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Generalized (σ,τ)-Revrse Derivations in non ideal on Prime Rings

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Abstract: Let R be a prime ring, I be a non-zero ideal on R, and σ,τ be a automorphisms of R. Suppose that F is a generalized (σ,τ)-reverse derivation on R associated with (σ,τ)-reverse derivation d on R respectively $\sigma(\mathrm{I})\neq 0$ and $\tau(\mathrm{I})\neq 0$. In this paper, we studied the following identities in prime rings: (i) $F(xy)+d(x)F(y)=0$; (ii) $F(xy)+d(x)F(y)+\sigma(xy)=0$; (iii) $F(xy)+d(x)F(y)+\sigma(yx)=0$; (iv) $F(xy)+d(x)F(y)+\sigma(xoy)=0$; (v) $F(xy)+d(y)F(x)=0$; (vi) $F(xy)+d(y)F(x)+\sigma(xy)=0$; (vii) $F(xy)+d(y)F(x)+\sigma(yx)=0$; (viii) $F(xy)+d(y)F(x)+\sigma(xoy)=0$; (ix) $F(xy)+F(x)F(y)=0$; (x) $F(xy)+F(y)F(x)=0$; for all x,y in some suitable sub sets of R.

Keywords: *Prime ring, Derivation, Reverse derivation, Generalized derivation, (σ,τ)-derivation, Generalized (σ,τ)-derivation, (σ,τ)-reverse derivation, Generalized (σ,τ)-reverse derivation.*

Introduction:

In 1994, Yenigul and Argac in [8], obtained the some result for α derivation on prime rings. In 1999, Ashraf , Nadeem and Quadri in [3], extended the result for (θ, φ) derivation in pime and semiprime rings. Further Chirag Garg et al. in [5] studied on generalized (α,β)-derivations in prime rings. The notion of reverse derivation has been introduced by Bresar and Vukman in [4] and the reverse derivations on semi prime rings have been studied by Samman and Alyamani in [7]. Aboubakr and Gonzalez in [1] studied the relationship between generalized reverse derivation and generalized derivation on an ideal in semi prime rings, and C. Jaya subbareddy et.al in [6] is proved that in case R is a prime ring with a non-zero right reverse derivation d and U is the left ideal of R then R is commutative. In 2011, the concepts of (θ, φ)-reverse derivation, and generalized (θ, φ)-reverse derivation has been introduced by Anwar Khaleel Faraj in [2]. In this paper, we inspire of Chirag Garg et al. in [5], we proved some results on generalized (σ,τ)-reverse derivations in prime rings.

Preliminaries: Throughout this paper R denote an associative ring with center Z. Recall that a ring R is prime if $xRy=\{0\}$ implies $x=0$ or $y=0$. For any $x,y\in R$, the symbol $[x,y]$ stands for the commutator $xy-yx$ and the symbol (xoy) denotes the anticommutator $xy+yx$. An additive mapping $d:R\rightarrow R$ is called a derivation if $d(xy)=d(x)y+xd(y)$, for all $x,y\in R$. An additive mapping $d:R\rightarrow R$ is called a reverse derivation if $d(xy)=d(y)x+yd(x)$, for all $x,y\in R$. An additive mapping $d:R\rightarrow R$ is called a (σ,τ)-derivation if $d(xy)=d(x)\sigma(y)+\tau(x)d(y)$, for all $x,y\in R$. An additive mapping

$d: R \to R$ is called a (σ, τ)-reverse derivation if $d(xy) = d(y)\sigma(x) + \tau(y)d(x)$, for all $x, y \in R$. An additive mapping $F: R \to R$ is called a generalized derivation, if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. An additive mapping $F: R \to R$ is called a generalized reverse derivation, if there exists a reverse derivation $d: R \to R$ such that $F(xy) = F(y)x + yd(x)$, for all $x, y \in R$. An additive mapping $F: R \to R$ is said to be a generalized (σ, τ)-derivation of R, if there exists a (σ, τ)-derivation $d: R \to R$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$, for all $x, y \in R$. An additive mapping $F: R \to R$ is said to be a generalized (σ, τ)-reverse derivation of R, if there exists a (σ, τ)-reverse derivation $d: R \to R$ such that $F(xy) = F(y)\sigma(x) + \tau(y)d(x)$, for all $x, y \in R$, where σ and τ be a automorphisms of R.

Throughout this paper, we shall make use of the basic commutator identities:

$[x, yz] = y[x, z] + [x, y]z$; $[xy, z] = [x, z]y + x[y, z]$.

