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Evolutionary Dynamic of Quarter Car model: Regular and Chaotic Evolution

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Abstract: Suspension systems in cars is very important for comfort driving and safety of passengers. Automobile industries design various suspension systems depending on the type of vehicle. Quarter-car models have been in use for years to study car dynamics. The objective of this article is to study the dynamics of evolution of a quarter-car model with each suspension system subjected to linear damping. The equation of motion of the system is described with a single degree of freedom. Stability criteria of steady states discussed analytically for the dimensionless equation in detail. Numerical simulation was performed to explore bifurcation phenomena and to obtain various regular and chaotic attractors. Lyapunov exponents, for chaotic and regular motions, calculated and presented through graphics. Numerical calculations extended to obtain Poincaré surfaces of sections and Poincaré maps which are significant to analyze evolutionary motion and clear indicators of regularity and chaos.

Keywords- Chaos, Lyapunov Exponents, Bifurcation, Attractors

Introduction

Suspension system in cars is very important ingredient for automobiles for comfort safety and comfort driving of the vehicle. The automobile industries design various suspension systems depending on the type of vehicle. Quarter-car models, with variant design of suspension system, are in use for years to study the car dynamics. Such vehicle models have applications in problems related to ride dynamics. The quarter car model comprises of a car body (sprung mass) attached to four wheels. Each suspension system is presumptively equipped with an active linear spring, linear 3force actuator, and linear viscous damper. Wide description of quarter car model is given in recent articles (Agrawal et al. 2013 [1]; Al-Mutar and Abdalla 2015 [2]; Viswanath & Allam, 2016 [3]; Verros et al., 2000 [4]; Gobbi and Mastinu, 2001[5]; Von Wagner, 2004 [6]; Turkay and Akcay, 2005 [7]; Verros et al., 2005 [8]; Litak et al., 2007a, b, c [9], [10], [11]; Li et al., 2004 [12]). These pioneer articles relating to vehicle vibrations due to a rough road profile are of significant interest. These motivate researchers to investigate more on dynamics of such models and to obtain every possibility of appearance of regular and chaotic responses.

Regular and chaotic motions of a dynamical system can be identified by various tools such as phase plots, power spectrums, time series curves etc. But confirm identification of such motion given by Lyapunov exponents, (LCEs), which actually provides a measure of chaos and regularity. For LCE <0 a motion is regular and that for chaos LCE >0. Thus, Lyapunov exponents are perfect indicator of regularity and chaos (Benettin et al., 1980 [13]; Bryant et al., 1990 [14]; Grassberger and Procaccia, 1983a, b [15], [16]; Saha et al., 2014 [17]; Sandri 1996 [18]; Kumra, Neha [19]; Saha, L. M., Das, M. K. 2021) [20].

The objective of the present article is to perform extended investigations on the dynamics of evolution of a quarter car model and to discuss various aspects of regular and chaotic motion. We analyze such observed motions analytically as well as numerically. The quarter-car model presented here is subject to linear damping and the equation of motion is described by a second order equation with a single degree of freedom. The equilibrium points, (fixed points), of the model obtained and their stability criteria have been established. In the processes of numerical simulation work, we obtained plots of regular and chaotic attractors for set of different parameter values. Also, Poincaré surfaces of sections and Poincare maps are drawn for different cases and plots of Lyapunov exponents obtained for proper identification of regular and chaotic motion within certain ranges of a particular parameter. In the final conclusion, we summarized the investigated results.

2. Description of Quarter Car Model: Equilibrium solutions

Following earlier descriptions, (Agrawal et al. 2013 [1]; Al-Mutar and Abdalla 2015[2]; Viswanath & Allam, 2016 [3]; Verros et al., 2000 [4]; Gobbi and Mastinu, 2001 [5]; Von Wagner, 2004 [6]; Turkay and Akcay, 2005 [7]; Verros et al., 2005) [8], a brief description presented in this section for the proposed quarter-car model. The load imposed to a car when moving forward and describes the reaction of this car when a torque is imposed on one of its tractive forces. The effects of different forces opposed to the vehicle displacement are given by (i) the aerodynamic drag, (ii) the anti-rolling resistance and (iii) the slope resistance. Kinematic excitation with nonlinear damping and stiffness is shown in figure 1.

