{s}}{a+s}, a_{21} = \hat{\imath}, a_{22} = 0,$

$a_{23} = \frac{b\hat{\imath}}{a+i}, a_{31} = 0, a_{32} = 0, a_{33} = \frac{c\alpha\hat{s}}{a+\hat{s}} - d_2 + \frac{cb\hat{\imath}}{a+\hat{\imath}}.$

Here, the characteristic equation of the above variational matrix is provided by

$$\lambda^3 + X\lambda^2 + Y\lambda + Z = 0.$$

Where,

$$X = a_{11} - a_{33},$$

$$Y = a_{21}a_{12} + a_{33}a_{11},$$

$$Z = a_{12}a_{21}a_{33}.$$

According to Routh-Hurwitz Criteria,

Hence, E_2 is locally asymptotically stable.

Theorem:6

The endemic critical point $E^*(s^*, i^*, p^*)$ is locally asymptotically stable.

Proof: At an critical point E^*, the variational matrix is given by

$$J(E_1) = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

Where,

$$g_{11} = \frac{r}{1+kp^*}(1-2s^*-i^*) - i^* - \theta_1 - \frac{\alpha p^*}{(1+\zeta s^*)(1+\eta p^*)} + \frac{\alpha\zeta s^* p^*}{(1+\zeta s^*)^2(1+\eta p^*)},$$

$$g_{12} = -s^*\left(\frac{r}{1+kp^*}+1\right),$$

$$g_{13} = (1-s^*-i^*) - \frac{\alpha s^*}{(1+\zeta s^*)(1+\eta p^*)} + \frac{\alpha\eta s^* p^*}{(1+\zeta s^*)(1+\eta p^*)^2}$$

$$g_{21} = i^*, g_{22} = s^* - d_1 - \frac{ab(1-g)p^*}{(a+(1-g)i^*)^2}, g_{23} = -\frac{b(1-g)i^*}{(a+(1-g)i^*)^2},$$

$$g_{31} = \frac{ac\alpha p^*}{(a+i^*)^2}, g_{32} = \frac{acb(1-g)p^*}{(a+(1-g)i^*)^2}, g_{33} = -d_2 + \frac{cb(1-g)i^*}{(a+(1-g)i^*)}.$$

Here, the characteristic equation of the above variational matrix is provided by

$$\lambda^3 + E\lambda^2 + F\lambda + G = 0.$$

Where,

$$E = -g_{11} - g_{33},$$

$$F = g_{21}g_{12} + g_{22}g_{11} - g_{13}g_{31} + g_{23}g_{32},$$

$$G = g_{13}(-g_{22}g_{31} + g_{21}g_{32}) + g_{23}(g_{12}g_{31} - g_{11}g_{32}).$$

According to Routh-Hurwitz Criteria,

Hence, E^* is locally asymptotically stable.

Global Stability Analysis:

Theorem:7

The endemic critical point E^* is globally asymptotically stable.

Proof:

Consider a Lyapunov function

$$V(S,I,P) = [s - s^* - s^*\ln\frac{s}{s^*}] + q_1[i - i^* - i^*\ln\frac{i}{i^*}] + q_2[p - p^* - p^*\ln\frac{p}{p^*}]$$

Applying the caputo fractional derivative, we obtain

$$\leq \left(\frac{s-s^*}{s}\right) {}^{C}_{D}{}^{\beta} s(t) + q_1\left(\frac{i-i^*}{i}\right) {}^{C}_{D}{}^{\beta} i(t) + q_2\left(\frac{p-p^*}{p}\right) {}^{C}_{D}{}^{\beta} p(t)$$

$$\leq \left(\frac{s-s^*}{s}\right)\left(\frac{rs}{1+kp}(1-s-i)-is-\frac{\alpha sp}{(1+\zeta s)(1+\eta p)}-\theta_1 s\right) + q_1\left(\frac{i-i^*}{i}\right)\left(is-d_1 i-\frac{b(1-g)ip}{a+(1-g)i}-\theta_2 i\right) + q_2\left(\frac{p-p^*}{p}\right)\left(-d_2 p+\frac{cb(1-g)ip}{a+(1-g)i}+\frac{c\alpha sp}{(1+\zeta s)(1+\eta p)}\right)$$

