Analyzing Potential Source Response of the Stochastic Differential Equation of RLC Circuit driven by Lyapunov Techniques

M. Maheswari [1, 2, a)], D. Piriadarshani[1, b)] N. Geetha Lakshmi[3,c)] and N. Daniya Nishi[4,d)]
[1)] *Department of Mathematics, Hindustan Institute of Technology and Science, Padur, Chennai, India.*
[2)] *Department of Mathematics, Anna Adarsh College for Women, Anna nagar, Chennai, India.*
[3)] *Department of BCA, Anna Adarsh College for Women, Anna nagar, Chennai, India.*
[4)] *Transportation BU, L& T Technology and Services*, India.

Abstract: In this paper, we delve into the investigation of the necessary condition for the stochastic stability of an RLC circuit by employing the Lyapunov method, a rigorous mathematical approach widely utilized in dynamical systems analysis. Beyond merely identifying stability criteria, we delve deeper into understanding the behavioral intricacies inherent in the circuit's dynamics. To provide a comprehensive analysis, we manually derive the range of parameter values that satisfy the stability criterion, ensuring accuracy and reliability. Additionally, we implement PYTHON coding to computationally confirm our findings, thereby validating the analytical results with numerical simulations. The culmination of our efforts is visually represented through a graph, effectively showcasing the consistent outcomes obtained from both analytical and computational methodologies.

Keywords: *Stochastic Differential Equations, Brownian motion, Lyapunov Method, stability, RLC Circuit, PYTHON.*

1. Introduction

The concept of stability of dynamic systems which was introduced by A.M. Lyapunov is crucial in analyzing their behavior. He introduced two approaches: the indirect method, which assesses stability through the system's time response, and the direct method, which does not rely on the system's time response for analysis. The Lyapunov direct method is particularly effective for studying nonlinear and time-varying systems. This method focuses on the equilibrium points of a system and ensures that future states of the system can remain arbitrarily close to equilibrium by choosing initial conditions close enough. In [1] Mao, X. provides a concise overview of the contents and significance of the work. It likely touches upon the fundamental concepts of stochastic differential equations (SDEs) and their practical applications. SDEs are mathematical models used to describe systems subject to random fluctuations, making them crucial in various fields including finance, physics, biology, and engineering. Xuerong Mao [2] analyzed the mathematical theory behind the concept of exponential stability for SDEs concerning semimartingales and established conditions under which such stability can be achieved. Exponential stability is a crucial property indicating the rapid decay of solutions towards equilibrium or desired behavior. These equations can be seen as stochastic perturbed systems and exhibit exponential stability almost surely. C.Zeng, et.al.,[3] focus on two essential stability aspects: almost sure stability, which refers to the convergence of asset prices to a stable equilibrium with probability one, and moment stability, which involves the convergence of higher moments of the asset price distribution. Examining these stability properties aims to enhance our understanding of financial market dynamics and provide insights into option pricing under fractional calculus. Caibin Zeng et al. [4] have contributed to the development of stochastic Lyapunov techniques tailored for stochastic differential equations (SDEs) propelled by Fractional

Brownian Motion (FBM). Their research explores more into the theory of stochastic stabilization and destabilization and on many issues that arise in systems governed by such SDEs. In the investigation of the stability characteristics of SDEs, Marie Kilmesova [5] Marie Kilmesova aims to comprehend the circumstances in which the solutions to these equations either converge to a steady state over time or always stay constrained. An essential component of understanding dynamic behavior is stability analysis which sheds light on the behavior and performance of these systems over the long run. J.F.G. Aguilar [6] examined the properties and actions of several electrical components, including cap-resistors, memcapacitors, and memristors. The response from electrical circuits with resistor-capacitor (RC) and RL that are represented by fractional differential equations serves as the foundation for this study. Gomez-Aguilar et.al, [7] examine how fractional operators with bi-order affect the dynamic responses and stability characteristics of RC and RL circuits. The work of Atangana et al. [8] advances our theoretical knowledge of fractional calculus as it relates to electrical engineering. The authors shed light on the intricate dynamics RC, LC, and RL circuits by examining the impact of Atangana-Baleanu fractional derivatives. They also suggest potential remedies for reducing unfavourable consequences including instability and signal distortion. Sene et.al, [9] seeks to provide insights into the behavior of these circuits under various operating conditions and external perturbations. The study aims to analyze the stability properties of RLC circuits described by the Caputo-Liouville generalized fractional derivative, and it will provide their dynamic behavior and ensure their reliable operation in practical systems. P.Uma maheswari et.al, [10] aims to establish the existence of solutions for Caputo fractional SDEs with Lévy noise and analyze their stability properties, and it also providing insights into their long-term behavior and robustness against perturbations. For an understanding of the paper we present below the preliminaries in Section 2, a necessary condition for the existence of Stochastic Stability of RLC Circuit in Section 3, and concluding the analysis in section 4. Readers are also requested to utilize the PYTHON CODING to determine the range of V(t), given in Appendix section.