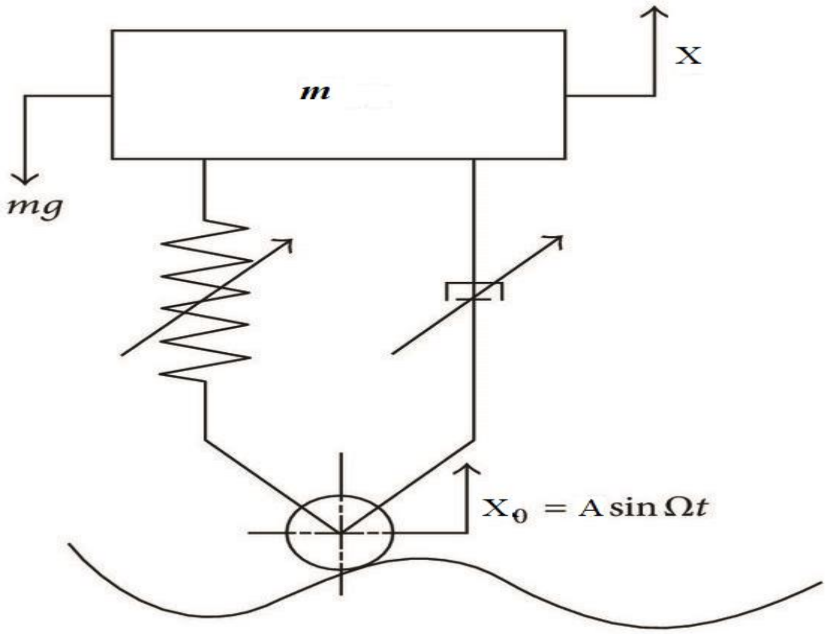


Figure 1: Description of Quarter-car model.

The equation of motion of the quarter-car model with a single degree of freedom, (Litak et al., 2007a), is

$$m\frac{d^2x}{dt^2} + k_1(x - x_0) + mg + F_h\left(\frac{d}{dt}(x - x_0), (x - x_0)\right) = 0. \quad (1)$$

In this instance F_h denoted an additional hysteretic suspension damping and stiffness force, is given by

$$F_h\left(\frac{d}{dt}(x - x_0), (x - x_0)\right) = k_2\,(x - x_0)^3 + c_1\frac{d}{dt}(x - x_0) + c_2\left(\frac{d}{dt}(x - x_0)\right)^3 \quad (2)$$

and

$$x_0 = A\,sin(\Omega\, t) \quad (3)$$

Put a relative displacement as

$$y = x - x_0 \quad (4)$$

then Equations (1) – (4) implies equation

$$\frac{d^2y}{dt^2} + \omega^2 y + B_1\,y^3 + B_2\frac{dy}{dt} + B_3\left(\frac{dy}{dt}\right)^3 = -g + A\Omega^2 sin\Omega t \quad (5)$$

where $\omega^2 = \frac{k_1}{m}$, $B_1 = \frac{k_2}{m}$, $B_2 = \frac{c_1}{m}$, $B_3 = \frac{c_2}{m}$.

Parameter estimation of equation (5), already provided by Li et al. 2004, [12] as

$k_1 = 160000\ N/m$, $k_2 = -300000\ N/m^3$, $m = 240\ kg$, $c_1 = -250\ N_s/m$, $c_2 = 25\ N_s^3/m^3$.

Introducing without dimension form with time scale re-scaled $\tau = \omega t$, we make changes

$$k = \frac{B_1}{\omega^2} = \frac{k_2}{k_1},\ \alpha = \frac{B_2}{\omega} = \frac{c_1}{\sqrt{k_1 m}},\ \beta = B_3\omega = c_2\sqrt{\frac{k_1}{m^3}},\ g' = \frac{g}{\omega^2} \text{ and } \Omega' = \frac{\Omega}{\omega},$$

Then, writing Ω'as Ω and replacing variable y by x, equation (5) takes the dimensionless form

$$\ddot{x} + x + kx^3 + \alpha\dot{x} + \beta\dot{x}^3 = -g' + A\ \Omega^2 sin\Omega\tau \tag{6}$$

Where, over dots indicating derivatives with respect to dimension time τ.