$$\leq -(s-s^*)\left[r\left\{\frac{(s+i)-(s^*+i^*)}{(1+kp)(1+kp^*)}\right\}\right]+\alpha\left[\frac{p}{(1+\zeta s)(1+\eta p)}-\frac{s^*}{(1+\zeta s^*)(1+\eta p^*)}\right] - q_1(i-i^*)\left[(s-s^*)-\frac{p}{a+i}-\frac{p^*}{a+i^*}\right]-q_2(p-p^*)\left[cb\left(\frac{i(a+i^*)-i^*(a+i)}{(a+i)(a+i^*)}\right)\right] + c\alpha\left[\frac{(1+\zeta s^*)(1+\eta p^*)s-(1+\zeta s)(1+\eta p)s^*}{(1+\zeta s)(1+\eta p)(1+\zeta s^*)(1+\eta p^*)}\right]$$

Obviously, $C_D^{\beta}V(s,i,p)\leq 0$.

We conclude that E^* is globally asymptotically stable.

Hopf-Bifurcation Analysis:

In this section, we discuss about the Hopf-bifurcation analysis of system (2.4). Define a function, with respect to β by

$$m(\beta)=\frac{\beta\pi}{2}-\min_{1\leq i\leq 3}|\arg(\lambda_i)|$$

Theorem:8

The fractional-order system (2.4) experiences a Hopf bifurcation at the endemic equilibrium point $E^*$when bifurcation parameter θ_1 passes through the critical value $\theta_1^*\in(0,1)$, provided that the following conditions are satisfied:

(i) the corresponding characteristic equation (5.3) of system (2.4) has a pair of complex conjugates $\lambda_{1,2}=\delta+i\omega\ (where\ \delta>0)$ and one negative real root λ_3.

(ii) $m(\theta_1^*)=\frac{\beta^*\pi}{2}-\min_{1\leq i\leq 3}|\arg(\lambda_i)|=0$.

(iii) $\left|\frac{dm(\beta)}{d\beta}\right|_{\beta=\beta^*}\neq 0$.

Here, we give the conditions under which a Hopf bifurcation would exist as the derivative's order approaches a critical value at the interior equilibrium point E^*.

Numerical Analysis :

In this section, we demonstrate a numerical simulation to examine the fractional-order system's dynamics. While there are other numerical methods available for solving nonlinear fractional differential equations, the Adams-type predictor corrector method [20] is a more suitable and practical approach for solving the dynamic behaviour of fractional differential equation solutions.

This study examines a set of parametric values:

$r=1.05,\alpha=0.25,a=0.3,d_1=0.1,b=0.5,d_2=0.1,c=0.5,g=0.2,k=0.4,$

$\theta_1=0.1,\theta_2=0.3,\zeta=0.4,\eta=0.2.$

In this case, we have investigated the use of fear factor in the integer order system. When the model system (2.4) approaches its interior equilibrium point, the dynamic behaviour of its solutions is seen to change from an unstable steady state to a stable steady state.

The time series and phase portrait of the system (2.4) solutions for k = 0.7 are shown in Figs. 1, respectively. These figures clearly show the unstable steady state of the interior equilibrium point $E^*(0.859953,0.0114115,0.591645)$ for $\beta = 1$. The time series and phase portrait of the system (2.4) solutions around the interior equilibrium point $E^*(0.859953,0.0114115,0.591645)$ for $\beta = 0.92$ are shown in Figs. 2, respectively. Next, we plot the solution of system (2.4) by choosing the same set of parameters. Figures 3 illustrate the impact of β on each population, highlighting its crucial role in the system's dynamics due to its fractional order derivative. We can see from this diagram that β significantly affects each population.