2. Preliminaries

Definition 2.1: The m-dimensional Brownian motion denoted by $W(t) = (W_1(t), W_2(t), \ldots, W_m(t))$ describes the stochastic process where each component $W_i(t)$ represents the movement of a particle in the i^{th} dimension over time t and let $b: [0,T] \times R^s \to R^s$ and $\rho : [0,T] \times R^s \to R^{s\times m}$ be measurable functions. Then the process $Y(t) = (Y_1(t), Y_2(t), \ldots . Y_m(t)), t \in [0,T]$ is the solution of SDE,

$$dY_t = b(t,Y_t)dt + \sigma(t,Y_t)dW_s \quad -------- (2.1)$$

Where $b(t, Y_t) \in R$, $\sigma(t, Y_t)W_s \in R$

$$Y_t = Y_0 + \int_0^t b(t,Y_s)ds + \int_0^t \sigma(s,Y_s)dW_s --(2.2)$$

For any initial value $Y_t(0) = Y \in R^n$, given the assumption of a unique global solution denoted by $Y(t; t_0, X_0)$, the equilibrium position of the stochastic process defined by equation (2.1) is the solution corresponding to $Y_t(0) = 0$.

Definition 2.2: For a function V (t, x) defined over the domain D, it is considered positive definite, if there exists a positive definite function W: D → R such that W(x) is lower bound for V (t, x) for all (t, x) ∈ R × D.

Definition 2.3: A function V (t,x) defined on domain D is characterized as negative definite, if there exists a positive definite function W: D → R such that W(x) is upper bound for V (t, x) for all (t, x) ∈ R × D.

Definition2.4: For any real parameter μ, a continuous non negative function V(x,t) defined over the domain S_h and for $t \geq t_0$ is deemed to be decreasing if it adheres to the inequality

$$V(x,t) \leq \mu(|x|), \quad \forall (x,t) \; in \; S_h \times [t_0, \infty)$$

Where S_h is the domain and t_0 is the initial time.

Definition 2.5: A mapping V(y, t) defined on the space $R^m \times [t_0, \infty)$ is considered radically unbounded if,

$$\lim_{|y|\to\infty} \inf_{t\geq t_0} V(y,t) = \infty$$

Definition 2.6: For any chosen values of δ ∈ (0, 1) and r > 0, there exists a corresponding ε = ε (δ, r, t_0) > 0 such that the trivial solution of equation (2.1) demonstrates stochastically stability or instability depending upon the following condition holds,

$$P\{|y(t, t_0, y_0| < r\} \geq 1 - \delta \quad \forall\, t \geq t_0 \ whenever \ |y_0| < \varepsilon$$

Definition 2.7: For every δ ∈ (0, 1) there exists an ε = ε (δ, r, t_0) > 0 ensuring the trivial solution of equation (2.1) is said to be stochastically asymptotically stable then,

$$P\left\{\lim_{t\to\infty} y(t: t_0, y_0) = 0\right\} \geq 1 - \delta \quad whenever \ |y_0| < \varepsilon$$

Definition 2.8: The trivial solution of equation (1) is considered stochastically asymptotically stable on a large scale if it maintains stochastic stability and converges to zero for any initial condition $y_0 \in R^m$, then

$$P\left\{\lim_{t\to\infty} y(t: t_0, y_0) = 0\right\} = 1 \ \ whenever \ |y_0| < \varepsilon$$

Definition 2.9: In order to access stochastic stability, we replace the inequality $\dot{V}(x,t) \leq 0 \ with \ LV(x,t) \leq 0$
Where L associated with equation (2.1) and is defined as

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial y_i}(t, Y_t) b_i(y,t) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial y_i \partial y_j} [\sigma(y,t)\sigma^T(y,t)]_{ij}$$