Homogeneous part of equation (6) is

$$\ddot{x} + x + kx^3 + \alpha\dot{x} + \beta\dot{x}^3 = 0 \tag{7}$$

Let us re-write equation (7) as

$$\frac{dx}{dt} = y \equiv f(y,u)$$

$$\frac{dy}{dt} = -x - kx^3 - \alpha y - \beta y^3 \equiv g(x,y)\,. \tag{8}$$

The equilibrium points of this system given by

$$(0,0) \text{ and } \left(\pm\sqrt{-\frac{1}{k}}\,,0\right).$$

Corresponding Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 - 3kx^2 & -\alpha - 3\beta y^2 \end{pmatrix} \tag{9}$$

(i) Eigenvalues corresponding to equilibrium point $(0,0)$ given by

$$\lambda_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4}}{2}.$$

Thus, stability of $(0,0)$ depends on values of α and is stable if $\alpha \geq 0$ and unstable if $\alpha < 0$.

(ii) Eigenvalues corresponding to equilibriums $\left(\pm\sqrt{-\frac{1}{k}}\,,0\right)$ given by

$$\lambda_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 + 8}}{2}.$$

Thus, both the points of equilibriums are unstable saddle.

Dynamic, regular and chaotic, evolution of the quarter car model depends on the display of orbits initiating nearby an equilibrium point. In this regard criteria of stability are very significant.

3. Numerical Simulations

3. 1 Bifurcations and Attractors

Fixing parameters as $\alpha = -0.04034, \beta = 3.68957, g = 0.014715, k = -1.875$ and $A = 0.5$ of equation (6) and varying parameter Ω, set of regular and chaotic attractors obtained, figure 2. As Ω increases from value 0.5, system bifurcates and orbits of periods $1, 2, 4, 3, 6$ and then chaotic orbits visible here.

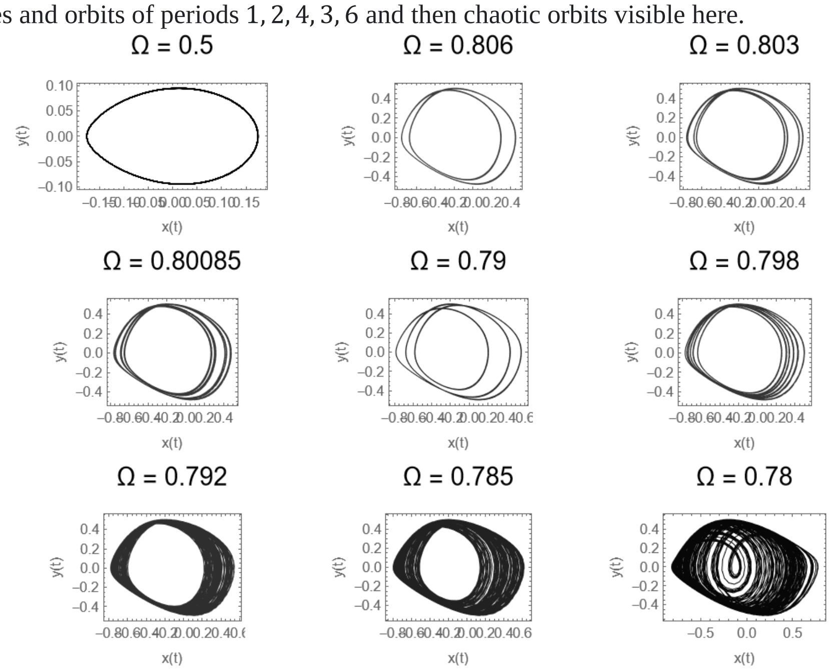


Figure 2: Regular and chaotic attractors obtained for different values of Ω. Values of other parameters are $\alpha = -0.04034, \beta = 3.68957, g = 0.014715, k = -1.875$ and $A = 0.5$.

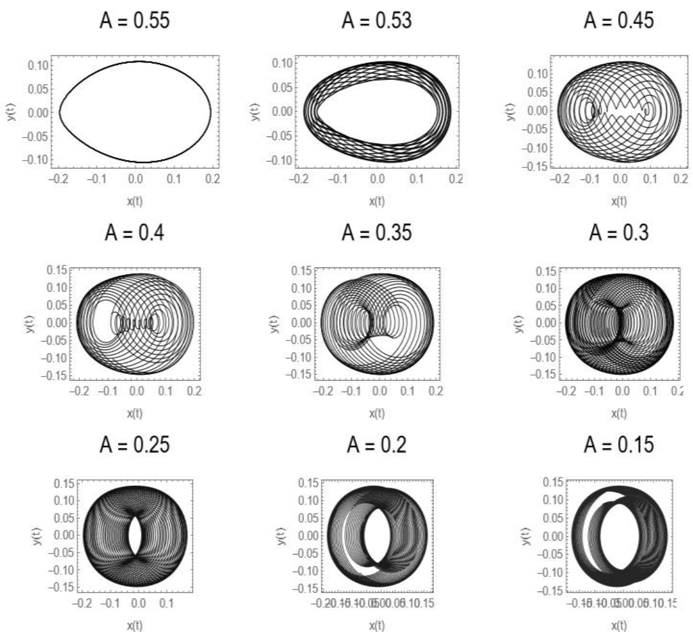


Figure 3: Attractors obtained for different values of A. values of other parameters are $\alpha = -0.04034, \beta = 3.68957, k = -1.875,\ g = 0.014715$ and $\Omega = 0.5$.