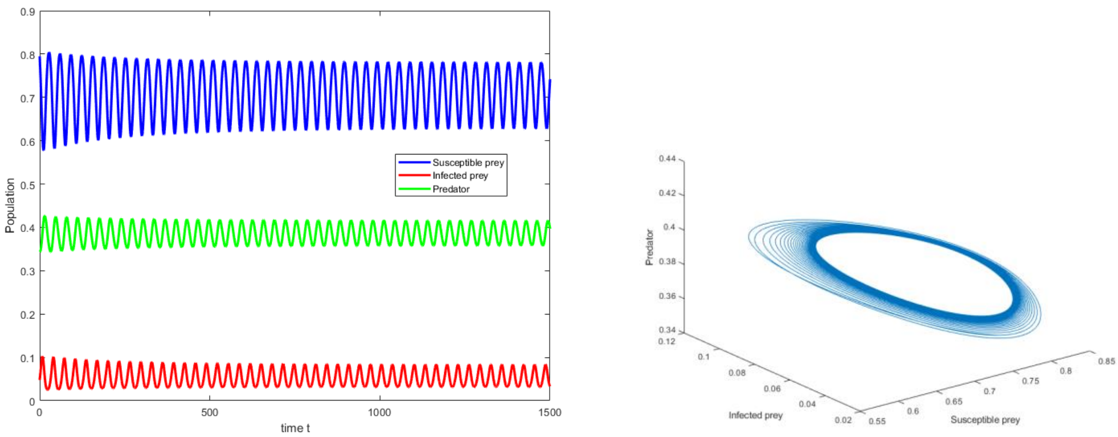


Figure 1: Time series and Phase portrait solution of system (2.4) for the interior equilibrium point when β = 1

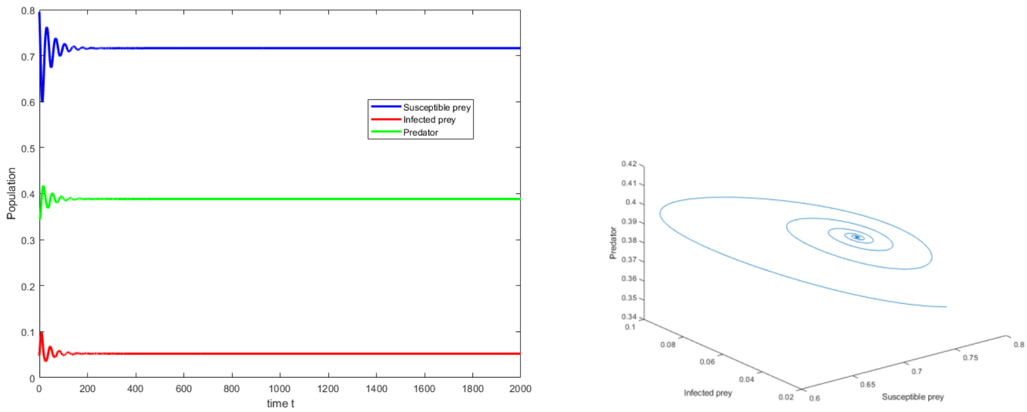


Figure 2: Time series and Phase portrait solution of system (2.4) for the interior equilibrium point when β = 0.95

Tuijin Jishu/Journal of Propulsion Technology
ISSN: 1001-4055
Vol. 45 No. 2 (2024)