Here, the differential operator L will allows us to analyze the stability of the system.
If $LV(x,t) \leq 0$, then the system is stochastically stable.
Definition 2.10: Lyapunov quadratic function V is given by V (Y_t) = Y_t^T P Y_t
Here, P is the positive definite symmetric matrix.
Lemma 2.1: Let v (t) be a differentiable scalar signal for which $\dot{v}(t) \leq \mu v(t), for\ every\ t \geq t_0$
For some constant μ ∈ ℜ, then

$$v(t) \leq e^{\mu(t-t_0)}\, v(t_0), \ \ for\ every\ t \geq t_0$$

3. Necessary Condition For Existence Of Stochastically Stability Of Rlc Circuit:
Theorem 3.1:
If V(X (t)) constitutes a Lyapunov quadratic function and $V(x,t) \in C^{2,1}(S_h \times [t_0, \infty); R_+)$ such that $LV(X_t) \leq 0$ then the trivial solution of (3.3) attains stochastic stability.
Proof:
The general equation of RLC Circuit is,

$$LQ''(t) + RQ'(t) + \frac{1}{C} Q(t) = U(t) \text{ -------- } (3.1)$$

At the time t, Q (t) denotes the charge at a fixed point in an electric circuit, while U (t) represents the potential source. The initial conditions are Q (0) = Q_0, Q ′(0) = I_0, Where Q_0 is the initial charge and I_0 is the initial derivative of charge.
By adding a noise terms in equation (3.1), it becomes,

$$U^*(t) = U(t) + \text{noise}$$

$$L\frac{d^2Q}{dt} + R\frac{dQ}{dt} + \frac{1}{R} Q(t) = U(t) + \propto \varepsilon_t \ ----(3.2)$$

Where $\propto \varepsilon_t$ represents the intensity of noise.
Now, the matrix form of equation (3.2) is,

$$dX(t) = (AX(t)\, H(t))\, dt + K\, dB(t) \text{---------} (3.3)$$

Where, $dX = \begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ \frac{-1}{LC} & \frac{-R}{L} \end{bmatrix}$, $H(t) = \begin{bmatrix} 0 \\ \frac{V(t)}{L} \end{bmatrix}$, $K(t) = \begin{bmatrix} 0 \\ \frac{\alpha}{L} \end{bmatrix}$ --------(3.4)
Here, B (t) is 1-dimensional Brownian motion.
According to definition (2.10) of system (3.3), we compute the Lyapunov function,
Let $V(X_t) = X_t^T P X_t$
dV(X_t) = V(X_t + dX_t) - V(X_t)

$$= (X_t^T + dX_t^T)P(X_t + dX_t) - X_t^T P X_t$$

$$= [X_t^T + (AX(t)dt + H(t)dt + KdB(t))^T]P\big[X_t + \big(AX(t)dt + H(t)dt + KdB(t)\big)\big] - X_t^T P X_t$$

$$= X_t^T PX_t + X_t^T PAXdt + X_t^T PH(t)dt + X_t^T KPdB(t) + A^T X^T PX_t dt + A^T X^T PAX\, dt.dt + A^T X^T PH(t)dt\, dt + A^T X^T KP\, dt\, dB(t) + H^T P\, X_t\, dt + H^T P\, AX\, dt\, dt + H^T P\, H(t)dt\,.dt + H^T KP\, dB(t)dt + K^T P\, X_t\, dB(t) + K^T P\, AX\, dt\, dB(t) + K^T P\, H(t)dt\, dB(t) + K^T KP\, dB(t).dB(t) - X_t^T PX_t$$

We know that, dt. dt = dB(t). dt = dt. dB(t) = 0, dB(t).dB(t) = dt and also E[dB(t)]=0

$$dV(X_t) = X_t^T P\, AXdt + X_t^T PH(t)dt + X_t^T KPdB(t) + A^T X^T P\, X_t\, dt + H^T P\, X_t\, dt + K^T PX_t\, dB(t) + K^T KP\, dt$$

Now, taking expectation we get,

$$E[dV(X_t)] = [X_t^T P\, AX + X_t^T PH(t) + A^T X^T P\, X_t + H^T P\, X_t + K^T KP]\, dt = LV(X_t)dt$$

$$LV(X_t) = [X_t^T P\, AX + X_t^T PH(t) + A^T X^T P\, X_t + H^T P\, X_t + K^T KP] --- (3.5)$$

Equation (3.5) represents the Lyapunov stability equation of RLC Circuit.

