Fractional Order Differentiators and Integrators Design using Transform Techniques

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Abstract:- The design of fractional order differentiators and integrators (FODI) using is the main objective of this paper. In this indirect discretization scheme based Al-Alaoui and Tustin operators used to design a novel FODIs. Firstly, fifth ordr rational approximations in the s-domain are used and finally, discretized using more popular s-to-z transforms to acheive the new FODIs. The performances of proposed FODIs are well compared with the continued fractionan expansion (CFE) approximation based ones. Fractional order differentiators and integrators of order 0.5 and 0.25 are designed by using indirect discretization. Matlab simulation software is used to obtain the magnitude, phase responses and normalized magnitude error (NME) plots of the digital differentiators and integrators.It is noted that the designed digital integrators and differentiators are accurate ntemd of magnitude and NME.

Keywords: *Discretization, Al-Alaoui operator, Continued frcation expansion (CFE), NME.*

1. Introduction

Digital differentiators and integrators plays an important role in all engineering areas including digital image processing, control systems, communications, Radar, Bio-medical engineering and signal processing applications, etc[1]-[3]. Fractional differentiators and integrators are used to determine the time derivative and integral of a signal to an arbitrary order. The fractional order system contains infinite memory, whereas regular integer order system involve with finite memory. So, these are used in the modeling electrical and mechanical properties of materials, transmission line theory, rheology of soils and quantum mechanical calculations etc. The frequency response of fractional order differ-integrators are defined as [3],[9]

$$H(j\omega) = (j\omega)^{\pm\alpha} \tag{1}$$

Where $j^2 = -1$ is a complex constant and$\alpha$is the fractional order. The positive sign for differentiator and negative sign used for integrator operations. Discretization is the key factor in the design of fractional order differ-integrators. Basically, direct and indirect discretization techniques are used. Direct discretization means direct expansion of s^α by using any one of series expansion techniques like Taylor series expansion (TSE), binomial series expansion (BSE) and continued fraction expansion (CFE) etc. Two step procedure involved in the indirect discretization. Firstly obtain the rational transfer function in the continuous domain and truncated to some finite order. Finally use s-to-z transform to obtain the digital transfer function.
Direct discretization of integer order s-to-z transformation like Al-Alaoui, reduced order and interpolated tarnsformsusing continued fraction expansion presented in [12]-[14]. Fractional order differentiators and integrators are designed by using indirect discretization technique are explained in [3],[11]. Drect discretization with series expansion and CFE schemes are proposed by Chen and et al.,[13] to design the fractional order differentiators and integrators . Both particle swarm optimization (PSO) and indirect discretization are used in the design of fractional order integrators and differentiators are presented in [10]. The fractional order differentiators design using model order procedure presented in [5], which are suitable to the low frequency applications. Low frequency suitable fractional differ-integratirs presented with the application of reducdeorder s-to-z transforms presented in [12],[15]. Sparten 3E field programmable implementation (FPGA) of fractional differ-integrators of order 1/2, 1/3 and 1/4 are explained in [14].

In the literature several s-to-z transforms are available, which are used in the discretization process. The more popular wide band recursive integrators and differentiators are bilinear [2], Al-Alaoui [4], interpolated transforms [6],[7], reduced order tansforms [9] and etc. Based on the Newton- Cotes integration rules, Al-Alaoui proposed different transform namely two segment rule, three segment rule and four segment rule [7].

Now a days most of the researchers concentrayes on the fractional calculas due to its wide applications in microwave theory, biomedical image processing, biomedical signal processing, control theory and etc. Recently, SaeedBalochian et al., applied a Prewitt operator with fractional order differentiator for detection of edges in an image. The fractional order differentitor not only calculate the derivative of an image but also eliminate the noise. The proposed method compared with the existing traditional approaches by performing experimentation on sample images. So, the fractional order differentiators are promising potentiality for edge detection of medical images [8]. In [9], developed the mathematical models of DFODs of half, one third and one-fourth fractional orders, which are well worked at the low frequency region. The proposed designs are also applied to the images for edge detection and were superior in terms of mean square error (MSE) compared to the well existed conventional approaches as Prewitt and Canny operators .