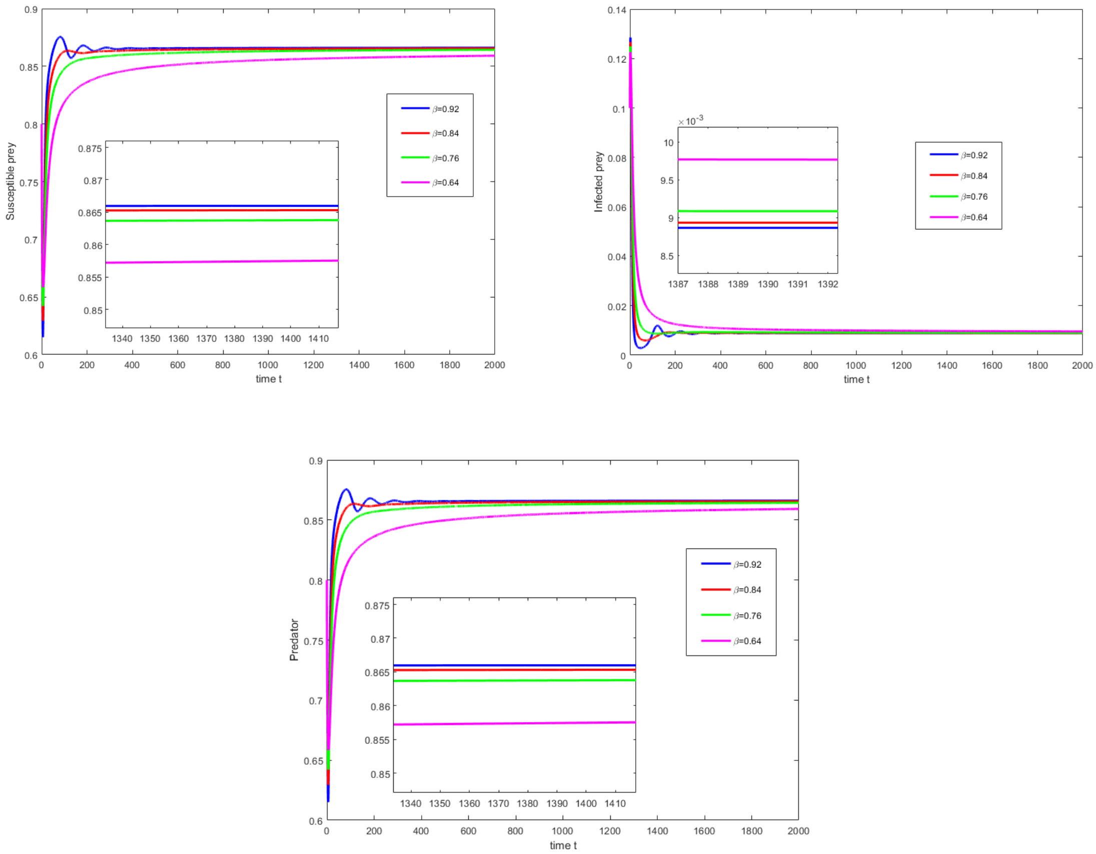
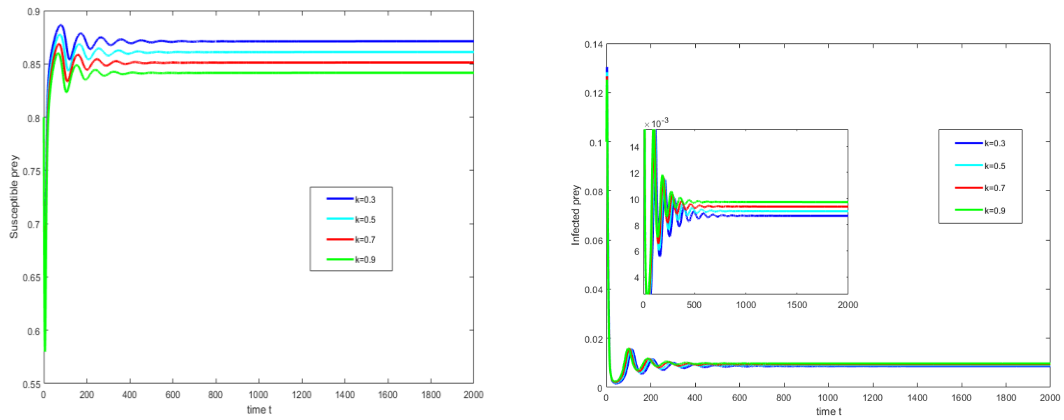


Figure:3 Effectiveness of fractional order on each population of system (2.4) when β = 0.92, 0.84, 0.72, 0.64

Effect of varying Fear Effect:

Figure 4 shows that the density of the susceptible prey falls as the fear effect increases. As the fear impact k increases from 0.3 to 0.9, the number of infected prey increases (Figure 4)).As the impact of fear increases, Figure 4 illustrates that the density of predators decreases. Figure 4 shows that the fractional-order derivative has a major impact on the stability of our suggested system.