Lemma 1: [3, Lemma 2] Let R be a 2-torsion free prime ring and U be a non-zero square-closed Lie ideal of R. If $[\alpha(x), \beta(y)] = 0$, for all $x, y \in U$, where α, β are automorphisms on R, then $U \subseteq Z$.

Lemma 2: Let R be a prime ring and I a nonzero lie ideal of R. If d is a non zero (σ, τ)- reverse derivation of R such that $d(I) = O$, then $I \subseteq Z$.

Proof: We have $d(u) = 0$, for all $u \in I$. (1)

We replacing u by $[u, r]$ in equation (1), we get

$d([u, r]) = 0$

$d(ur - ru) = 0$

$d(ur) - d(ru) = 0$

$d(r)\sigma(u) + \tau(r)d(u) - d(u)\sigma(r) - \tau(u)d(r) = 0$, for all $u \in I, r \in R$.

Using equation (1) in the above equation, we get

$d(r)\sigma(u) - \tau(u)d(r) = 0$, for all $u \in I, r \in R$. (2)

We replacing r by rv in the above equation, we get

$d(rv)\sigma(u) - \tau(u)d(rv) = 0$

$d(v)\sigma(r)\sigma(u) + \tau(v)d(r)\sigma(u) - \tau(u)d(v)\sigma(r) - \tau(u)\tau(v)d(r) = 0$, for all $u, v \in I, r \in R$.

Using equation (i) in the above equation, we get

$\tau(v)d(r)\sigma(u) - \tau(u)\tau(v)d(r) = 0$, for all $u, v \in I, r \in R$. (3)

Left multiplying equation (2) by $\tau(v)$, we get

$\tau(v)d(r)\sigma(u) - \tau(v)\tau(u)d(r) = 0$, for all $u, v \in I, r \in R$. (4)

We subtracting equation (4) from equation (3), we get

$\tau[u, v]d(r) = 0$, for all for all $u, v \in I, r \in R$. (5)

We replacing v by sv, $s \in R$ in equation (5), we get

$\tau[u, sv]d(r) = 0$

$\tau(s)\tau[u,v]d(r) + \tau[u,s]\tau(v)d(r) = 0$

Using equation (5) in the above equation, we get

$\tau[u,s]\tau(v)d(r) = 0$, for all $u, v \in I, r, s \in R$.

We replacing v by tv, $t \in R$ in the above equation, we get

$\tau[u,s]\tau(tv)d(r) = 0$, for all $u, v \in I, r, s, t \in R$.

$\tau[u,s]R\tau(v)d(r) = 0$, for all $u, v \in I$, $s \in R$. Since R is a prime ring and I is a nonzero lie ideal of R, we get either $\tau[u,s] = 0$ or $d(r) = 0$. If $d(r) = 0$, is contradiction to our assumption. So we get $[u,s] = 0$, for all $u \in I$, $s \in R$. Then $I \subseteq Z$.

Theorem 1: Let R be a prime ring and I be a non-zero ideal on R. Suppose that F is a generalized (σ,τ)-reverse derivation on R associated with (σ,τ)-reverse derivation d on R respectively, $\tau(I) \neq 0$ and $\sigma(I) \neq 0$. If $F(xy) + d(x)F(y) = 0$, for all $x, y \in I$, then $I \subseteq Z$.

Proof: We have $F(xy) + d(x)F(y) = 0$, for all $x, y \in I$. (6)

We replacing y by xy in equation (6), we obtain

$F(xxy) + d(x)F(xy) = 0$, for all $x, y \in I$

$F(xy)\sigma(x) + \tau(xy)d(x) + d(x)(F(y)\sigma(x) + \tau(y)d(x)) = 0$

$(F(xy) + d(x)F(y))\sigma(x) + \tau(xy)d(x) + d(x)\tau(y)d(x) = 0$, for all $x, y \in I$.

Using equation (6), it reduces to

$\tau(xy)d(x) + d(x)\tau(y)d(x) = 0$, for all $x, y \in I$. (7)

We replacing y by xy in equation (7), we get

$\tau(xxy)d(x) + d(x)\tau(xy)d(x) = 0$, for all $x, y \in I$. (8)

Left multiplying equation (7) by $\tau(x)$, we get

$\tau(x)\tau(xy)d(x) + \tau(x)d(x)\tau(y)d(x) = 0$, for all $x, y \in I$. (9)

We subtracting equation (9) from equation (8), we get

$d(x)\tau(x)\tau(y)d(x) - \tau(x)d(x)\tau(y)d(x) = 0$

$[d(x), \tau(x)]\tau(y)d(x) = 0$, for all $x, y \in I$. (10)

We replacing y by sy, $s \in R$ in equation (10), we get

$[d(x), \tau(x)]\tau(sy)d(x) = 0$

$[d(x), \tau(x)]R\tau(y)d(x) = 0$, for all $x, y \in I$, $s \in R$. (11)

Since R is prime, we get either $[d(x),\tau(x)]=0$, for all $x\in I$ or $\tau(y)d(x)=0$, for all $x,y\in I$. Since τ is an automorphism of R and $\tau(I)\neq 0$, we have either $[d(x),\tau(x)]=0$, for all $x\in I$ or $d(x)=0$, for all $x\in I$.