Set P = I, where the positive symmetric matrix P is replaced by identity matrix I with order 2x2.

$$LV(X_t)dt = [X_t^T AX + X_t^T H(t) + A^T X^T X_t + H^T X_t + K^T K]\, dt$$

$$LV(X_t) = [X_t^T AX + X_t^T H(t) + A^T X^T X_t + H^T X_t + K^T K]$$

$$LV(X_t)dt = [A + X_t^T H(t) + A^T + H^T X_t + K^T K] --- (3.6)$$

Using equation (3.4) we get,

$$LV(X_t) = \begin{bmatrix} 0 & 1 - \frac{1}{CL} \\ 1 - \frac{1}{CL} & \frac{-2R}{L} \end{bmatrix} + \left[\frac{\alpha^2}{L^2}\right] + 2[V(t)] --- (3.7)$$

If LV(X_t) is negative definite, i.e LV (X_t) ≤ 0

$$\begin{bmatrix} 0 & 1 - \frac{1}{CL} \\ 1 - \frac{1}{CL} & \frac{-2R}{L} \end{bmatrix} + \left[\frac{\alpha^2}{L^2}\right] + 2[V(t)] \le 0 ---- (3.8)$$

Equation (3.8) represents the necessary condition for the existence of stochastically stability of the RLC circuit using Lyapunov method. Using (3.8) we can find the potential source of the RLC Circuit,

$$2[V(t)] \le -\begin{bmatrix} 0 & 1 - \frac{1}{CL} \\ 1 - \frac{1}{CL} & \frac{-2R}{L} \end{bmatrix} - \left[\frac{\alpha^2}{L^2}\right]$$

$$[V(t)] \le -\frac{1}{2}\left[\begin{bmatrix} 0 & 1 - \frac{1}{CL} \\ 1 - \frac{1}{CL} & \frac{-2R}{L} \end{bmatrix} + \left[\frac{\alpha^2}{L^2}\right]\right]$$

$$|V(t)| \le -\frac{1}{2}\left[\begin{vmatrix} 0 & 1 - \frac{1}{CL} \\ 1 - \frac{1}{CL} & \frac{-2R}{L} \end{vmatrix} + \left|\frac{\alpha^2}{L^2}\right|\right]$$

We get,

$$|V(t)| \le \frac{1}{2C^2L^2}[C^2(\alpha^2 - L^2) + 2CL - 1] ---- (3.9)$$

Example:

Suppose L = 3H, C = 2F, α = 4,

Then RHS of equation (3.9) becomes,

$$|V(t)| \le \frac{1}{2C^2L^2}[C^2(\alpha^2 - L^2) + 2CL - 1]$$

$$|V(t)| \le \frac{1}{2\times 4\times 9}[4(4^2 - 3^2) + 2 \times 2 \times 3 - 1]$$

$$|V(t)| \leq \frac{1}{72}[4(16-9)+12-1]$$
$$|V(t)| \leq \frac{1}{72}[39]$$
$$|V(t)| \leq 0.54166 \leq 1$$

Therefore, the inequality holds only for V (t) = 0, V (t) = 0.5 & V (t) = 1 and it is not satisfied for V(t) = 1.5 consistent, with the specified conditions. Therefore, the inequality (3.9) has been examined for an electrical circuit characterized by L = 3H, C = 2F, α = 4 for various values of voltage V (t) ensuring that the absolute value of the voltage remains within specified bounds. However, it is not satisfied for V (t) = 1, where the absolute value of the voltage exceeds the RHS of the inequality. Therefore, the inequality is valid for certain ranges of voltage values, ensuring that the circuit's behavior remains within specified bounds under given conditions.