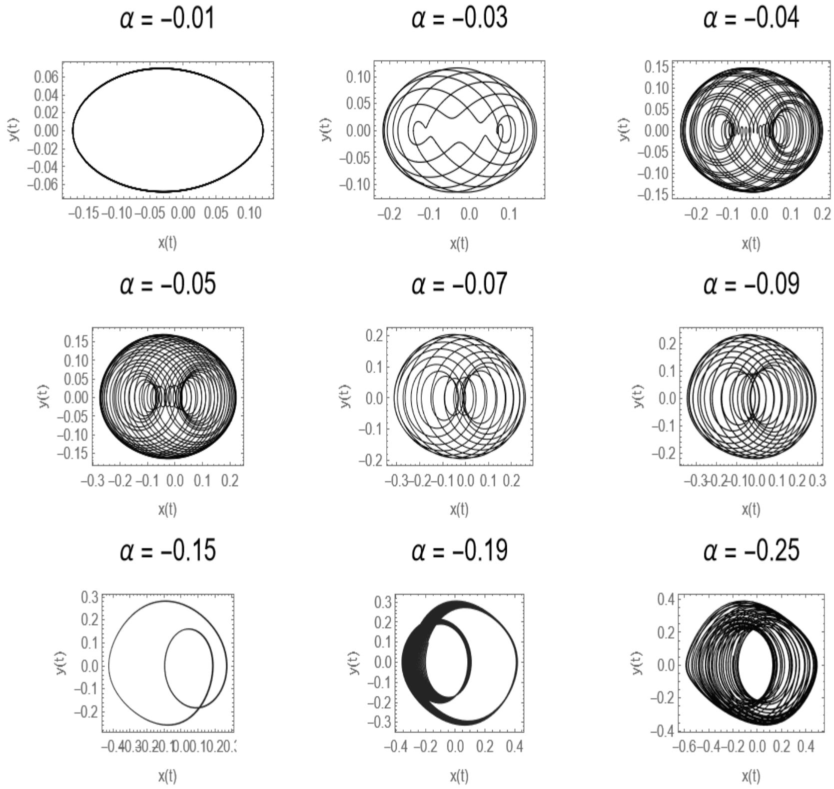
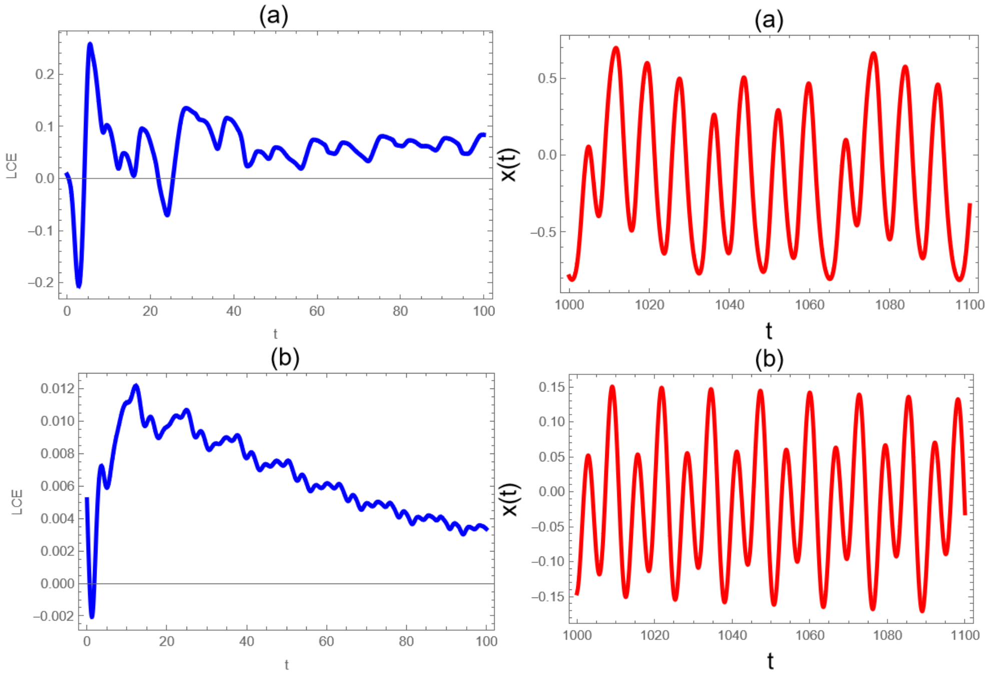