Now let $A=\{x\in I/[d(x),\tau(x)]=0\}$ and $B=\{x\in I/d(x)=0\}$. Clearly, A and B are additive proper subgroups of I whose union is I. Since a group cannot be the set theoretic union of two proper subgroups. Hence either $A=I$ or $B=I$.

If $B=I$, then $d(x)=0$, for all $x\in I$, by lemma 2 implies that $I\subseteq Z$.

On the other hand if $A=I$, then $[d(x),\tau(x)]=0$, for all $x\in I$.

If $[d(x),\tau(x)]=0$, for all $x\in I$. (12)

We replacing x by $x+y$ in equation (12), we get

$[d(x+y),\tau(x+y)]=0$

$[d(x),\tau(x)]+[d(x),\tau(y)]+[d(y),\tau(x)]+[d(y),\tau(y)]=0$, for all $x,y\in I$.

Using equation (12) in the above equation, we get

$[d(x),\tau(y)]+[d(y),\tau(x)]=0$, for all $x,y\in I$. (13)

We replacing y by yx in equation (13), we get

$[d(x),\tau(yx)]+[d(yx),\tau(x)]=0$

$[d(x),\tau(y)]\tau(x)+\tau(y)[d(x),\tau(x)]+[d(x)\sigma(y)+\tau(x)d(y),\tau(x)]=0$

$[d(x),\tau(y)]\tau(x)+\tau(y)[d(x),\tau(x)]+[d(x)\sigma(y),\tau(x)]+[\tau(x)d(y),\tau(x)]=0$

$[d(x),\tau(y)]\tau(x)+\tau(y)[d(x),\tau(x)]+[d(x),\tau(x)]\sigma(y)+d(x)[\sigma(y),\tau(x)]+$
$\tau(x)[d(y),\tau(x)]+[\tau(x),\tau(x)]d(y)=0$
, for all $x,y\in I$.

Using equation (12) in the above equation, we get

$[d(x),\tau(y)]\tau(x)+d(x)[\sigma(y),\tau(x)]+\tau(x)[d(y),\tau(x)]=0$, for all $x,y\in I$. (14)

Right multiplying equation (13) by $\tau(x)$, we get

$[d(x),\tau(y)]\tau(x)+[d(y),\tau(x)]\tau(x)=0$, for all $x,y\in I$. (15)

We subtracting equation (15) from equation (14), we get

$d(x)[\sigma(y),\tau(x)]+\tau(x)[d(y),\tau(x)]-[d(y),\tau(x)]\tau(x)=0$, for all $x,y\in I$.

We replacing $d(y)$ by $\tau(x)$ in the above equation, we get

$d(x)[\sigma(y),\tau(x)]=0$, for all $x,y\in I$. (16)

We replacing y by ys in equation (16), we get

$d(x)[\sigma(ys),\tau(x)] = 0$

$d(x)[\sigma(y),\tau(x)]\sigma(s) + d(x)\sigma(y)[\sigma(s),\tau(x)] = 0$, for all $x, y, s \in I$.

Using equation (16) in the above equation, we get

$d(x)\sigma(y)[\sigma(s),\tau(x)] = 0$, for all $x, y, s \in I$.

We replacing y by yv, $v \in R$ in the above equation, we get

$d(x)\sigma(yv)[\sigma(s),\tau(x)] = 0$, for all $x, y, s \in I, v \in R$.

$d(x)\sigma(y)R[\sigma(s),\tau(x)] = 0$, for all $x, y, s \in I$.

Since R is prime, we get either $d(x)\sigma(y) = 0$, for all $x, y \in I$ or $[\sigma(s),\tau(x)] = 0$, for all $x, s \in I$. Since σ is an automorphism of R and $\sigma(I) \neq 0$, we have either $[\sigma(x),\tau(y)] = 0$, for all $x, y \in I$ or $d(x) = 0$, for all $x \in I$. If $d(x) = 0$, for all $x \in I$, by lemma 2 implies that $I \subseteq Z$. If $[\sigma(x),\tau(y)] = 0$, for all $x, y \in I$, by lemma 1 implies that $I \subseteq Z$.