In this context, an attempt made to design the digital fractional differentiators and integrators with the help of indirect discretization approach. The efficacy compared with the help of designed dfifferentiators magnitude and normalized magnitude error (NME) plots.

The brief of this paper is as follows. Section 2 presents the fifth order analog rational approximations using continued fraction expansion (CFE) methods. Design of fractional order digital differentiators and integrators of order 1/2 and 1/4 are given in section 3. The simulation results of proposed schemes are explained in section 3. Fiallyconculsion remarks are explained in section 5.

2. Rational Approximations of s^{α}

The fifth order analog rational approximation of $s^{1/2}$ and $s^{1/4}$with the application of continued fraction expansion (CFE) given as [3]

$$s^{1/2}=\frac{11s^5+165s^4+462s^3+330s^2+55s+1}{s^5+55s^4+330s^3+462s^2+165s+1} \tag{2}$$

$$s^{1/4}=\frac{663s^5+12597s^4+41990s^3+35530s^2+7315s+209}{209s^5+7315s^4+35530s^3+41990s^2+12597s+663} \tag{3}$$

The proposed $s^{1/2}$ and $s^{1/4}$ approximation as

$$s^{1/2}=\frac{55.38s^5+1.33X10^4s^4+1.18X10^5s^3+6.151X10^4s^2+1533s+1}{s^5+1533s^4+6.151X10^4s^3+1.18X10^5s^2+1.33X10^4s+5538} \tag{4}$$

$$s^{1/4}=\frac{7.058s^5+2627s^4+3.415X10^4s^3+2.484X10^4s^2+909.5s+1}{s^5+909.5s^4+2.484X10^4s^3+3.415X10^4s^2+2627s+7.058} \tag{5}$$

3. Fractional order Differentiators and Integrators

The s-to-z transform plays a vital role in the design of fractional order differentiators as well integrators. The well known bilinear operator expressed in Eqn. (6) as

$$s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \tag{6}$$

Where T is the sampling period. The interpolation of rectangular differentiator and Tustin tansformation are well known as Al-Alaoui operator, which is defined as

$$s=\frac{8}{7T}\frac{1-z^{-1}}{1+\frac{1}{7}z^{-1}} \tag{7}$$

The half order fractional digital differentiator obtained by substituting bilinear (BLT) and Al-Alaoui operators (Eqns. (6) and (7)) in Eqns. (2) and (4). The transfer function of half order differentiators can be expressed as

$$H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3} + p_4 z^{-4} + p_5 z^{-5}}{q_0 + q_1 z^{-1} + q_2 z^{-2} + q_3 z^{-3} + q_4 z^{-4} + q_5 z^{-5}} \qquad (8)$$

Where p_0, p_1,…,p_5 and q_0, q_1,…,q_5 are the numerator and denominator coefficients respectively. The coeffients for the half order differentiators for the designed methods are listed in Table 1. The half order integrator coefficients obtained by interchanding the roles of nememerator and denominator coefficients.

Table 1. Coeffients of half order digital differentiatos

Coefficients	CFE+BLT	CFE+Al-Alaoui	TF2+BLT	TF2+Al-Alaoui
p_0	8119	24999391	1.386472536	1.065266379
p_1	-11721	-63334707	-1.31589075120437	-2.62743326630872
p_2	-1002	55367574	-1.90153174971432	1.97450157477320
p_3	6302	-18716230	1.77059448490759	-0.317278199173139
p_4	-1469	1617211	0.543002372488472	-0.0941854068477530
p_5	-197	99529	-0.482615374555945	-0.000863693255846634
q_0	5741	23384789	1	1
q_1	-2547	-45881457	-0.0135400005464419	-1.92554361644233
q_2	-6126	27482610	-1.79753620074584	0.900190090852817
q_3	2474	-4184962	0.0357036983660851	0.0719090153005021
q_4	1073	-492343	0.798913218732459	-0.0421314314943641
q_5	-263	51811	-0.0217952533183002	-0.00401477996511390

Similarly, the one-fourth order fractional digital differentiator obtained by substituting bilinear (BLT) and Al-Alaoui operators (Eqns. (6) and (7)) in Eqns. (3) and (5). The transfer function of one-fourth order differentiators can be expressed as

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4} + b_5 z^{-5}}{c_0 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + c_4 z^{-4} + c_5 z^{-5}} \qquad (9)$$

Where b_0, b,…,b_5 and c, c_1,…,c_5 are the numerator and denominator coefficients respectively. The coeffients for the one-fourth order differentiators for the designed methods are listed in Table 2. The one-fourth order integrator coefficients obtained by interchanding the roles of nememerator and denominator coefficients.