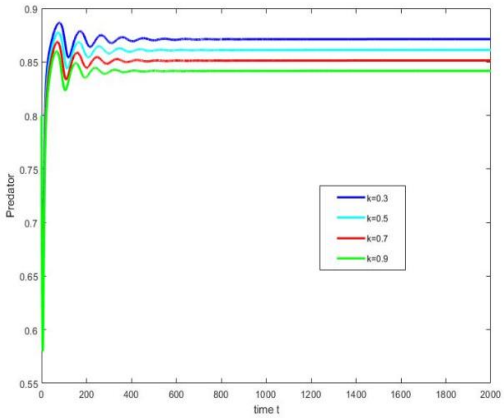


Figure:4 Time series solutions on each population of system (2.4) for k = 0.2, 0.4, 0.6, 0.8

Effect of varying Refuge:

Figure 5 shows that the density of the susceptible prey falls as the refuge value increases. As Figure 5 illustrates, the quantity of infected prey grows when the refuge g rises from 0.3 to 0.9. As the effect of refuge increases, Figure 5 illustrates that the density of predators decreases. Figure 5 shows that the fractional-order derivative has a major impact on the stability of our suggested system.

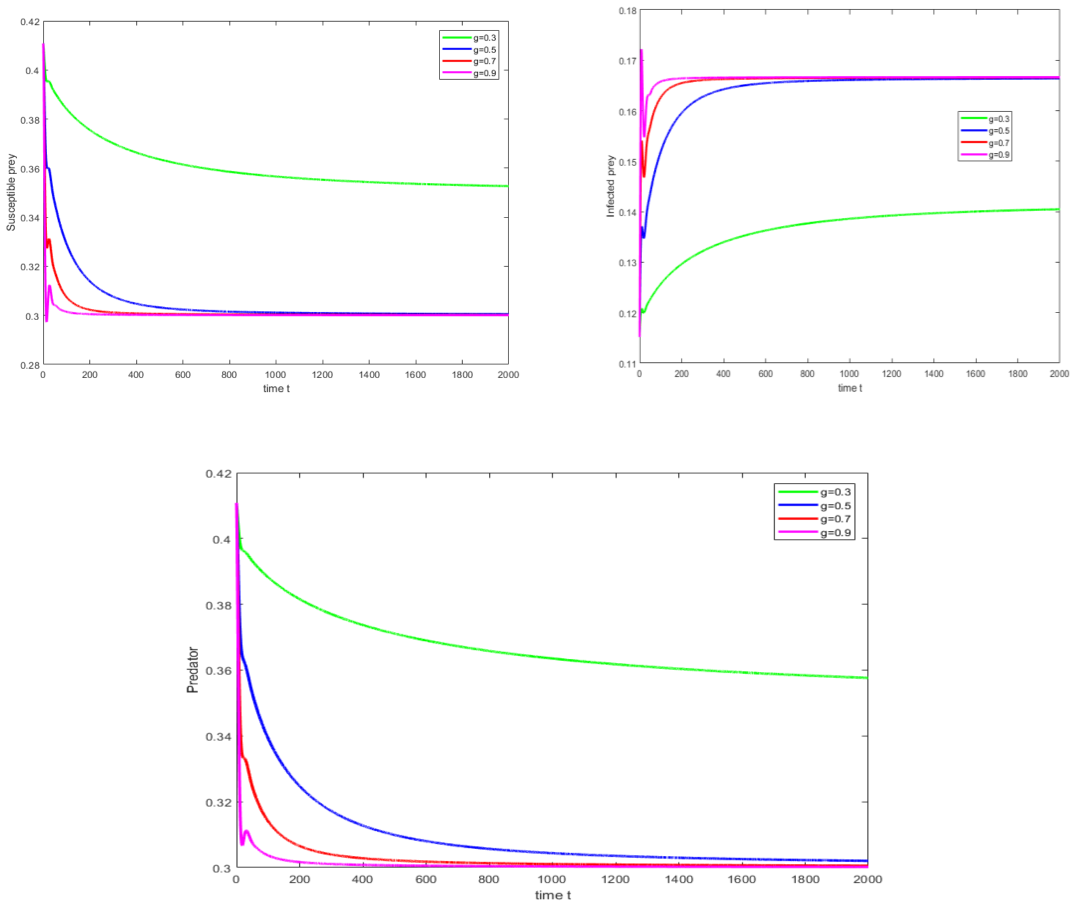


Figure 5: Different values of refuge on each population of system (2.4) for g = 02, 0.4, 0.6, 0.8

Effect of varying Harvesting:

Figure 6 shows that the density of the susceptible prey falls as the harvesting rate increases.As θ_1 increases from 0.3 to 0.9, the number of infected prey also increases (Figure 6).As the affect of harvesting increases, Figure 6 illustrates how the density of predators decreases.Figure 4 shows that the fractional-order derivative has a major impact on the stability of our suggested system.

