New Operations for Linear Pentagonal Fuzzy Numbers

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Abstract: - Fuzzy predicates as pentagonal fuzzy variable increases the precision and considered to be more appropriate and effective translational fuzzy set model in characterizing decision variables. This paper aims at revisiting the theoretical developments on pentagonal fuzzy variable along with the existing operational properties. Also, the algebraic properties for linear symmetric pentagonal fuzzy number have been introduced with suitable illustrations. Further, the significance of proposed algebraic properties in the decision-making models has been discussed with numerical illustration.

Keywords*:* Pentagonal Fuzzy Number, Extension Principle, Interval Method2020 Mathematics Subject Classifcation: 03E72

1. Introduction

In the year 1962, Lotfi A. Zadeh argued that insufficiency that exists in conventional mathematics which specifies impreciseness by probability distributions [23]. Moreover, he first hinted the term mathematics of fuzzy in his article *"Circuit Theory to System Theory"* in the year 1962 where he advocated the necessity of new mathematics of fuzzy or cloudy quantities in describing biological systems, which are more complex than man-made systems [22]. Further in the year 1965, he proposed the idea of fuzzy subsets as an alternative to conventional crisp sets [23]. People use natural language to communicate their opinions, ideas, and decisions. Opinions are of two kinds: objective and subjective. Objective statements or propositions are very simple to quantify; whereas, subjective opinions, statements or propositions are hard to quantify. Fuzzy sets and its arithmetic is a widely used tool to represent such subjectivity in a larger sense. Human rational thinking reflects as a vague, imprecise natural language sentences or statements and these imprecise reflect human rational opinions with diverge meanings [4]. Quantifying the opinions that are subjective in nature involves large sensibility and is very much reliant on contextual nature. Fuzzy number is a tool to represent such impreciseness in a larger sense [6], [7],[8].

Translational fuzzy set models are defined over the real line as a real valued function that depicts the imprecise fuzzy predicates. Nahmias and Dubios et.al. synthesized the concept of fuzzy variable for the first time and argued that fuzzy variables, as the effective tool to characterize vagueness in a deeper sense [6], [7], [8], [17] [18]. Arnold Kaufmann and Madan M. Gupta elaborated the nuances in fuzzy numbers furthermore [2]. Translational fuzzy set model like triangular and trapezoidal be found to be linear and represent vagueness to certain extent. Trapezoidal and triangular fuzzy numbers have a kind of sharp increase and decrease on the sides of the membership function. Always there is a need for such generalized piecewise linear representation. It’s always challenging to represent ill defined variables using such functions. Generalizations have been made on the linear translational fuzzy set model and it has served the purpose of quantifying vague predicates in any real time application. Decision making theories are driven by experts’ opinion. Fuzzy numbers play a significant role in quantifying the expert’s opinion by fuzzifying the linguistic information. The process of doing so is called as Fuzzification. Fuzzy numbers are inevitable in a situation where impreciseness exists. Various arithmetic operations on fuzzy numbers have been developed in order to combine the imprecise information. Again, in decision making theories, aggregating two or more information is common and arithmetic operations are widely used to combine the opinions. Further, it is essential to study the importance of arithmetic operations and its functions in combining two or more linguistic information. In this research paper, various arithmetic operations have been introduced to the notion Pentagonal Fuzzy Number.

In 2015, Pathinathan, Ponnivalavan and Mike Dison conceptualized the idea of pentagonal fuzzy arithmetic with real time application [20]. The conceptual significance of Pentagonal Fuzzy Number as an alternative for the existing triangular and trapezoidal fuzzy numbers is to study the imprecise vague circumstances with the variations observed in α -level. Further they argued the importance and validation of Pentagonal Fuzzy Number by associating it to the vague fuzzy predicate "indeed *x*", whereas "about *x*" and "approximately between" are the vague fuzzy predicate largely associated with triangular and trapezoidal fuzzy numbers respectively. The vague term "indeed *x*" is rather difficult to characterize into a single linear type of fuzzy function, which eventually forces the curve that which observed to has fluctuations in α -level. [9]

The key objective of this research paper is to introduce various arithmetic operations based on Extension Principle and Interval Method for the pentagonal fuzzy number. Arithmetic operations such as addition, subtraction, multiplication, scalar multiplication and division are introduced based on the Extension Principle and α -cut empowered Interval Method. Further, the significance of each arithmetic operations has been emphasized withnumerical illustrations and insights into real time application.

2. Translational Pentagonal Fuzzy Variable

2.1 Fuzzy Variable [17], [18]

A Fuzzy variable $\underset{\sim}{A}$ or Fuzzy number defined in the real number line R is a Fuzzy set with the degree of belongingness $\mu_{\underset{\sim}{A}}(x)$ is continuous or piecewise continuous with atleast one $x_0 \in R$ such that $\mu_{\underset{\sim}{A}}(x_0) = 1$. Also $\underset{\sim}{A}$ is convex and normal and support of $\underset{\sim}{A}$ must be bounded.