Theorem 2: Let R be a prime ring and I be a non-zero ideal on R. Suppose that F is a generalized (σ,τ)-reverse derivation on R associated with (σ,τ)-reverse derivation d on R respectively, $\tau(I) \neq 0$ and $\sigma(I) \neq 0$. If $G(xy) + d(x)F(y) + \sigma(xy) = 0$, for all $x, y \in I$, then $I \subseteq Z$.

Proof: We replacing F by $F + \sigma$ in theorem 1, we get the required result.

Theorem 3: Let R be a prime ring and I be a non-zero ideal on R. Suppose that F is a generalized (σ,τ)-reverse derivation on R associated with (σ,τ)-reverse derivation d on R respectively, $\tau(I) \neq 0$ and $\sigma(I) \neq 0$. If $F(xy) + d(x)F(y) + \sigma(yx) = 0$, for all $x, y \in I$, then $I \subseteq Z$.

Proof: We have $F(xy) + d(x)F(y) + \sigma(yx) = 0$, for all $x, y \in I$. (17)

We replacing y by xy in equation (17), we obtain

$F(xxy) + d(x)F(xy) + \sigma(xyx) = 0$, for all $x, y \in I$

$F(xy)\sigma(x) + \tau(xy)d(x) + d(x)\big(F(y)\sigma(x) + \tau(y)d(x)\big) + \sigma(xyx) = 0$

$\big(F(xy) + d(x)F(y)\big)\sigma(x) + \tau(xy)d(x) + d(x)\tau(y)d(x) + \sigma(xyx) = 0$, for all $x, y \in I$.

Using equation (17), it reduces to

$\tau(xy)d(x) + d(x)\tau(y)d(x) + \sigma(xyx) - \sigma(yx)\sigma(x) = 0$

$\tau(xy)d(x) + d(x)\tau(y)d(x) + \sigma[x,y]\sigma(x) = 0$, for all $x, y \in I$. (18)

We replacing y by xy in equation (18), we get

$\tau(xxy)d(x) + d(x)\tau(xy)d(x) + \sigma[x,xy]\sigma(x) = 0$

$\tau(xxy)d(x) + d(x)\tau(xy)d(x) + \sigma(x)\sigma[x,y]\sigma(x) = 0$, for all $x, y \in I$. (19)

Left multiplying equation (18) by $\tau(x)$, we get

$\tau(x)\tau(xy)d(x)+\tau(x)d(x)\tau(y)d(x)+\tau(x)\sigma[x,y]\sigma(x)=0$, for all $x,y,z\in I$. (20)

We subtracting equation (20) from equation (19), we get

$d(x)\tau(x)\tau(y)d(x)-\tau(x)d(x)\tau(y)d(x)+\sigma(x)\sigma[x,y]\sigma(x)-\tau(x)\sigma[x,y]\sigma(x)=0$

$[d(x),\tau(x)]\tau(y)d(x)+(\sigma(x)-\tau(x))\sigma[x,y]\sigma(x)=0$, for all $x,y\in I$.

We replacing $\tau(x)$ by $\sigma(x)$ in the above equation, we get

$[d(x),\sigma(x)]\tau(y)d(x)=0$, for all $x,y\in I$. (21)

We replacing $\sigma(x)$ by $\tau(x)$ and y by sy, $s\in R$ in equation (21), we get

$[d(x),\tau(x)]\tau(sy)d(x)=0$

$[d(x),\tau(x)]R\tau(y)d(x)=0$, for all $x,y\in I$, $s\in R$. (22)

The equation (22) is same as equation (11) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result $I\subseteq Z$.

Theorem 4: Let R be a prime ring and I be a non-zero ideal on R. Suppose that F is a generalized (σ,τ)-reverse derivation on R associated with (σ,τ)-reverse derivation d on R respectively, $\tau(I)\neq 0$ and $\sigma(I)\neq 0$. If $F(xy)+d(x)F(y)+\sigma(xoy)=0$, for all $x,y\in I$, then $I\subseteq Z$.

Proof: We replacing F by $F+\sigma$ in theorem 3, we get the required result.

Theorem 5: Let R be a prime ring and I be a non-zero ideal on R. Suppose that F is a generalized (σ,τ)-reverse derivation on R associated with (σ,τ)-reverse derivation d on R respectively, $\tau(I)\neq 0$ and $\sigma(I)\neq 0$. If $F(xy)+d(y)F(x)=0$, for all $x,y\in I$, then $I\subseteq Z$.