Table 2. Coeffients of one fourth order digital differentiatos

Coefficients	CFE+BLT	CFE+Al-Alaoui	TF2+BLT	TF2+Al-Alaoui
b_0	715647	2360325231	1.173082336	1.031421994
b_1	-859601	-5644455299	-0.832298316722478	-2.40181596013697
b_2	-309466	4553637686	-1.84463371351520	1.63364971078860
b_3	551374	-1341809894	1.19904092564786	-0.170610164321364
b_4	-63381	68561931	0.681771121595898	-0.0889159929976232
b_5	-27885	10588857	-0.376872254052727	-0.00370698231342771
c_0	601785	2282831529	1	1
c_1	-421943	-4806900149	-0.253793857164856	-2.06399406115084
c_2	-546422	3217087162	-1.77561785934471	1.11785317287386
c_3	356002	-651189386	0.401465675714079	0.0114667543194370
c_4	62253	-27613587	0.777377401805107	-0.0604331088071246
c_5	-30459	7509615	-0.148795442596720	-0.00473320499688730

4. Results

The magnitude, phase responses and normalized magnitude error plots for the fractional order differentiaators for an order 1/2 are shown in Figs.1-3. The fractional differentiator of order 1/2 by using connued fraction expansion (CFE) and transfer function (TF2) approximation with Al-Alaoui transforms are more effective compared to the bilear (BLT) based designs. Moreover, the transfer function 2 based one is more accurate compared to the CFE based design upto the freqncy range of 2 rad. Figures 4 to 6 shows the 1/4 fractional order differentiators magnitude, phase and NME plots. From Fig. 6, it is clear that the transfer function 2 (TF2) based Al-Alaoui operator are more effective than the other designs upto to the frequency range of 2 rad.