2.2 Pentagonal Fuzzy Variable [19],[20],[21]

Generalized Pentagonal Fuzzy Number or Variable is defined as $\underset{\sim}{A}_P = (a,\ b,\ c,\ d,\ e : w)$ and w defined to be the maximum height of the pentagonal pentagonal membership function with the variations below 1. The pentagonal fuzzy membership function is given by,

$$\mu_{\underset{\sim}{A}_P}(x) = \begin{cases} 0 & x < a \\ \frac{1}{2}\frac{x-a}{b-a} & a \le x \le b \\ \frac{1}{2} + (w - \frac{1}{2})\frac{x-b}{c-b} & b \le x \le c \\ w & x = c \\ \frac{1}{2} + (w - \frac{1}{2})\frac{d-x}{d-c} & c \le x \le d \\ \frac{1}{2}\frac{e-x}{e-d} & d \le x \le e \\ 0 & x > e \end{cases}$$

with $a \le b \le c \le d \le e$ and $0 < w < 1$ which must be continuous in [0,1]. $\mu_{\underset{\sim}{A}_P}(x)$ is continuous and increasing from *a* to *c* and decreasing from *c* to *e*. The Generalised Pentagonal Fuzzy Variable turns to Normalised Pentagonal Fuzzy Variable if *w* is 1.

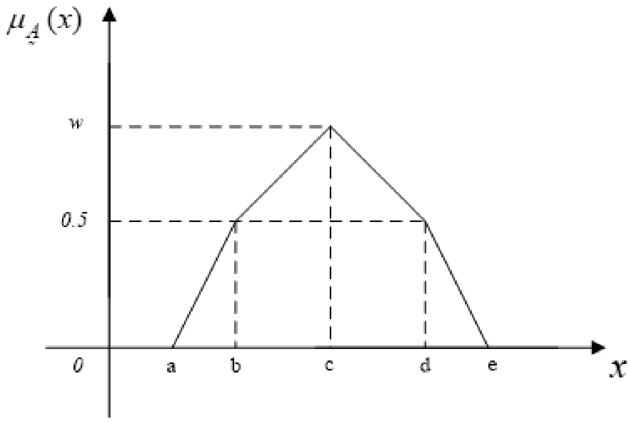


Figure 1: Pentagonal Fuzzy Number

3. Arithmetic Operations of Generalized Pentagonal Fuzzy Variable

Lee and Yuin (2014) defined pentagonal fuzzy set and the results of arithmetic operations [4]. T. Pathinathan and K. Ponnivalavan (2015) introduced the idea of reverse notion in triangular, trapezoidal and Pentagonal Fuzzy Number [20]. Apurba Panda and Madhumangal Pal (2015) studied the concept of pentagonal fuzzy variable and its generalisation along with the representation of pentagonal fuzzy matrices (PFMs) [2]. Abbasi introduced various operations on pseudopentagonal fuzzy numbers induced by transmission average [9]. Sankar Prasad Mondal and Manimohan Mandal summarized the formation of different types of Pentagonal Fuzzy Number [15]. Avinash Kamble discussed the notion of canonical Pentagonal Fuzzy Number and various arithmetic operations by means of α -cut [1]. Jesintha Rosline and Mike Dison (2018) introduced the notion of symmetric Pentagonal Fuzzy Number and quadratic Pentagonal Fuzzy Number with various arithmetic operations [11].

Let $\tilde{A}=(a_1,a_2,a_3,a_4,a_5:w_1)$ and $\tilde{B}=(b_1,b_2,b_3,b_4,b_5:w_2)$ be two generalized Pentagonal Fuzzy Numbers with the membership functions

$$\mu_A(x)=\max\left(\min\left(\frac{1}{2}\frac{x-a_1}{a_2-a_1},\frac{1}{2}+(w_1-\frac{1}{2})\frac{x-a_2}{a_3-a_2},w_1,\frac{1}{2}+(w_1-\frac{1}{2})\frac{a_4-x}{a_4-a_3},\frac{1}{2}\frac{a_5-x}{a_5-a_4}\right)\right)$$

$$\mu_B(y)=\max\left(\min\left(\frac{1}{2}\frac{y-b_1}{b_2-b_1},\frac{1}{2}+(w_2-\frac{1}{2})\frac{y-b_2}{b_3-b_2},w_2,\frac{1}{2}+(w_2-\frac{1}{2})\frac{b_4-y}{b_4-b_3},\frac{1}{2}\frac{b_5-y}{b_5-a_4}\right)\right)$$

and their parametric forms are

$$A_{1L}(\alpha)=\left[a_1+2\alpha(a_2-a_1),a_5-2\alpha(a_5-a_4)\right],\ \forall\alpha\in[0,w_1],0<w_1<0.5$$

$$A_{2R}(\alpha)=\left[a_2+\frac{2\alpha-1}{2w_1-1}(a_3-a_2),a_4-\frac{2\alpha-1}{2w_1-1}(a_4-a_3)\right]\forall\alpha\in[0.5,w_1],0.5<w_1<1$$

$$B_{1L}(\alpha)=\left[b_1+2\alpha(b_2-b_1),b_5-2\alpha(b_5-b_4)\right],\ \forall\alpha\in[0,w_2],0<w_2<0.5$$

$$B_{2R}(\alpha)=\left[b_2+\frac{2\alpha-1}{2w_1-1}(b_3-b_2),b_4-\frac{2\alpha-1}{2w_1-1}(b_4-b_3)\right]\forall\alpha\in[0.5,w_2],0.5<w_2<1$$

where, $A_{1L}(\alpha),A_{2R}(\alpha)$ and $B_{1L}(\alpha),B_{2R}(\alpha)$ are increasing and decreasing functions in α - respectively.
Throughout this paper, the operations defned using Extension Principle and Interval Method have been verifed by the set of pentagonal fuzzy variables $\tilde{A}=(0.1,0.3,0.5,0.7,0.9)$ and $\tilde{B}=(0.2,0.4,0.6,0.8,1.0)$.