Proof: We have $F(xy)+d(y)F(x)=0$, for all $x,y\in I$. (23)

We replacing x by xw in equation (23), we obtain

$F(xwy)+d(y)F(xw)=0$

$F(wy)\sigma(x)+\tau(wy)d(x)+d(y)(F(w)\sigma(x)+\tau(w)d(x))=0$

$(F(wy)+d(y)F(w))\sigma(x)+\tau(wy)d(x)+d(y)\tau(w)d(x)=0$, for all $x,y,w\in I$.

Using equation (23), it reduces to

$\tau(wy)g(x)+d(y)\tau(w)d(x)=0$, for all $x,y,w\in I$. (24)

We replacing y by zy in equation (24), we get

$\tau(wzy)d(x)+d(zy)\tau(w)d(x)=0$

$\tau(wzy)d(x)+d(y)\sigma(z)\tau(w)d(x)+\tau(y)d(z)\tau(w)d(x)=0$, for all $x,y,z,w\in I$. (25)

We replacing y by z in equation (24), we get

$\tau(wz)d(x)+d(z)\tau(w)d(x)=0$, for all $x,z,w\in I$. (26)

Left multiplying equation (26) by $\tau(y)$, we get

$\tau(y)\tau(wz)d(x) + \tau(y)d(z)\tau(w)d(x) = 0$, for all $x, y, z \in I$. (27)

We subtracting equation (27) from equation (25), we get

$\big(\tau(wzy) - \tau(ywz)\big)d(x) + d(y)\sigma(z)\tau(w)d(x) = 0$

$\tau[wz, y]d(x) + d(y)\sigma(z)\tau(w)d(x) = 0$

$\tau([w, y]z + w[z, y])d(x) + d(y)\sigma(z)\tau(w)d(x) = 0$, for all $x, y, z, w \in I$.

We replacing z by y and w by y in the above equation, we get

$d(y)\sigma(y)\tau(y)d(x) = 0$, for all $x, y \in I$. (28)

We replacing x by sx in equation (28), we get

$d(y)\sigma(y)\tau(y)d(sx) = 0$

$d(y)\sigma(y)\tau(y)d(x)\sigma(s) + d(y)\sigma(y)\tau(y)\tau(x)d(s) = 0$, for all $x, y, s \in I$.

Using equation (28) in the above equation, we get

$d(y)\sigma(y)\tau(y)\tau(x)d(s) = 0$, for all $x, y, s \in I$.

We replacing x by rx, $r \in R$ in the above equation, we get

$d(y)\sigma(y)\tau(y)\tau(rx)d(s) = 0$

$d(y)\sigma(y)\tau(y)R\tau(x)d(s) = 0$, for all $x, y, s \in I$.

Since R is prime, we get either $d(y)\sigma(y)\tau(y) = 0$, for all $y \in I$ or $\tau(x)d(s) = 0$, for all $x, s \in I$. Since τ is an automorphism of R and $\tau(I) \neq 0$, we have either $d(x)\sigma(x) = 0$, for all $x \in I$ or $d(x) = 0$, for all $x \in I$. If $d(x) = 0$, for all $x \in I$, by lemma 2 implies that $I \subseteq Z$.

If $d(x)\sigma(x) = 0$, for all $x \in I$.Since σ is an automorphism of R and $\sigma(I) \neq 0$ then $d(x) = 0$, for all $x \in I$, by lemma 2 implies that $I \subseteq Z$.

Theorem 6: Let R be a prime ring and I be a non-zero ideal on R. Suppose that F is a generalized (σ, τ)-reverse derivation on R associated with (σ, τ)-reverse derivation d on R respectively, $\tau(I) \neq 0$ and $\sigma(I) \neq 0$. If $F(xy) + d(y)F(x) + \sigma(xy) = 0$, for all $x, y \in I$, then $I \subseteq Z$.

Proof: We replacing F by $F + \sigma$ in theorem 5, we get the required result.

Theorem 7: Let R be a prime ring and I be a non-zero ideal on R. Suppose that F is a generalized (σ, τ)-reverse derivation on R associated with (σ, τ)-reverse derivation d on R respectively, $\tau(I) \neq 0$ and $\sigma(I) \neq 0$. If $F(xy) + d(y)F(x) + \sigma(yx) = 0$, for all $x, y \in I$, then $I \subseteq Z$.

