Linear Stability Analysis of Onset of Convection in a Couple Stress Fluid Saturated Porous Layer Using Thermal Non-Equilibrium Model

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Abstract: Linear stability ofcouple stress fluid saturated porous medium heated from below is studied when the fluid and solid phases are not in local thermal equilibrium. The Darcy model which includes couples stress parameter and permeability is employed as a momentum equation. The critical Rayleigh number for the onset of convection using linear stability analysis is found numerically as a function interphase heat transfer coefficient, aspect ratio. It is found that a small interphase heat transfer coefficient has significant effect on the stability of the system. The effect of porosity modified conductivity ratio, diffusivity ratio,couple stress parameter, interphase heat transfer coefficient on the stability of the system is investigated.

Keyword: Darcy model, thermal non–equilibrium model,critical Rayleigh number,Solutal Rayleigh number.

Introduction

Fluids with suspended particles are a working medium in many an industrial application. The fluids most often do not subscribe to a Newtonian description and the advent of micro momentum field theories in such a situation threw open new fields of application, such as polymeric suspensions, animal blood, liquid crystals which have very small sized suspended particles of different shapes has been reported in last few years. These particles may change shape, may shrink and expand, and moreover, they may execute rotation independent of the rotation and movement of the fluid. Most practical problems involving these types of working fluids are non-isothermal and these thermally responding fluids have uncovered new application areas. Convection is a dominant and important mode of heat transport in many such applications.

Sharma and Sharma (2004) have studied the Rayleigh-Benard convection in a couple stress fluid saturated porous layer. The effect of thermal and gravity modulation on the onset of thermal convection in a Boussinesq couple stress fluid has been studied by Malashetty and Basavaraja (2005) using linear theory. They have used a regular perturbation method to obtain the correction Rayleigh number and wave number, which characterize the stability of the system. They found that the couple stress parameter has a significant effect on the regulation of convection.

In modeling a fluid-saturated porous medium most of the investigations performed assumed a state of local thermal equilibrium (LTE) between the fluid and the solid phase at any point in the medium. This is a common practice for most of the studies where the temperature gradient at any location between the two phases is assumed to be negligible. For many practical applications, involving high-speed flows or large temperature differences between the fluid and solid phases, the assumption of local thermal equilibrium is inadequate and it is important to take account of the thermal non-equilibrium effects. Due to applications of porous media theory in drying, freezing of foods and other mundane materials and applications in everyday technology such as

microwave heating, rapid heat transfer from computer chips via use of porous metal foams and their use in heat pipes, it is believed that local thermal non-equilibrium theory will play a major role in future developments.
Recently, attention has been given to the LTNE model in the study of convection heat transfer in porous media. Much of this work has been reviewed in recent books by Ingham and Pop (1998) and Nield and Bejan (2006) Criteria for heat and mass transfer models in metal hydride packed beds has been investigated by Kuznetsov and Vafai (1995) and effects of non-equilibrium were suggested to be more significant at high Reynolds number and for high porosity. Kuznetsov (1996) studied a perturbation solution for a thermal non-equilibrium fluid flow through a three-dimensional sensible storage packed bed. Vafai and Amiri (1998) give detailed information about the work on thermal non-equilibrium effects of fluid flow through a porous packed bed. The review of Kuznetsov (1998) gives detailed information about the works on thermal non-equilibrium effects on internal forced convection flows.
Nield and Bejan (2006) have discussed a two-field model for energy equation. Instead of having a single energy equation, which describes the common temperature of the saturated porous media, two equations are used for fluid and solid phases separately. In two-field model, the energy equations are coupled by the terms, which account for the heat lost to or gained from the other phase. Rees and co-workers (Rees and Pop,2005, 2000 Banu and Rees, 2002) in a series of studies have investigated thermal non-equilibrium (LTNE) effect on free convective flows in a porous medium.
In this chapter we study the onset of convection in a couple-stress fluid saturated porous layer heated from below with emphasis on how the condition for the onset of convection is modified when the solid and fluid phase are not in local thermal equilibrium (LTNE). When non-equilibrium effects are included in the problem the linear analysis is slightly modified and it is still possible to proceed analytically to find the condition for the onset of convection. We have also carried out the asymptotic analysis for very small and very large values of the inter phase heat transfer coefficient. This work is more general in the sense that we recover their results in the limit as the couple-stress parameter C tends to zero.