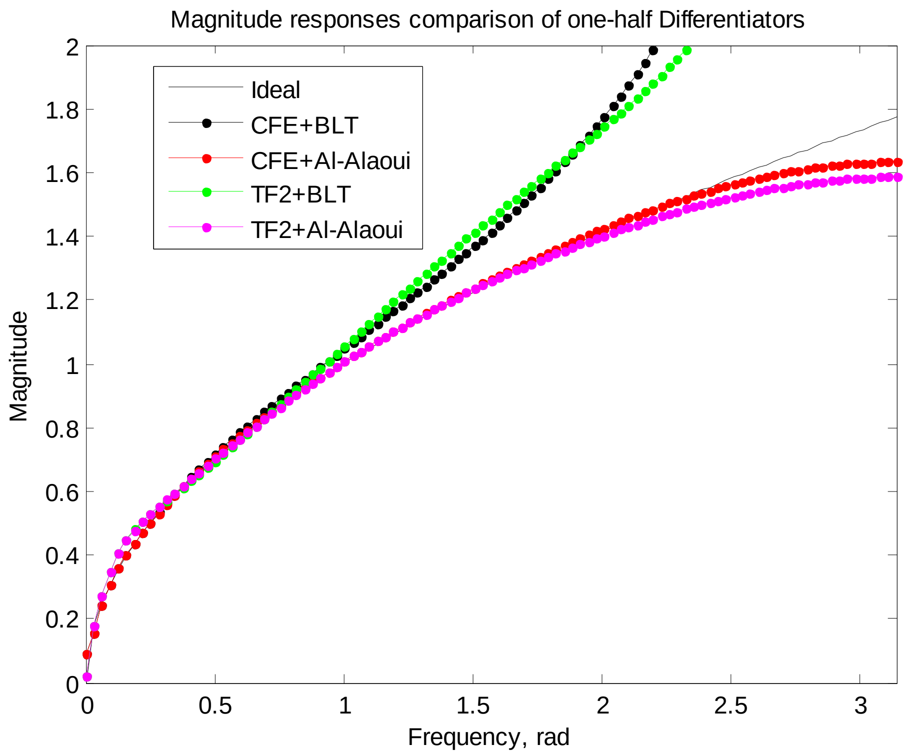

Fig. 1. Magnitude response of half order differentiators

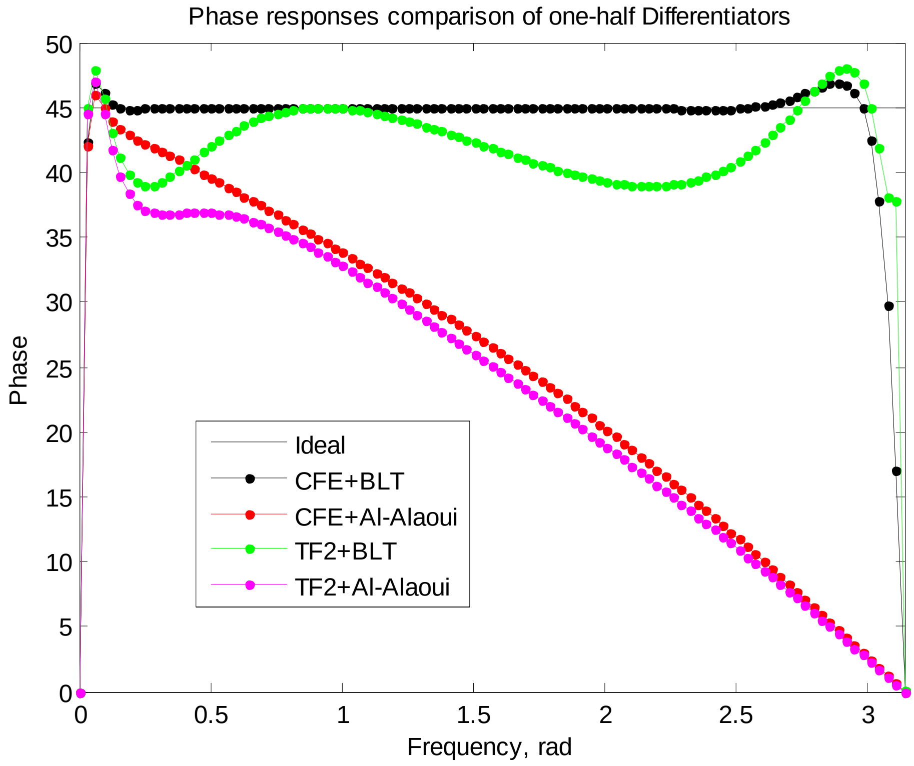

Fig. 2. Phase response of half order differentiators

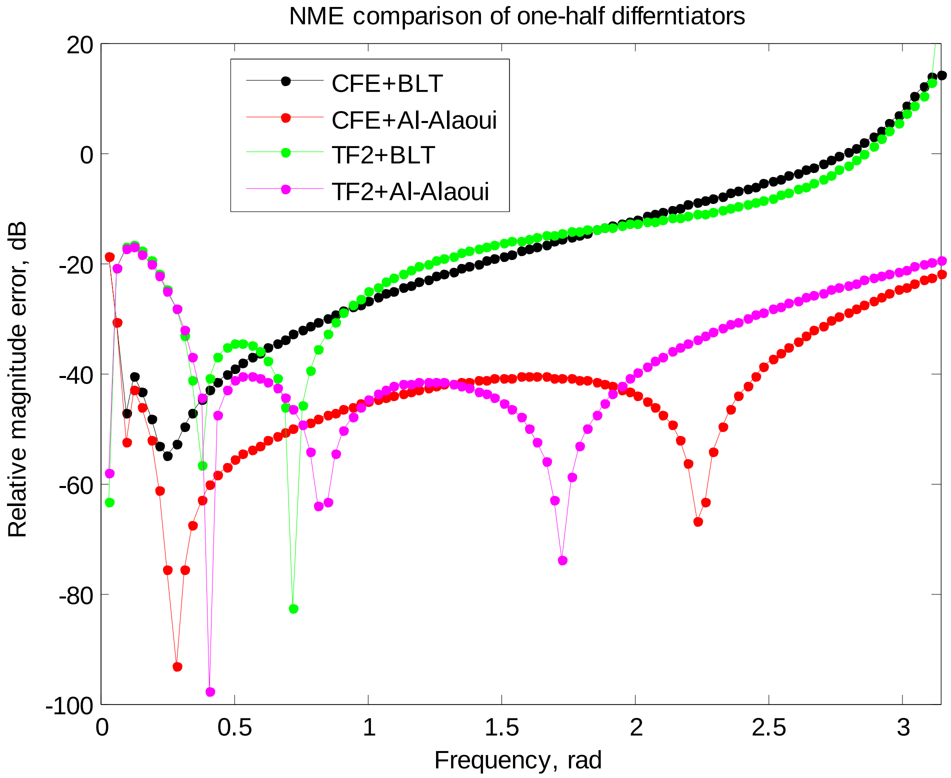

Fig. 3. NME plot of the half differentiators

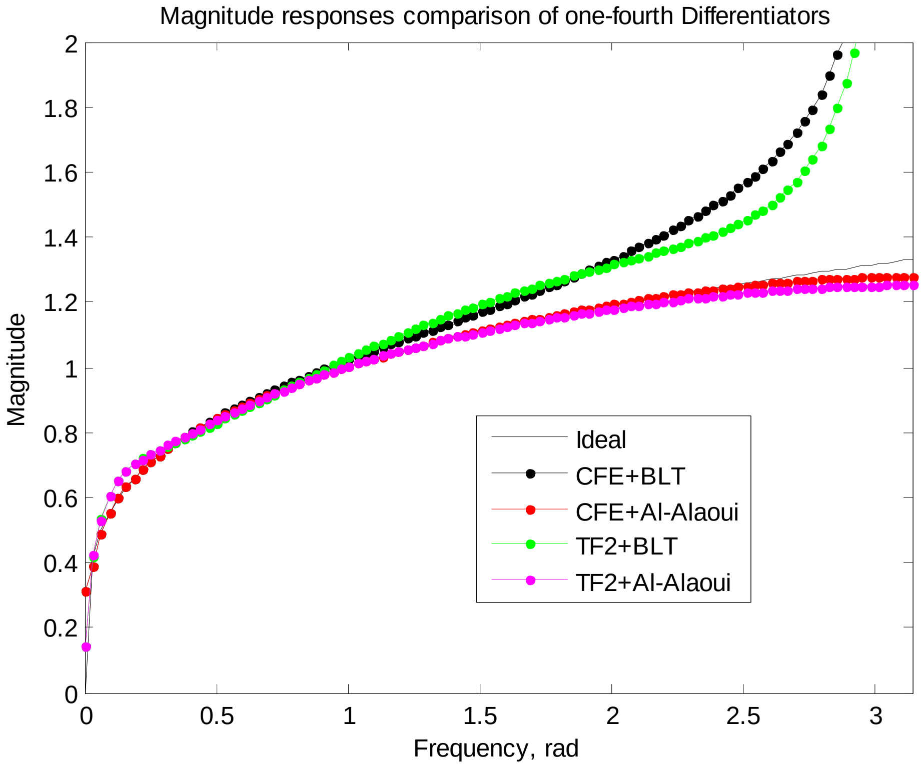

Fig. 4. Magnitude response of one-fourth differentiators