Proof: We have $F(xy) + d(y)F(x) + \sigma(yx) = 0$, for all $x, y \in I$. (29)

We replacing x by xw in equation (29), we obtain

$F(xwy)+d(y)F(xw)+\sigma(yxw)=0$, for all $x,y,w\in I$

$F(wy)\sigma(x)+\tau(wy)d(x)+d(y)(F(w)\sigma(x)+\tau(w)d(x))+\sigma(yxw)=0$

$(F(wy)+d(y)F(w))\sigma(x)+\tau(wy)d(x)+d(y)\tau(w)d(x)+\sigma(yxw)=0$

Using equation (29), it reduces to

$\tau(wy)d(x)+d(y)\tau(w)d(x)+\sigma(yxw)-\sigma(ywx)=0$

$\tau(wy)d(x)+d(y)\tau(w)d(x)+\sigma(y)\sigma[x,w]=0$, for all $x,y,w\in I$. (30)

We replacing y by zy in equation (30), we get

$\tau(wzy)d(x)+d(zy)\tau(w)d(x)+\sigma(zy)\sigma[x,w]=0$, for all $x,y,z,w\in I$.

$\tau(wzy)d(x)+d(y)\sigma(z)\tau(w)d(x)+\tau(y)d(z)\tau(w)d(x)+\sigma(zy)\sigma[x,w]=0$, for all $x,y,z,w\in I$. (31)

We replacing y by z in equation (30), we get

$\tau(wz)d(x)+d(z)\tau(w)d(x)+\sigma(z)\sigma[x,w]=0$, for all $x,z,w\in I$. (32)

Left multiplying equation (32) by $\tau(y)$, we get

$\tau(y)\tau(wz)d(x)+\tau(y)d(z)\tau(w)d(x)+\tau(y)\sigma(z)\sigma[x,w]=0$, for all $x,y,z,w\in I$. (33)

We subtracting equation (33) from equation (31), we get

$\big(\tau(wzy)-\tau(ywz)\big)d(x)+d(y)\sigma(z)\tau(w)d(x)+\sigma(zy)\sigma[x,w]-\tau(y)\sigma(z)\sigma[x,w]=0$

$\tau[wz,y]d(x)+d(y)\sigma(z)\tau(w)d(x)+\sigma(zy)\sigma[x,w]-\tau(y)\sigma(z)\sigma[x,w]==0$

$\tau([w,y]z+w[z,y])d(x)+d(y)\sigma(z)\tau(w)d(x)+\sigma(zy)\sigma[x,w]-\tau(y)\sigma(z)\sigma[x,w]=0$, for all $x,y,z,w\in I$.

We replacing z by y and w by y in the above equation, we get

$d(y)\sigma(y)\tau(y)d(x)+\sigma(yy)\sigma[x,y]-\tau(y)\sigma(y)\sigma[x,y]=0$, for all $x,y\in I$.

We replacing $\tau(y)$ by $\sigma(y)$ in the above equation, we get

$d(y)\sigma(y)\sigma(y)d(x)+\sigma(yy)\sigma[x,y]-\sigma(y)\sigma(y)\sigma[x,y]=0$, for all $x,y\in I$.

$d(y)\sigma(y)\sigma(y)d(x)=0$, for all $x,y\in I$. (34)

The equation (34) is same similar equation (28) in theorem 5. Thus, by same argument of theorem 5, we can conclude the result $I\subseteq Z$.

Theorem 8: Let R be a prime ring and I be a non-zero ideal on R. Suppose that F is a generalized (σ,τ)-reverse derivation on R associated with (σ,τ)-reverse derivation d on R respectively, $\tau(I)\neq 0$ and $\sigma(I)\neq 0$. If $F(xy)+d(y)F(x)+\sigma(xoy)=0$, for all $x,y\in I$, then $I\subseteq Z$.

Proof: We replacing F by $F+\sigma$ in theorem 7, we get the required result.

Theorem 9: Let R be a prime ring and I be a non-zero ideal on R. Suppose that F is a generalized (σ,τ)-reverse derivation on R associated with (σ,τ)-reverse derivation d on R respectively, $\tau(I)\neq 0$ and $\sigma(I)\neq 0$. If $F(xy)+F(x)F(y)=0$, for all $x,y\in I$, then $I\subseteq Z$.

Proof: We have $F(xy)+F(x)F(y)=0$, for all $x,y\in I$. (35)

We replacing y by xy in equation (35), we obtain

$F(xxy)+F(x)F(xy)=0$, for all $x,y\in I$

$F(xy)\sigma(x)+\tau(xy)d(x)+F(x)(F(y)\sigma(x)+\tau(y)d(x))=0$

$(F(xy)+F(x)F(y))\sigma(z)+\tau(xy)d(x)+F(x)\tau(y)d(x)=0$, for all $x,y\in I$.