Figure 4: Attractors obtained for different values of α. Values of other parameter values are as $A = 0.55, k = -1.875, \beta = 3.68957, g = 0.014715$ and $\Omega = 0.45$.

3. 2 Lyapunov Exponents

Lyapunov exponents for three chaotic cases are calculated and displayed in figure 5.



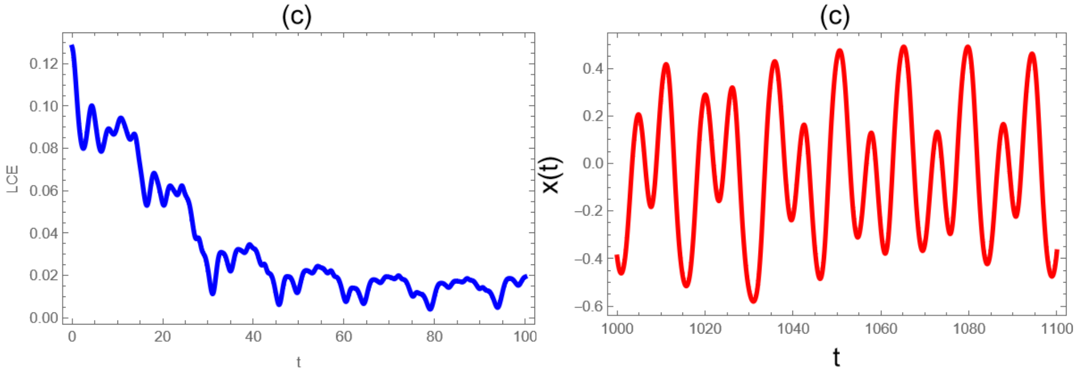
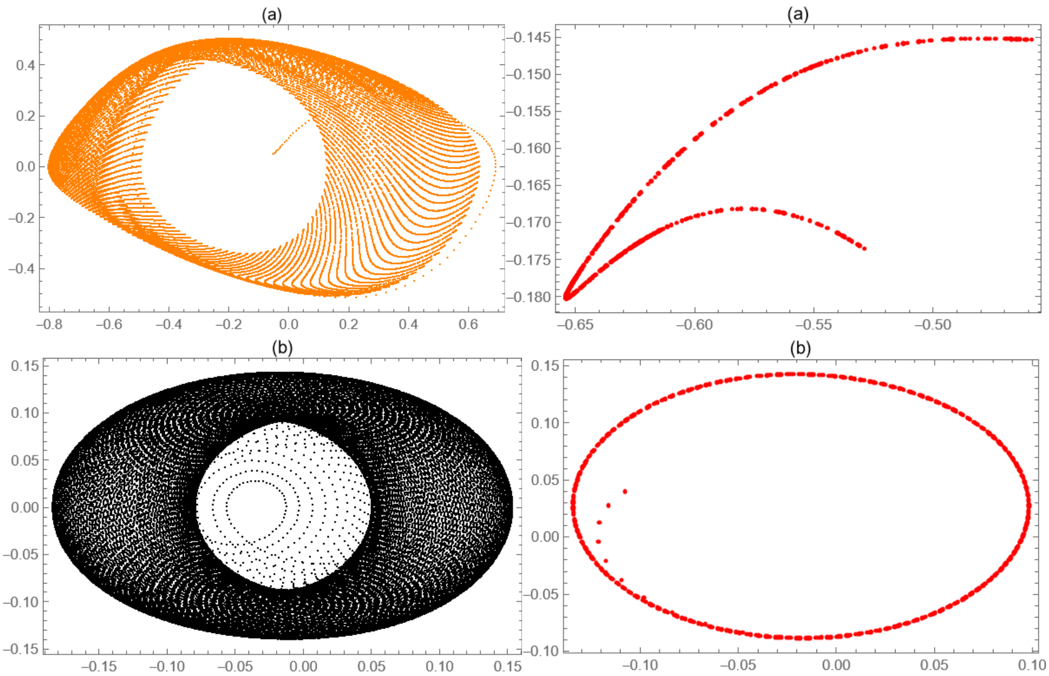


Figure 5: Plots of Lyapunov exponents (left figures) and time series for three chaotic cases. Parameters values are $\alpha = -0.04034, \beta = 3.68957, g = 0.014715, k = -1.875$ and for case (a) $A = 0.5, \Omega = 0.78$, $\alpha = -0.04034$; case (b) $A = 0.15, \Omega = 0.5$, $\alpha = -0.s04034$; case (c) $A = 0.55, \Omega = 0.45$, $\alpha = -0.25$.