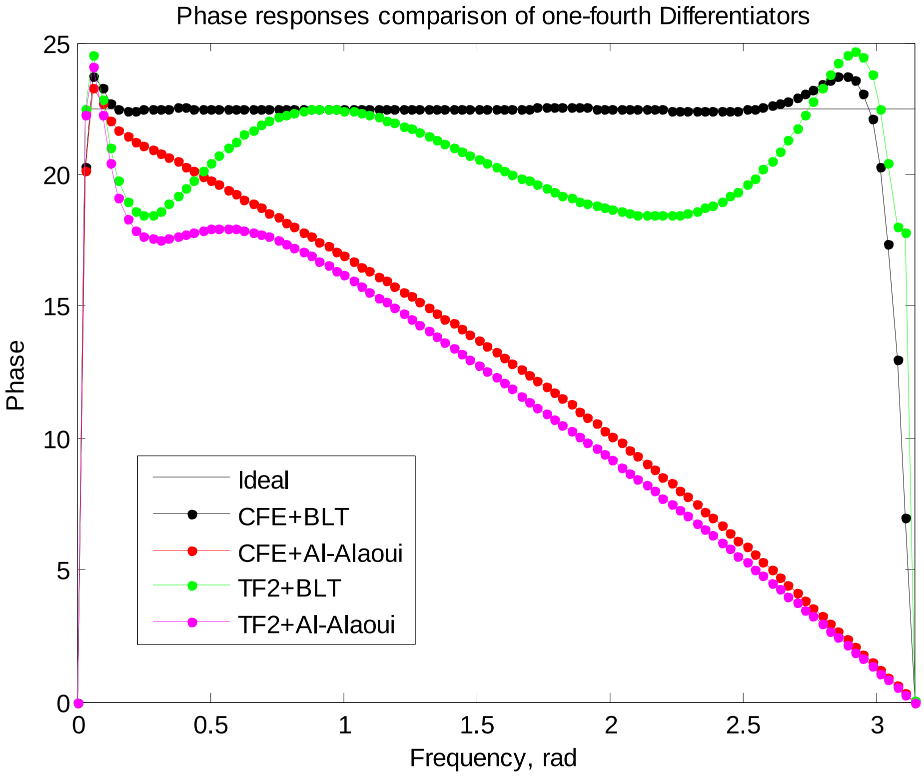

Fig. 5. Phase response of one-fourth differentiators

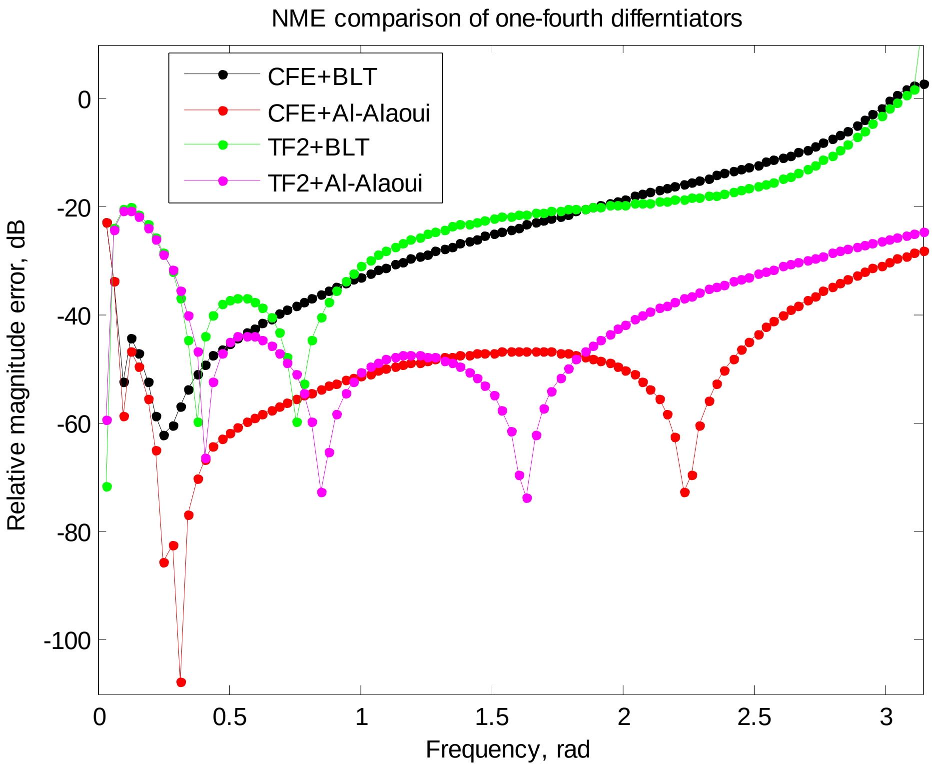

Fig. 6. NME plots of one-fourth differentiators

4. Discussion

The fractional order digital differentiators of order 1/2 and 1/4 are obtained by use of indirect discretization using Tustin and Al-Aloui operators. The magnitude, phase and normalized magnitude error plots are obtained by using MATLAB simulation software. The transfer function (TF2) approximation with the application of Al-Aloui transform are more accurate upto the fraequency range of 2 rad compared to the Tustin ans CFE with Al-Aloui based ones.

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