Using equation (35), it reduces to

$\tau(xy)d(x)+F(x)\tau(y)d(x)=0$, for all $x,y\in I$. (36)

We replacing y by wy in equation (36), we get

$\tau(xwy)d(x)+F(x)\tau(wy)d(x)=0$, for all $x,y,w\in I$. (37)

Left multiplying equation (36) by $\tau(w)$, we get

$\tau(w)\tau(xy)d(x)+\tau(w)F(x)\tau(y)d(x)=0$, for all $x,y,z,w\in I$. (38)

We subtracting equation (38) from equation (37), we get

$\tau(xwy)g(x)-\tau(wxy)g(x)+F(x)\tau(w)\tau(y)d(x)-\tau(w)F(x)\tau(y)d(x)=0$

$\big(\tau(xwy)-\tau(wxy)\big)g(x)+[F(x),\tau(w)]\tau(y)d(x)=0$, for all $x,y,w\in I$. (39)

We replacing w by x and y by sy, $s\in R$ in equation (39), we get

$[F(x),\tau(x)]\tau(sy)d(x)=0$

$[F(x),\tau(x)]R\tau(y)d(x)=0$, for all $x,y\in I$, $s\in R$. (40)

Since R is prime, we get either $[F(x),\tau(x)]=0$, for all $x\in I$ or $\tau(y)d(x)=0$, for all $x,y\in I$. Since τ is an automorphism of R and $\tau(I)\neq 0$, we have either $[F(x),\tau(x)]=0$, for all $x\in I$ or $d(x)=0$, for all $x\in I$.

Now let $A=\{x\in I/[F(x),\tau(x)]=0\}$ and $B=\{x\in I/d(x)=0\}$. Clearly, A and B are additive proper subgroups of I whose union is I. Since a group cannot be the set theoretic union of two proper subgroups. Hence either $A=I$ or $B=I$.

If $B=I$, then $d(x)=0$, for all $x\in I$, by lemma 2 implies that $I\subseteq Z$.

On the other hand if $A=I$, then $[F(x),\tau(x)]=0$, for all $x\in I$.

If $[F(x),\tau(x)]=0$, for all $x\in I$. (41)

We replacing x by $x+y$ in equation (12), we get

$[F(x+y),\tau(x+y)]=0$

$[F(x),\tau(x)]+[F(x),\tau(y)]+[F(y),\tau(x)]+[F(y),\tau(y)]=0$, for all $x,y\in I$.

Using equation (41) in the above equation, we get

$[F(x),\tau(y)]+[F(y),\tau(x)]=0$, for all $x,y\in I$. (42)

We replacing y by yx in equation (42), we get

$[F(x),\tau(yx)]+[F(yx),\tau(x)]=0$

$[F(x),\tau(y)]\tau(x)+\tau(y)[F(x),\tau(x)]+[F(x)\sigma(y)+\tau(x)d(y),\tau(x)]=0$

$[F(x),\tau(y)]\tau(x)+\tau(y)[F(x),\tau(x)]+[F(x)\sigma(y),\tau(x)]+[\tau(x)d(y),\tau(x)]=0$

$[F(x),\tau(y)]\tau(x)+\tau(y)[F(x),\tau(x)]+[F(x),\tau(x)]\sigma(y)+F(x)[\sigma(y),\tau(x)]+$
$\tau(x)[d(y),\tau(x)]+[\tau(x),\tau(x)]d(y)=0$
, for all $x,y\in I$.

Using equation (41) in the above equation, we get

$[F(x),\tau(y)]\tau(x)+F(x)[\sigma(y),\tau(x)]+\tau(x)[d(y),\tau(x)]=0$, for all $x,y\in I$.

We replacing $\sigma(y)$ by $\tau(x)$ in the above equation, we get

$[F(x),\tau(y)]\tau(x)+\tau(x)[d(y),\tau(x)]=0$, for all $x,y\in I$.

We replacing y by x in the above equation, we get

$[F(x),\tau(x)]\tau(x)+\tau(x)[d(x),\tau(x)]=0$, for all $x,y\in I$.

Using equation (41) in the above equation, we get

$\tau(x)[d(x),\tau(x)]=0$, for all $x,y\in I$.

Since τ is an automorphism of R and $\tau(I)\neq 0$, we get $[d(x),\tau(x)]=0$, for all $x,y\in I$. (43)

The equation (43) is same as equation (12) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result $I\subseteq Z$.

Theorem 10: Let R be a prime ring and I be a non-zero ideal on R. Suppose that F is a generalized (σ,τ)-reverse derivation on R associated with (σ,τ)-reverse derivation d on R respectively, $\tau(I)\neq 0$ and $\sigma(I)\neq 0$. If $F(xy)+F(y)F(x)=0$, for all $x,y\in I$, then $I\subseteq Z$.