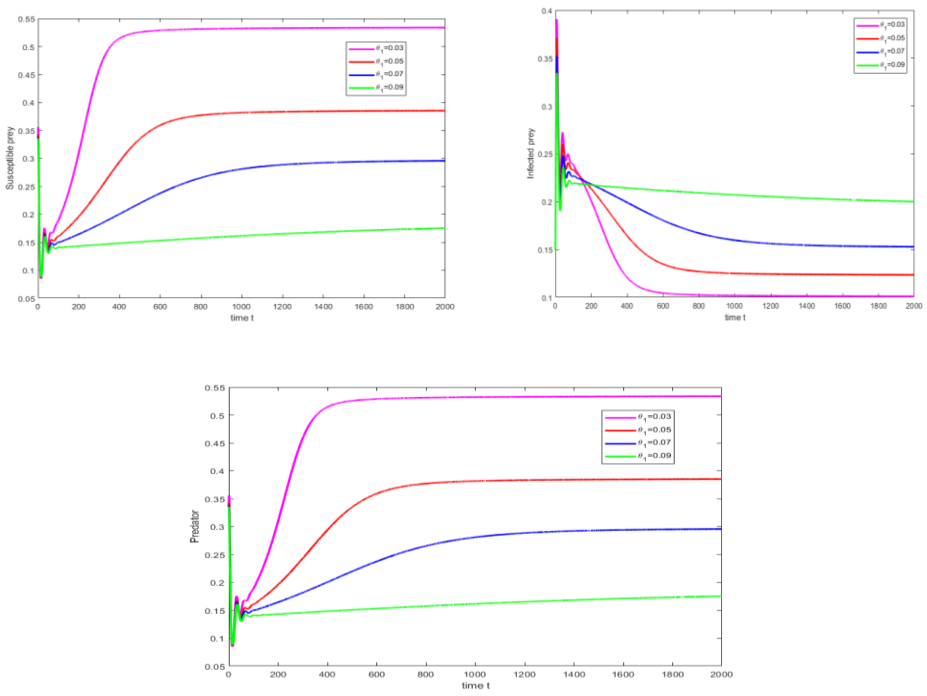
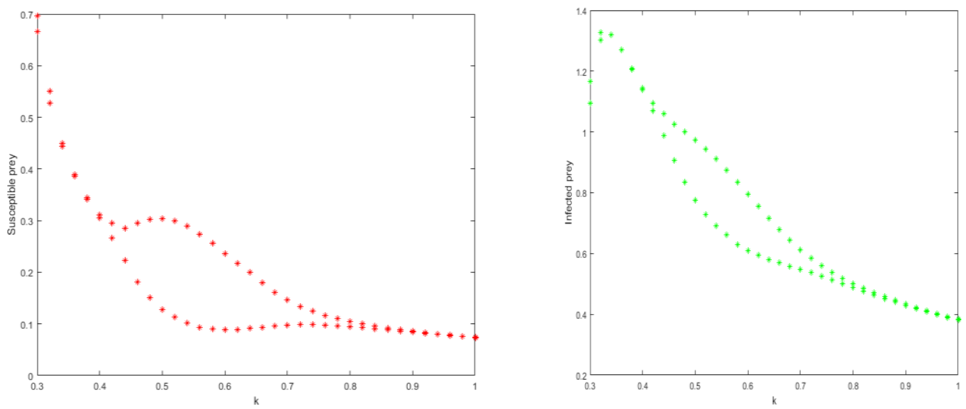


Figure 6: Harvesting effect of population for interior equilibrium point for the fractional order β = 0.92

Bifurcation of harvesting:

We now need to examine the impact of the harvesting rate on our system. A bifurcation diagram of system (2.4) is shown in Fig. 5 at the level of harvesting θ_1 when $\beta = 0.92$.For $\beta \in (0,1)$, system (2.4) exhibits an interior equilibrium, as we have demonstrated.The system becomes unstable nevertheless when β approaches the critical value β^*. Therefore, compared to the fractional-order model, the integer-order model is less stable.When $\beta = 0.92$ and the harvesting rate θ_1 is greater than the threshold value of $\theta_1 = 0.41$. Hopf-bifurcation is used to change the dynamics from an unstable to a stable steady state.