Determination of Range of V (t):

The range of V (t) that satisfies the given inequality, we first need to solve the inequality for $|V(t)|$. let's rearrange the inequality

$$|V(t)| \leq \frac{1}{2C^2L^2}[C^2(\alpha^2-L^2)+2CL-1]$$
$$|V(t)| \leq \frac{[C^2(\alpha^2-L^2)+2CL-1]}{2C^2L^2}$$

Now, we know that $|V(t)|$ must be less than or equal to the right-hand side of the inequality. Therefore, the range of V (t) that satisfies the inequality is:

$$0 \leq |V(t)| \leq \frac{[C^2(\alpha^2-L^2)+2CL-1]}{2C^2L^2}$$

Substituting the values for C = 2, L = 3 and α = 4, we can calculate the upper bound for V (t).

$$|V(t)| \leq \frac{[2^2(4^2-3^2)+2\times2\times3-1]}{2\times3^2\times2^2}$$
$$|V(t)| \leq \frac{[4(16-9)+12-1]}{2\times9\times4}$$
$$|V(t)| \leq \frac{[4\times7+11]}{72}$$
$$|V(t)| \leq 0.5417$$

So, the range of V (t) that satisfies the given inequality is 0 ≤ V (t) ≤ 0.5417. Within this range, the inequality holds, ensuring that the absolute value of V (t) remains within the specified bounds which are also shown in the Figure.1.

4. Conclusion:

This paper establishes the systematically explored stochastic stability of RLC circuits through rigorous mathematical analysis and computational simulations. By utilizing the Lyapunov method, we have derived the necessary conditions for the stochastic stability of the circuit, as demonstrated by Equation (3.9). The Lyapunov stability equation (3.5) provides insight into the behavior of the system under stochastic influences, indicating the range of voltage values that ensure stability. Furthermore, by setting the positive definite matrix P to

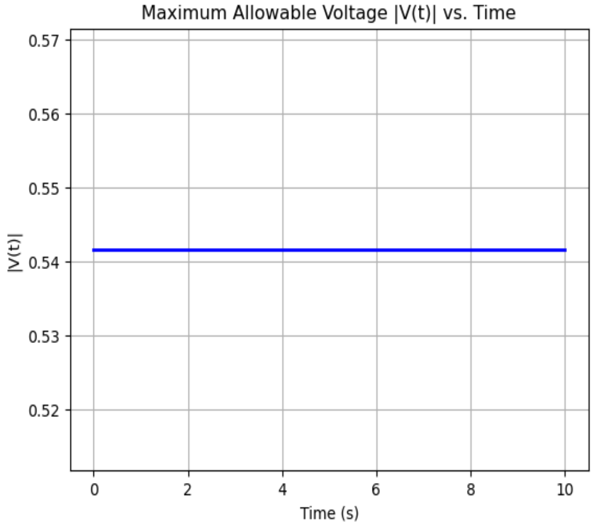

Figure 1: Range of V (t)

the identity matrix I, we have simplified the stability analysis, as shown in equation (3.9). Through an illustrative example with specific parameter values, we have verified the validity of the derived inequality and determined the range of voltages that guarantee stability. The PYTHON coding further supports our analytical findings by computationally confirming the maximum allowable voltage over time. Overall, this study contributes to the understanding of stochastic stability in electrical circuits, offering valuable insights for designing reliable and robust systems. Future research could extend these findings to more complex circuits and explore applications in various engineering domains. By enhancing our understanding of stochastic dynamics, we can pave the way for advancements in system reliability and efficiency, ultimately benefiting diverse industries and technological innovations.

4.1. Real time Applications:

Electrical Grid Stability, Electronic circuit design, Renewable Energy Systems, Smart Grid Technologies and Control systems etc.

4.2. Future Enhancement:

By implementing these enhancements, you can make the code more robust, reliable, and user-friendly, ultimately improving its utility and effectiveness in practical applications.

Appendix: PYTHON Coding to Determine the Range of V(t):

```
import numpy as np
import matplotlib.pyplot as plt

# Define the parameters
C = 2 # Capacitance (in Farads)
L = 3  # Inductance (in Henrys)
alpha = 4  # Alpha value

# Calculate the right-hand side (RHS) of the inequality
RHS = 1 / (2 * C**2 * L**2) * (C**2 * (alpha**2 - L**2) + 2 * C * L - 1)

# Display the RHS value
print('RHS of the inequality', RHS)
```

```
# Define the time range for plotting (optional)
t = np.linspace(0, 10, 1000)  # Time vector (0 to 10 seconds)

# Calculate the maximum allowable voltage |V(t)|
V_max = RHS * np.ones_like(t)

# Plot the maximum allowable voltage |V(t)| over time
plt.figure()
plt.plot(t, V_max, 'b-', linewidth=2)
plt.xlabel('Time (s)')
plt.ylabel('|V(t)|')
plt.title('Maximum Allowable Voltage |V(t)| vs. Time')
plt.grid(True)  # Turn on the grid
plt.show()
```

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