3.3 Poincaré Map

The Poincaré surfaces of sections and Poincaré map, named on the French mathematician Henri Poincaré, are the transversal intersection of an n-dimensional state space with an (n-1)-dimensional subspace. These maps are known as the perfect indicators of regular and chaotic evolutions of a dynamical system. The state space of these maps can be compared to the state space of the original continuous dynamical systems, which are discrete dynamical systems with one fewer dimension. A discrete dynamical system with a state space that is one dimension smaller than the initial continuous dynamical system can be viewed as such a map. In case of three-dimensional orbit such a map is transversal intersection by a plane. On Poincaré map, for a one periodic orbit only one point appears, for a two periodic orbit only two points appear, for an n −periodic orbit n points appear and for quasi-periodic orbit, a dotted closed curve appears. For chaotic motion, one observes randomly distributed points on Poincaré map.

For three chaotic cases of evolution of quarter-car system, equation (8), Poincaré surfaces of sections and their corresponding Poincaré maps obtained and presented shown in Figure 6.



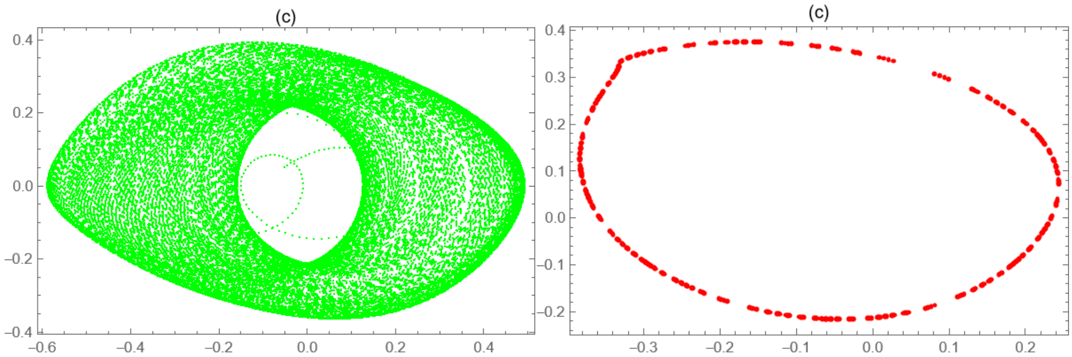


Figure 6: Poincaré surfaces of sections and Poincaré maps obtained for three cases of chaotic motion. Parameters values are $\alpha = -0.04034, \beta = 3.68957, g = 0.014715, k = -1.875$ and case (a) $A = 0.5, \Omega = 0.78$, $\alpha = -0.04034$; case (b) $A = 0.15, \Omega = 0.5$, $\alpha = -0.04034$; case (c) $A = 0.55, \Omega = 0.45$, $\alpha = -0.25$.

4. Conclusion

Results obtained on evolutionary dynamics on evolution of a quarter-car system are significant. Stability of equilibrium solutions for governing non-dimensional of equation (6) discussed which provides behavior of evolution as parameter of the system changes. The sets of regular and chaotic attractors drawn by varying parameters Ω, A and α are shown through figures 2 to 4. The value of other parameters was kept fixed while changing a particular parameter. Numerical simulations with proper codes applied to calculate Lyapunov exponents (LCEs) for three different chaotic cases as shown in figure 5. To have more clarity of motions, (regular or chaotic), Poincaré surfaces of sections and Poincaré maps are drawn and presented in figure 6. These two are perfect indicatiors for identification of regular and chaotic motions. All numerical simulation work was performed by using Mathematica.

The problem may be more interesting if it is described with the suspension system subject to nonlinear damping instead of linear damping. In order to control chaos in the quarter-car model as well as in other nonlinear models, one must adopt an appropriate chaos control strategy.

Conflict of Interest

The authors have no conflict of interest to declare for publication of this paper.

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