Proof: We have $F(xy)+F(y)F(x)=0$, for all $x,y\in I$. (44)

We replacing x by xw in equation (44), we obtain

$F(xwy)+F(y)F(xw)=0$

$F(wy)\sigma(x)+\tau(wy)d(x)+F(y)(F(w)\sigma(x)+\tau(w)d(x))=0$

$(F(wy)+F(y)F(w))\sigma(x)+\tau(wy)d(x)+F(y)\tau(w)d(x)=0$, for all $x,y,w\in I$.

Using equation (44), it reduces to

$\tau(wy)d(x) + F(y)\tau(w)d(x) = 0$, for all $x, y, w \in I$. (45)

We replacing y by zy in equation (45), we get

$\tau(wzy)d(x) + F(zy)\tau(w)d(x) = 0$

$\tau(wzy)d(x) + F(y)\sigma(z)\tau(w)d(x) + \tau(y)d(z)\tau(w)d(x) = 0$, for all $x, y, z, w \in I$. (46)

We replacing y by z in equation (45), we get

$\tau(wz)d(x) + F(z)\tau(w)d(x) = 0$, for all $x, z, w \in I$. (47)

Left multiplying equation (47) by $\tau(y)$, we get

$\tau(y)\tau(wz)d(x) + \tau(y)F(z)\tau(w)d(x) = 0$, for all $x, y, z, w \in I$. (48)

We subtracting equation (48) from equation (46), we get

$\big(\tau(wzy) - \tau(ywz)\big)d(x) + F(y)\sigma(z)\tau(w)d(x) + \tau(y)d(z)\tau(w)d(x) -$
$\tau(y)F(z)\tau(w)d(x) = 0$

$\tau[wz, y]d(x) + (F(y)\sigma(z) + \tau(y)d(z))\tau(w)d(x) - \tau(y)F(z)\tau(w)d(x) = 0$

$\tau([w, y]z + w[z, y])d(x) + (F(zy) - \tau(y)F(z))\tau(w)d(x) = 0$, for all $x, y, z, w \in I$.

We replacing z by y and w by y in the above equation, we get

$(F(yy) - \tau(y)F(y))\tau(y)d(x) = 0$, for all $x, y \in I$. (49)

We replacing x by sx, $s \in R$ in equation (49), we get

$(F(yy) - \tau(y)F(y))\tau(y)d(sx) = 0$

$\big(F(yy) - \tau(y)F(y)\big)\tau(y)d(x)\sigma(s) + (F(yy) - \tau(y)F(y))\tau(y)\tau(x)d(s) = 0$, for all $x, y, s \in I$.

Using equation (49) in the above equation, we get

$(F(yy) - \tau(y)F(y))\tau(y)\tau(x)d(s) = 0$, for all $x, y, s \in I$.

We replacing x by rx, $r \in R$ in the above equation, we get

$(F(yy) - \tau(y)F(y))\tau(y)\tau(rx)d(s) = 0$, for all $x, y, s \in I$.

$(F(yy) - \tau(y)F(y))\tau(y)R\tau(x)d(s) = 0$, for all $x, y, s \in I$.

Since R is prime, we get either $(F(yy) - \tau(y)F(y))\tau(y) = 0$, for all $y \in I$ or $\tau(x)d(s) = 0$, for all $x, s \in I$. Since τ is an automorphism of R and $\tau(I) \neq 0$, we have either $(F(yy) - \tau(y)F(y)) = 0$, for all $y \in I$ or $d(x) = 0$, for all $x \in I$.

Now let $A = \{x \in I/(F(x^2) - \tau(x)F(x)) = 0\}$ and $B = \{x \in I/d(x) = 0\}$. Clearly, A and B are additive proper subgroups of I whose union is I. Since a group cannot be the set theoretic union of two proper subgroups. Hence either $A = I$ or $B = I$.

If $B = I$, then $d(x) = 0$, for all $x \in I$, by lemma 2 implies that $I \subseteq Z$.

On the other hand if $A = I$, then $F(x^2) - \tau(x)F(x) = 0$, for all $x \in I$. (50)

We replacing y by xx in equation (44), we get

$G(xxx) = -F(xx)F(x)$, for all $x \in I$. (51)

We replacing x by xx and y by x in equation (44), we get

$G(xxx) = -F(x)F(xx)$, for all $x \in I$. (52)

From equation (51) and equation (52), we get

$F(x)F(x^2) = F(x^2)F(x)$, for all $x \in I$.

Using equation (50), it reduces to

$$F(x)\tau(x)F(x) = \tau(x)F(x)F(x)$$

$$\big(F(x)\tau(x) - \tau(x)F(x)\big)F(x) = 0$$

We conclude that $[F(x), \tau(x)] = 0$, for all $x \in I$. (53)

The equation (53) is same as equation (41) in theorem 9. Thus, by same argument of theorem 9, we can conclude the result $I \subseteq Z$.

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