3.1 Addition of two Generalized Pentagonal Fuzzy Variables by Extension Principle

The Addition of two Generalized Pentagonal Fuzzy Variables by Extension Principle is given as follows:

Let $\tilde{A}+\tilde{B}=\tilde{C}$ where $\mu_{\tilde{c}}(z)=\sup\left(\min(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(y):x+y=z\right)$

Let $w=\min(w_1,w_2)$

$$\mu_{\underset{\sim}{C}}(z)=\begin{cases}\sup(\min(\frac{1}{2}\frac{x-a_1}{a_2-a_1},\frac{1}{2}\frac{y-b_1}{b_2-b_1})) & a_1\le x\le a_2, b_1\le y\le b_2\\ \sup(\min(\frac{1}{2}+(w_1-\frac{1}{2})\frac{x-a_2}{a_3-a_2},\frac{1}{2}+(w_2-\frac{1}{2})\frac{y-b_2}{b_3-b_2})) & a_2\le x\le a_3, b_2\le y\le b_3\\ \sup(\min(w_1,w_2):x+y=z) & x=a_3, y=b_3\\ \sup(\min(\frac{1}{2}+(w_1-\frac{1}{2})\frac{a_4-x}{a_4-a_3},\frac{1}{2}+(w_2-\frac{1}{2})\frac{b_4-y}{b_4-b_3})) & a_3\le x\le a_4, b_3\le y\le b_4\\ \sup(\min(\frac{1}{2}\frac{a_5-x}{a_5-a_4},\frac{1}{2}\frac{b_5-y}{b_5-b_4})) & a_4\le x\le a_5, b_4\le y\le b_5\\ 0 & otherwise\end{cases}$$

$$\mu_{\underset{\sim}{C}}(z)=\begin{cases}\frac{1}{2}\frac{z-a_1-b_1}{a_2+b_2-a_1-b_1} & a_1+b_1\le z\le a_2+b_2\\ \frac{1}{2}+(w-\frac{1}{2})\frac{z-a_2-b_2}{a_3+b_3-a_2-b_2} & a_2+b_2\le z\le a_3+b_3\\ w & z=a_3+b_3\\ \frac{1}{2}+(w-\frac{1}{2})\frac{a_4+b_4-z}{a_4+b_4-a_3-b_3} & a_3+b_3\le z\le a_4+b_4\\ \frac{1}{2}\frac{a_5+b_5-z}{a_5+b_5-a_4-b_4} & a_4+b_4\le z\le a_5+b_5\\ 0 & otherwise\end{cases}$$

Therefore, the addition Addition of two Generalised Pentagonal Fuzzy Variables is another Generalised Pentagonal Fuzzy Variable with the above membership function.

Example

The sum $\underset{\sim}{A}+\underset{\sim}{B}$ with *w=0.8*, using Extension Principle is found to be

$$\mu_{\underset{\sim}{C}}(z)=\begin{cases}\frac{1}{2}\frac{z-0.3}{0.4} & 0.3\le z\le 0.7\\ \frac{1}{2}+0.3\frac{z-0.7}{0.4} & 0.7\le z\le 1.1\\ 0.8 & z=1.1\\ \frac{1}{2}+0.3\frac{1.5-z}{0.4} & 1.1\le z\le 1.5\\ \frac{1}{2}\frac{1.9-z}{0.4} & 1.5\le z\le 1.9\\ 0 & otherwise\end{cases}$$

which is the membership function for the sum of two Pentagonal Fuzzy Variables.

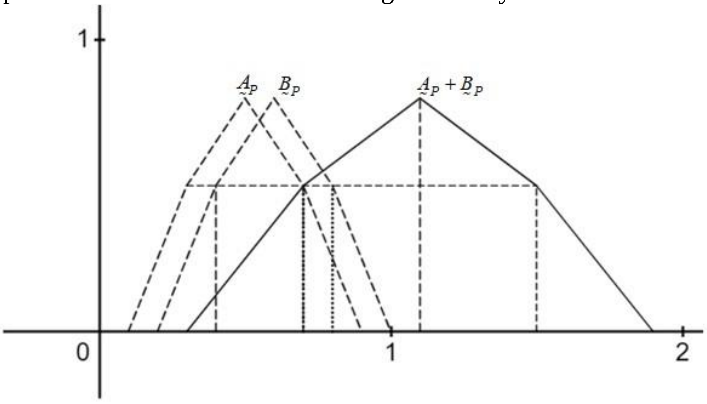


Figure 2: Addition of two Pentagonal Fuzzy Variables using Extension Principle

3.2 Addition of two Generalized Pentagonal Fuzzy Variables based on Interval Method

Given $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the two Generalized Pentagonal Fuzzy Variables and their addition based on Interval Method is obtained as follows:

$$\underset{\sim}{C}_\alpha = \underset{\sim}{A}_\alpha + \underset{\sim}{B}_\alpha \ \ \forall \alpha \in [0,w], 0<w<1, w=\min(w_1,w_2)$$

$$\underset{\sim}{C}_\alpha = \begin{cases} x: a_1+b_1+2\alpha(a_2+b_2-a_1-b_1) \le x \le a_5+b_5-2\alpha(a_5+b_5-a_4-b_4), 0\le\alpha\le 0.5 \\ x: a_2+b_2+\frac{2\alpha-1}{2w-1}(a_3+b_3-a_2-b_2) \le x \le a_4+b_4-\frac{2\alpha-1}{2w-1}(a_4+b_4-a_3-b_3), 0.5\le\alpha\le 1 \end{cases}$$