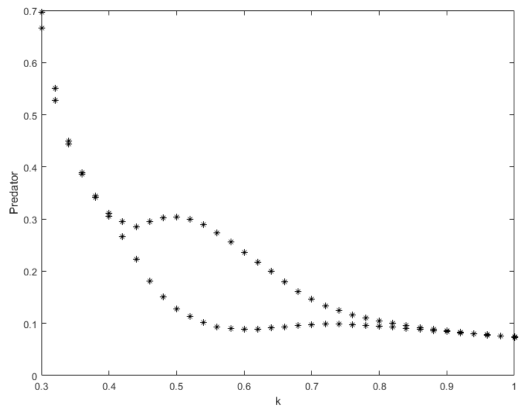


Figure 7: Bifurcation diagram for harvesting h_1 each population for the fractional order derivative β = 0.92

Conclusion:

In a fractional-order ecosystem under investigation, predator attacks on susceptible and infected prey cause fear and infection in the respective populations of susceptible and infected prey. It may also be shown that system (2.4) represents all potential biological states of equilibrium. The existence, uniqueness, boundedness, and local stability of the suggested model (2.4) were also investigated, and Hopf-bifurcation was observed. Our conclusion was based on these results, which showed that the stability (2.4) of the model is immediately affected by changing the cost of fear k in fractional order. We have investigated the equilibrium points of the fractional-order system with respect to their global asymptotic stability. These results imply that the fractional-order mathematical model can be helpful in explaining the dynamics of systems with practical memory.

References:

[1] Lotka, Alfred James, Elements of physical biology, Williams & Wilkins.

[2] Volterra, V., Variations and fluctuations in the number of individuals in cohabiting animal species Accidental Mem. Lincei Rome 2 31; Fluctuations in the abundance of a species consider mathematically, Nature (London), 118(558), 1926.

[3] Kermack, William Ogilvy and McKendrick, Anderson G, A contribution to the mathematical theory of epidemics, Proceedings of the royal society of london. Series A, Containing papers of a mathematical and physical character, 115 (772), 1927.

[4] Panja Prabir, Dynamics of a fractional order predator-prey model with intraguild predation, International Journal of Modelling and Simulation, Taylor & Francis, 39(4), 2019.

[5] Panigoro, Hasan S and Suryanto, Agus and Kusumawinahyu, Wuryansari Muharini and Darti, Isnani, Dynamics of an eco-epidemic predator--prey model involving fractional derivatives with power-law and Mittag--Leffler kernel, Symmetry, MDPI, 13(5), 2021.

[6] Magudeeswaran, S and Sathiyanathan, K and Sivasamy, R and Vinoth, S and Sivabalan M, Analysis on Dynamics of Delayed Intraguild Predation Model with Ratio-Dependent Functional Response, Discontinuity, Nonlinearity, and Complexity, L&H Scientific Publishing, 10(03), 2021.

[7] Ramesh, Perumal and Sambath, Muniyagounder and Mohd, Mohd Hafiz and Balachandran, Krishnan, Stability analysis of the fractional-order prey-predator model with infection, International Journal of Modelling and Simulation, 41(6), 2021.

[8] Melese, Dawit and Muhye, Ousman and Sahu, Subrata K, Dynamical behavior of an eco-epidemiological model incorporating prey refuge and prey harvesting, Applications and Applied Mathematics: An International Journal (AAM), 15(2), 2020.

[9] Mahardhika, Widowati, Sumanto, Routh-hurwitz criterion and bifurcation method for stability analysis of tuberculosis transmission model, Journal of Physics: Conference Series,1217(2019).