3.3 Subtraction of two Generalized Pentagonal Fuzzy Variables by Extension Principle

Given $\underset{\sim}{A}, \underset{\sim}{B}$ be the two generalized Pentagonal Fuzzy Variables and their subtraction by Extension Principle is obtained as follows:

$$\mu_{\underset{\sim}{C}}(z) = \begin{cases} \sup(\min(\frac{1}{2}\frac{x-a_1}{a_2-a_1}, \frac{1}{2}\frac{b_5-y}{b_5-b_4})) & a_1 \le x \le a_2, b_4 \le y \le b_5 \\ \sup(\min(\frac{1}{2}+(w_1-\frac{1}{2})\frac{x-a_2}{a_3-a_2}, \frac{1}{2}+(w_2-\frac{1}{2})\frac{b_4-y}{b_4-b_3})) & a_2 \le x \le a_3, b_3 \le y \le b_4 \\ \sup(\min(w_1,w_2): x-y=z) & x=a_3, y=b_3 \\ \sup(\min(\frac{1}{2}+(w_1-\frac{1}{2})\frac{a_4-x}{a_4-a_3}, \frac{1}{2}+(w_2-\frac{1}{2})\frac{b_2-y}{b_3-b_2})) & a_3 \le x \le a_4, b_2 \le y \le b_3 \\ \sup(\min(\frac{1}{2}\frac{a_5-x}{a_5-a_4}, \frac{1}{2}\frac{b_1-y}{b_2-b_1})) & a_4 \le x \le a_5, b_1 \le y \le b_2 \\ 0 & otherwise \end{cases}$$

$$\mu_{\underset{\sim}{C}}(z) = \begin{cases} \frac{1}{2}\frac{z-a_1+b_5}{a_2+b_5-a_1-b_4} & a_1-b_5 \le z \le a_2-b_4 \\ \frac{1}{2}+(w-\frac{1}{2})\frac{z-a_2+b_4}{a_3-b_3-a_2+b_2} & a_2-b_4 \le z \le a_3-b_3 \\ w & z=a_3-b_3 \\ \frac{1}{2}+(w-\frac{1}{2})\frac{a_4-b_2-z}{a_4-b_2-a_3+b_3} & a_3-b_3 \le z \le a_4-b_4 \\ \frac{1}{2}\frac{a_5-b_1-z}{a_5-b_1-a_4+b_2} & a_4-b_2 \le z \le a_5-b_1 \\ 0 & otherswise \end{cases}$$

Example

The Pentagonal Fuzzy Variable $\underset{\sim}{A}-\underset{\sim}{B}$ for *w=0.8* obtained by Extension Principle is found to be

$$\mu_{\underset{\sim}{C}}(z) = \begin{cases} \frac{1}{2}\frac{z+0.9}{0.4} & -0.9 \le z \le -0.5 \\ \frac{1}{2}+0.3\frac{z+0.5}{0.4} & -0.5 \le z \le -0.1 \\ 0.8 & z=-0.1 \\ \frac{1}{2}+0.3\frac{0.3-z}{0.4} & -0.1 \le z \le 0.3 \\ \frac{1}{2}\frac{0.7-z}{0.4} & 0.3 \le z \le 0.7 \\ 0 & otherwise \end{cases}$$

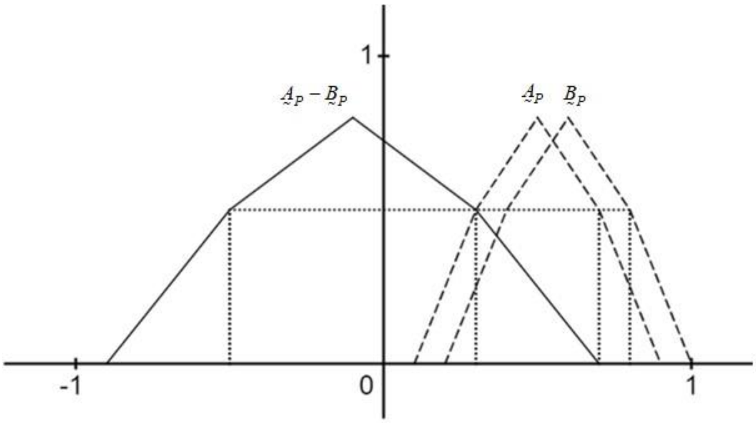


Figure 3: Subtraction of two Pentagonal Fuzzy Variables using Extension Principle

3.4 Subtraction of two Generalized Pentagonal Fuzzy Variables based on Interval Method

Given $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the two Generalized Pentagonal Fuzzy Variables and their subtraction by Interval Method is obtained as follows:

$$\underset{\sim}{C}_\alpha = \begin{cases} x : a_1 - b_5 + 2\alpha(a_2 - a_1 + b_5 - b_4) \le x \le a_2 - b_4 + \dfrac{2\alpha - 1}{2w-1}(a_3 - a_2 + b_4 - b_3), \; 0 \le \alpha \le 0.5 \\ x : a_4 - b_2 - \dfrac{2\alpha - 1}{2w-1}(a_4 - a_3 + b_3 - b_2) \le x \le a_5 - b_1 - 2\alpha(a_5 - a_4 + b_2 - b_1) - 2\alpha(a_5 - a_4 + b_2 - b_1), 0.5 \le \alpha \le 1 \end{cases}$$