[10] Alidousti, Javad, Stability and bifurcation analysis for a fractional prey--predator scavenger model, Applied Mathematical Modelling, Elsevier, 81(342-355), 2020.

[11] Xie, Binfeng and Zhang, Zhengce, Impact of Allee and fear effects in a fractional order prey--predator system incorporating prey refuge, Chaos: An Interdisciplinary Journal of Nonlinear Science, AIP Publishing, 33(1), 2023.

[12] Javidi, M and Nyamoradi, N, Dynamic analysis of a fractional order prey--predator interaction with harvesting, Applied mathematical modelling, Elsevier, 37(20-21), 2013.

[13] Sambath, M and Ramesh, P and Balachandran, K, Asymptotic behavior of the fractional order three species prey--predator model, International Journal of Nonlinear Sciences and Numerical Simulation, De Gruyter, 19(7-8), 2018.

[14] Purnomo, Anna Silvia and Darti, Isnani and Suryanto, Agus and Kusumawinahyu, Wuryansari Muharini, Fear Effect on a Modified Leslie-Gower Predator-Prey Model with Disease Transmission in Prey Population., Engineering Letters, 31(2), 2023.

[15] Siva Pradeep, Manickasundaram and Nandha Gopal, Thangaraj and Yasotha, Arunachalam, Dynamics and Bifurcation Analysis of an Eco-Epidemiological Model in a Crowley--Martin Functional Response with the Impact of Fear, Engineering Proceedings, MDPI, 56(1), 2023.

[16] Ahmed, E and El-Sayed, AMA and El-Saka, Hala AA, Equilibrium points, stability and numerical solutions of fractional-order predator--prey and rabies models, Journal of Mathematical Analysis and Applications, Elsevier.325(1), 2007.

[17] Mukherjee, D and Mondal, R, Dynamical analysis of a fractional order prey-predator system with a reserved area, Journal of Fractional calculus and Applications, 11(1), 2020.

[18] Arumugam, Divya and Muthurathinam, Sivabalan and Anbulinga, Ashwin and Manickasundaram, Siva Pradeep, Impact of Fear on a Crowley--Martin Eco-Epidemiological Model with Prey Harvesting, Engineering Proceedings, MDPI, 56(1), 2023.

[19] Matignon, Denis, Stability results for fractional differential equations with applications to control processing, Computational engineering in systems applications, 2(1), 1996.

[20] Diethelm, Kai and Ford, Neville J, Analysis of fractional differential equations, Journal of Mathematical Analysis and Applications, Elsevier, 265(2), 2002.

[21] Garrappa, Roberto, hort tutorial: Solving fractional differential equations by Matlab codes, Department of Mathematics University of Bari, Italy, 2014.

[22] Caputo, Michele, Linear models of dissipation whose Q is almost frequency independent—II, Geophysical Journal International, Blackwell Publishing Ltd Oxford, UK, 13(5), 1967.

[23] Choi, Sung Kyu and Kang, Bowon and Koo, Namjip, Stability for Caputo fractional differential systems, Abstract and applied analysis, 2014.

[24] Das, Meghadri and Maiti, Alakes and Samanta, GP, Stability analysis of a prey-predator fractional order model incorporating prey refuge, Ecological Genetics and Genomics, Elsevier, 7, 2018.

[25] Kilbas, Anatouli Aleksandrovich and Srivastava, Hari M and Trujillo, Juan J, Theory and applications of fractional differential equations, Elsevier, 204, 2006.

[26] Majeed, Salam J and Naji, Raid Kamel and Thirthar, Ashraf Adnan, The dynamics of an Omnivore-predator-prey model with harvesting and two different nonlinear functional responses, AIP Conference Proceedings, 2096(1), 2019.

[27] Jawad, Shireen Rasool and Al Nuaimi, Marwa, Persistence and bifurcation analysis among four species interactions with the influence of competition, predation and harvesting, Iraqi Journal of Science, 2023.

[28] Al Nuaimi, Marwa and Jawad, Shireen, Modelling and stability analysis of the competitional ecological model with harvesting, Commun. Math. Biol. Neurosci., 2022.