3.5 Scalar multiplication of a Generalized Pentagonal Fuzzy Variable based on Extension Principle method

Given $\underset{\sim}{A}$ be the generalized Pentagonal Fuzzy Variable and its scalar multiplication based on Extension Principle is obtained as follows:

Case i: $k \ge 0$

$$\mu_{\underset{\sim}{c}}(z) = \begin{cases} \sup(\min(\frac{1}{2}\frac{x - a_1}{a_2 - a_1}) : kx = z) & a_1 \le x \le a_2 \\ \sup(\min(\frac{1}{2} + (w_1 - \frac{1}{2})\frac{x - a_2}{a_3 - a_2}) : kx = z) & a_2 \le x \le a_3 \\ \sup(\min(w) : kx = z) & x = a_3 \\ \sup(\min(\frac{1}{2} + (w_1 - \frac{1}{2})\frac{a_4 - x}{a_4 - a_3}) : kx = z) & a_3 \le x \le a_4 \\ \sup(\min(\frac{1}{2}\frac{a_5 - x}{a_5 - a_4}) : kx = z) & a_4 \le x \le a_5 \\ 0 & otherwise \end{cases}$$

$$\mu_{\underset{\sim}{c}}(z) = \begin{cases} \frac{1}{2}\frac{z - ka_1}{k(a_2 - a_1)} & ka_1 \le z \le ka_2 \\ \frac{1}{2} + (w_1 - \frac{1}{2})\frac{z - ka_2}{k(a_3 - a_2)} & ka_2 \le z \le ka_3 \\ w & z = ka_3 \\ \frac{1}{2} + (w_1 - \frac{1}{2})\frac{ka_4 - z}{k(a_4 - a_3)} & ka_3 \le z \le ka_4 \\ \frac{1}{2}\frac{ka_5 - z}{k(a_5 - a_4)} & ka_4 \le z \le ka_5 \\ 0 & otherwise \end{cases}$$

Case ii: $k < 0$

$$\mu_{\varsigma}(z)=\begin{cases}\sup(\min(\frac{1}{2}\frac{a_1-x}{a_2-a_1}):kx=z) & a_1\le x\le a_2\\ \sup(\min(\frac{1}{2}+(w_1-\frac{1}{2})\frac{a_2-x}{a_3-a_2}):kx=z) & a_2\le x\le a_3\\ \sup(\min(w):kx=z) & x=a_3\\ \sup(\min(\frac{1}{2}+(w_1-\frac{1}{2})\frac{x-a_4}{a_4-a_3}):kx=z) & a_3\le x\le a_4\\ \sup(\min(\frac{1}{2}\frac{x-a_5}{a_5-a_4}):kx=z) & a_4\le x\le a_5\\ 0 & otherwise\end{cases}$$

$$\mu_{\varsigma}(z)=\begin{cases}\frac{1}{2}\frac{ka_1-z}{k(a_2-a_1)} & ka_1\le z\le ka_2\\ \frac{1}{2}+(w_1-\frac{1}{2})\frac{ka_2-z}{k(a_3-a_2)} & ka_2\le z\le ka_3\\ w & z=ka_3\\ \frac{1}{2}+(w_1-\frac{1}{2})\frac{z-ka_4}{k(a_4-a_3)} & ka_3\le z\le ka_4\\ \frac{1}{2}\frac{z-ka_5}{k(a_5-a_4)} & ka_4\le z\le ka_5\\ 0 & otherwise\end{cases}$$

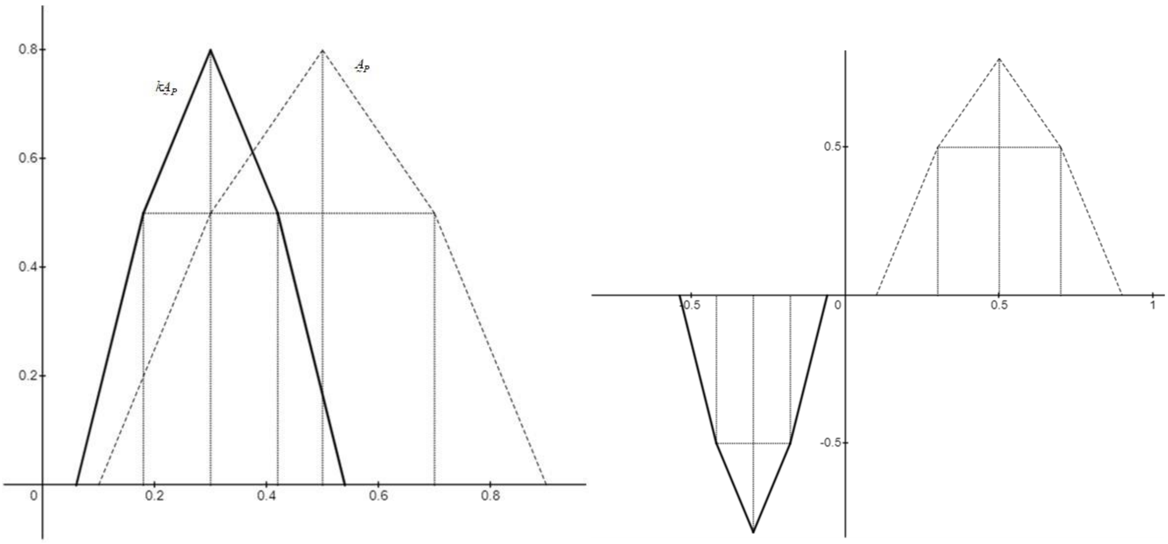
Example

Case i: $k=0.6$

$$\mu_{\varsigma}(z)=\begin{cases}\frac{1}{2}\frac{z-0.06}{0.12} & 0.06\le z\le 0.18\\ \frac{1}{2}+0.3\frac{z-0.18}{0.12} & 0.18\le z\le 0.3\\ 0.8 & z=0.3\\ \frac{1}{2}+0.3\frac{0.42-z}{0.12} & 0.3\le z\le 0.42\\ \frac{1}{2}\frac{0.54-z}{0.12} & 0.42\le z\le 0.54\\ 0 & otherwise\end{cases}$$

Case ii: $k=-0.6$

$$\mu_{\varsigma}(z)=\begin{cases}\frac{1}{2}\frac{z+0.06}{0.12} & -0.18\le z\le -0.06\\ -\frac{1}{2}+0.3\frac{z+0.18}{0.12} & -0.3\le z\le -0.18\\ 0.8 & z=-0.3\\ -\frac{1}{2}-0.3\frac{0.42+z}{0.12} & -0.42\le z\le -0.3\\ -\frac{1}{2}\frac{0.54+z}{0.12} & -0.54\le z\le -0.42\\ 0 & otherwise\end{cases}$$



(i) $k \geq 0$ (ii) $k < 0$

Figure 4: Scalar product of Pentagonal Fuzzy Variable using Extension Principle

3.6 Scalar multiplication of a generalized Pentagonal Fuzzy Variable by Interval Method

Let $\underset{\sim}{A}$ be the generalized Pentagonal Fuzzy Variable and the scalar multiplication based on Interval Method is obtained as follows:

Case i: $k \geq 0$

$$\underset{\sim}{C}_\alpha = \begin{cases} x : ka_1 + k\alpha(a_2 - a_1) \leq x \leq ka_5 - 2\alpha k(a_5 - a_4), & 0 \leq \alpha \leq 0.5 \\ x : ka_2 + \dfrac{2\alpha - 1}{2w - 1}k(a_3 - a_2) \leq x \leq ka_4 - \dfrac{2\alpha - 1}{2w - 1}k(a_4 - a_3), & 0.5 \leq \alpha \leq 1 \end{cases}$$

Case ii: $k < 0$

$$\underset{\sim}{C}_\alpha = \begin{cases} x : ka_5 - 2k\alpha(a_5 - a_4) \leq x \leq ka_1 + \alpha k(a_2 - a_1), & 0 \leq \alpha \leq 0.5 \\ x : ka_2 + \dfrac{2\alpha - 1}{2w - 1}k(a_3 - a_2) \leq x \leq ka_4 - \dfrac{2\alpha - 1}{2w - 1}k(a_4 - a_3), & 0.5 \leq \alpha \leq 1 \end{cases}$$

3.7 Multiplication of two Generalised Pentagonal Fuzzy Variables by Extension Principle method

Given $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the two Generalized Pentagonal Fuzzy Variables and their multiplication by Extension Principle is obtained as follows:

$$\mu_{\underset{\sim}{c}}(z) = \begin{cases} \sup(\min(\frac{1}{2}\frac{x - a_1}{a_2 - a_1}, \frac{1}{2}\frac{y - b_1}{b_2 - b_1})) & a_1 \leq x \leq a_2, b_1 \leq y \leq b_2 \\ \sup(\min(\frac{1}{2} + (w_1 - \frac{1}{2})\frac{x - a_2}{a_3 - a_2}, \frac{1}{2} + (w_2 - \frac{1}{2})\frac{y - b_2}{b_3 - b_2})) & a_2 \leq x \leq a_3, b_2 \leq y \leq b_3 \\ \sup(\min(w_1, w_2) : xy = z) & x = a_3, y = b_3 \\ \sup(\min(\frac{1}{2} + (w_1 - \frac{1}{2})\frac{a_4 - x}{a_4 - a_3}, \frac{1}{2} + (w_2 - \frac{1}{2})\frac{b_4 - y}{b_4 - b_3})) & a_3 \leq x \leq a_4, b_3 \leq y \leq b_4 \\ \sup(\min(\frac{1}{2}\frac{a_5 - x}{a_5 - a_4}, \frac{1}{2}\frac{b_5 - y}{b_5 - b_4})) & a_4 \leq x \leq a_5, b_4 \leq y \leq b_5 \\ 0 & otherwise \end{cases}$$

$$\mu_{\underset{\sim}{C}}(z)=\begin{cases} \sup(\min(\frac{1}{2}\frac{x-a_1}{a_2-a_1},\frac{1}{2}\frac{\frac{z}{x}-b_1}{b_2-b_1})) & a_1\le x\le a_2, a_1b_1\le z\le a_2b_2 \\ \sup(\min(\frac{1}{2}+(w_1-\frac{1}{2})\frac{x-a_2}{a_3-a_2},\frac{1}{2}+(w_2-\frac{1}{2})\frac{\frac{z}{x}-b_2}{b_3-b_2})) & a_2\le x\le a_3, a_2b_2\le z\le a_3b_3 \\ \sup(\min(w_1,w_2)) & x=a_3, z=a_3b_3 \\ \sup(\min(\frac{1}{2}+(w_1-\frac{1}{2})\frac{a_4-x}{a_4-a_3},\frac{1}{2}+(w_2-\frac{1}{2})\frac{b_4-\frac{z}{x}}{b_4-b_3})) & a_3\le x\le a_4, a_3b_3\le z\le a_4b_4 \\ \sup(\min(\frac{1}{2}\frac{a_5-x}{a_5-a_4},\frac{1}{2}\frac{b_5-\frac{z}{x}}{b_5-b_4})) & a_4\le x\le a_5, a_4b_4\le z\le a_5b_5 \\ 0 & otherwise \end{cases}$$

$$\mu_{\underset{\sim}{C}}(z)=\begin{cases} \frac{-B_1+\sqrt{B_1^{\,2}-4A_1(C_1-z)}}{2A_1} & a_1b_1\le z\le a_2b_2 \\ \frac{1}{2}+(w-\frac{1}{2})\frac{-B_2+\sqrt{B_2^{\,2}-4A_2(C_2-z)}}{2A_2} & a_2b_2\le z\le a_3b_3 \\ w & z=a_3b_3 \\ \frac{1}{2}+(w-\frac{1}{2})\frac{-B_3+\sqrt{B_3^{\,2}-4A_3(C_3-z)}}{2A_3} & a_3b_3\le z\le a_4b_4 \\ \frac{-B_4+\sqrt{B_4^{\,2}-4A_4(C_4-z)}}{2A_4} & a_4b_4\le z\le a_5b_5 \\ 0 & otherwise \end{cases}$$

where,

$A_1=4(a_2-a_1)(b_2-b_1), B_1=2[a_1(b_2-b_1)+b_1(a_2-a_1)], C_1=a_1b_1$

$A_2=(a_3-a_2)(b_3-b_2), B_2=[a_2(b_3-b_2)+b_2(a_3-a_2)], C_2=a_2b_2$

$A_3=(a_4-a_3)(b_4-b_3), B_3=[a_4(b_4-b_3)+b_4(a_4-a_3], C_3=a_4b_4$

$A_4=4(a_5-a_4)(b_5-b_4), B_4=2[a_5(b_5-b_4)+b_5(a_5-a_4)], C_4=a_5b_5$

3.8 Multiplication of two Generalised Pentagonal Fuzzy Variables by Interval Method

Given $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the two Generalized Pentagonal Fuzzy Variables and their multiplication based on Interval is obtained as follows

$$\underset{\sim}{C}_\alpha=\begin{cases} x: a_1b_1+2\frac{2\alpha-1}{2w-1}[a_1(b_2-b_1)+b_1(a_2-a_1)]+4\left(\frac{2\alpha-1}{2w-1}\right)^2(a_2-a_1)(b_2-b_1)\le x \\ \quad \le a_5b_5+2\frac{2\alpha-1}{2w-1}[a_5(b_5-b_4)+b_5(a_5-a_4)]+4\left(\frac{2\alpha-1}{2w-1}\right)^2(a_5-a_4)(b_5-b_4),\ 0\le\alpha\le 0.5 \\ x: a_4b_4+2\frac{2\alpha-1}{2w-1}[a_4(b_4-b_3)+b_4(a_4-a_3]+4\left(\frac{2\alpha-1}{2w-1}\right)^2(a_4-a_3)(b_4-b_3)\le x \\ \quad \le a_2b_2+2\frac{2\alpha-1}{2w-1}[a_2(b_3-b_2)+b_2(a_3-a_2)]+4\left(\frac{2\alpha-1}{2w-1}\right)^2(a_3-a_2)(b_3-b_2), 0.5\le\alpha\le 1 \end{cases}$$

3.9 Division of two Generalised Pentagonal Fuzzy Variables by ExtensionPrinciple method

Given $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the two Generalized Pentagonal Fuzzy Variables and their division based on Extension Principle method is defined as follows:

Let $\underset{\sim}{A}:\underset{\sim}{B}=\underset{\sim}{C}$ where $\mu_{\underset{\sim}{C}}(z)=\sup\left(\min(\mu_{\underset{\sim}{A}}(x),\mu_{\underset{\sim}{B}}(y)):\frac{x}{y}=z\right)$

$$\mu_{\underset{\sim}{c}}(z)=\begin{cases}\sup(\min(\frac{1}{2}\frac{x-a_1}{a_2-a_1},\frac{1}{2}\frac{b_5-y}{b_5-b_4})) & a_1\le x\le a_2, b_4\le y\le b_5\\ \sup(\min(\frac{1}{2}+(w_1-\frac{1}{2})\frac{x-a_2}{a_3-a_2},\frac{1}{2}+(w_2-\frac{1}{2})\frac{b_4-y}{b_4-b_3})) & a_2\le x\le a_3, b_3\le y\le b_4\\ \sup(\min(w_1,w_2)) & x=a_3, y=b_3\\ \sup(\min(\frac{1}{2}+(w_1-\frac{1}{2})\frac{a_4-x}{a_4-a_3},\frac{1}{2}+(w_2-\frac{1}{2})\frac{y-b_2}{b_3-b_2})) & a_3\le x\le a_4, b_2\le y\le b_3\\ \sup(\min(\frac{1}{2}\frac{a_5-x}{a_5-a_4},\frac{1}{2}\frac{y-b_1}{b_2-b_1})) & a_4\le x\le a_5, b_1\le y\le b_2\\ 0 & otherwise\end{cases}$$

$$\mu_{\underset{\sim}{c}}(z)=\begin{cases}\frac{1}{2}\frac{zb_5-a_1}{(a_2-a_1)-z(b_4-b_5)} & a_1/b_5\le z\le a_2/b_4\\ \frac{1}{2}+(w-\frac{1}{2})\frac{zb_4-a_1}{(a_4-a_3)+z(b_4-b_3)} & a_2/b_4\le z\le a_3/b_3\\ w & z=a_3/b_3\\ \frac{1}{2}+(w-\frac{1}{2})\frac{a_4-zb_2-z}{a_4-b_2-a_3+b_3} & a_3/b_3\le z\le a_4/b_4\\ \frac{1}{2}\frac{a_5-zb_1-z}{a_5-b_1-a_4+b_2} & a_4/b_2\le z\le a_5/b_1\\ 0 & otherswise\end{cases}$$

Example

$\underset{\sim}{A}:\underset{\sim}{B}$ for *w=0.8* by extension principle is found to be

$$\mu_{\underset{\sim}{c}}(z)=\begin{cases}\frac{1}{2}\frac{z-0.1}{0.2(1+z)} & 0.1\le z\le 0.375\\ \frac{1}{2}+0.3\frac{0.8z-0.3}{0.2(1+z)} & 0.375\le z\le 0.83\\ 0.8 & z=0.83\\ \frac{1}{2}+0.3\frac{0.7-0.4z}{0.2(1+z)} & 0.83\le z\le 1.75\\ \frac{1}{2}\frac{0.9-0.2z}{0.2(1+z)} & 1.75\le z\le 4.5\\ 0 & otherwise\end{cases}$$

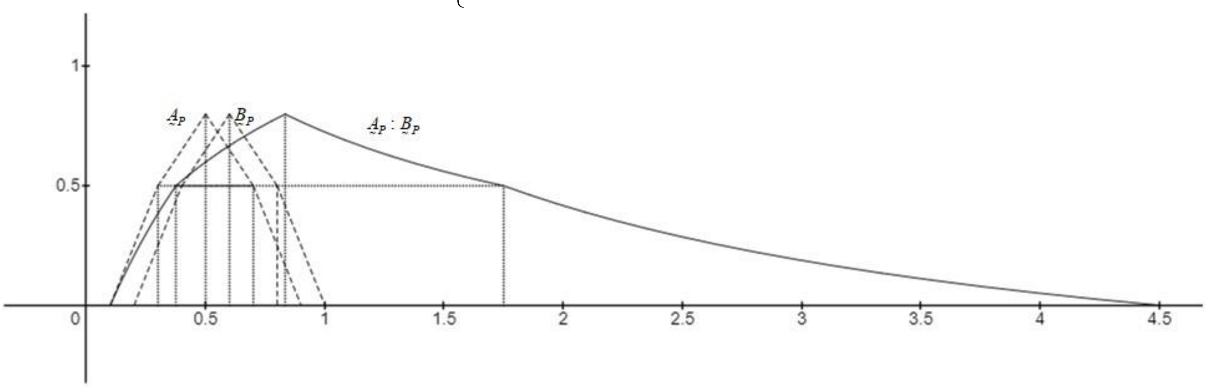


Figure 5: Division of two Pentagonal Fuzzy Varaibles using Extension Principle

3.10 Division of two generalised Pentagonal Fuzzy Variables by Interval Method

Given two Generalized Pentagonal Fuzzy Variables and their division based on Interval Method is defined as follows:

$\underset{\sim}{A}_\alpha : \underset{\sim}{B}_\alpha = \underset{\sim}{C}_\alpha$ is given by

$$\underset{\sim}{C}_\alpha = \begin{cases} x : \dfrac{a_1 + 2\alpha(a_2 - a_1)}{b_1 + 2\alpha(b_2 - b_1)} \le x \le \dfrac{a_5 - 2\alpha(a_5 - a_4)}{b_5 - 2\alpha(b_5 - b_4)}, \ 0 \le \alpha \le 0.5 \\ x : \dfrac{a_4 - \dfrac{2\alpha - 1}{2w_1 - 1}(a_4 - a_3)}{b_4 - \dfrac{2\alpha - 1}{2w_2 - 1}(b_4 - b_3)} \le x \le \dfrac{a_2 + \dfrac{2\alpha - 1}{2w_1 - 1}(a_3 - a_2)}{b_4 + \dfrac{2\alpha - 1}{2w_2 - 1}(b_3 - b_2)}, 0.5 \le \alpha \le 1 \end{cases}$$

4. Conclusion

In this Research Paper, various arithmetic operations such as Addition, Subtraction, Multiplication, Scalar Multiplication and Division have been derived for Generalized Pentagonal Fuzzy Variables based on Extension Principle and Interval Method. They have been discussed and verified with suitable illustrations. Further, the significant importance of α-cut on conceptualizing various operations has been discussed with detailed investigations. The knowledge shared will drive researchers from various background on implementing the derived arithmetic operations in their respective domains.

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