Mathematical Formulation

We consider a couple-stress fluid saturated porous layer of depth d, which is heated from below and cooled from above. The lower surface is held at a temperature T_1, while the upper surface is at T_u. We assume that the solid and fluid phases of the medium are not in local thermal equilibrium and use a two-field model for temperatures. It is assumed that at the bounding surfaces the solid and fluid phases have identical temperatures. The basic governing equations are

$$\nabla.\mathbf{q} = 0, \tag{1}$$

$$\rho_f\left[\frac{1}{\varepsilon}\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{\varepsilon^2}\mathbf{q}.\nabla\mathbf{q}\right] = -\nabla p - \frac{1}{K}(\mu_f - \mu_c\nabla^2)\mathbf{q} + \rho_f\mathbf{g}, \tag{2}$$

$$\varepsilon(\rho c)_f\frac{\partial T_f}{\partial t} + (\rho c)_f\mathbf{q}.\nabla T_f = \varepsilon k_f\nabla^2 T_f + h(T_s - T_f), \tag{3}$$

$$(1-\varepsilon)(\rho c)_s\frac{\partial T_s}{\partial t} = (1-\varepsilon)k_s\nabla^2 T_s - h(T_s - T_f), \tag{4}$$

$$\rho_f = \rho_o[1-\beta(T_f - T_u)]. \tag{5}$$

We confine ourselves to two-dimensional motions. Further, the boundaries are assumed to be free-free and isothermal.

Basic State

The basic state is assumed to be quiescent and we superimpose a small perturbation on it. Eqs.(9.1)–(9.5) are now made dimensionless using following transformations

$$(x,y)=d(x^*,y^*),\qquad (u,v)=\frac{\varepsilon k_f}{(\rho c\,)_f d}(u^*,v^*),\quad p=\frac{k_f \mu}{(\rho c\,)_f K}p^*, \tag{6}$$

$$T_f=(T_1-T_u)\theta+T_u,\qquad T_s=(T_1-T_u)\phi+T_u,\qquad t=\frac{(\rho c)_f d^2}{k_f}t^*.$$

We eliminate the pressure from the momentum transport equation (9.2) and arrive at the vorticity transport equation. The non-dimensional form of the vorticity and the heat transport equations are obtained in the form

$$\frac{Da}{\Pr}\left[\frac{\partial}{\partial t}\nabla^2\psi-J(\psi,\nabla^2\psi)\right]=Ra\frac{\partial\theta}{\partial x}-(1-C\nabla^2)\nabla^2\psi\ , \tag{7}$$

$$\frac{\partial\theta}{\partial t}-J(\psi,\theta)=\nabla^2\theta+H(\phi-\theta), \tag{8}$$

$$\alpha\frac{\partial\phi}{\partial t}=\nabla^2\phi+\gamma H(\theta-\phi)\ , \tag{9}$$

where ψ is the stream function, which is related to u and v by

$$u=-\frac{\partial\psi}{\partial y}\ ,\quad v=\frac{\partial\psi}{\partial x}$$

$\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the two dimensional Laplacian operator and J is the Jacobean. The asterisks have been removed for simplicity.

The non-dimensional groups that appear in the above equations are

$$Ra=\frac{\rho_0 g\beta(T_1-T_u)Kd}{\varepsilon\,\mu_f\kappa_f},\qquad \gamma=\frac{\varepsilon k_f}{(1-\varepsilon)k_s},\qquad H=\frac{hd^2}{\varepsilon k_f}, \tag{10}$$

$$\alpha=\frac{(\rho c)_s}{(\rho c)_f}\frac{k_f}{k_s}=\frac{\kappa_f}{\kappa_s},\qquad C=\frac{\mu_e}{\mu_f d^2},\qquad \Pr=\frac{\mu_f\varepsilon(\rho c)_f}{\rho_0 k_f}\ ,\qquad Da=\frac{K}{d^2}.$$

In Eq.(9.10), Ra is the Darcy-Rayleigh number which expresses the balance between buoyancy and viscous forces, γ is the porosity modified conductivity ratio, H is the non-dimensional inter phase heat transfer coefficient, α is the diffusivity ratio, Da is the Darcy number , Pr is the Prandtl number and C is the couple stress parameter.

We note that, the fluid and solid phases are not in thermal equilibrium, the use of appropriate boundary conditions for the temperature fields may pose a difficulty. However we assume that the phases have equal temperatures at the bounding surfaces. Therefore Eqs.(9.7)-(9.9) are solved for stress–free isothermal boundaries and hence the boundary conditions are

$$\psi=\frac{\partial^2\psi}{\partial y^2}=0\qquad \text{on}\quad y=0\ \text{ and }\ 1, \tag{11}$$

$$\theta=\phi=0\qquad \text{on}\quad y=0\ \text{ and }\ 1. \tag{12}$$

Linear Stability Theory.

We assume that the equilibrium state is subjected to infinitesimal perturbations. To study the linear theory we use the linearized version of Eqs.(9.7)–(9.9). The principle of exchange of stabilities may be proved easily so that the onset of convection is stationary.

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We seek the solutions to the linearized equations in the form

$$\psi = A_1 \sin \pi y \cos ax, \qquad (13)$$

$$\theta = A_2 \sin \pi y \sin ax, \qquad (14)$$

$$\phi = A_3 \sin \pi y \sin ax, \qquad (15)$$

where a is the horizontal wave number and the A's are constants. Substitution of Eq.(13) –(15) into the linearized version of the Eqs.(9.7)–(9.9) yields the following matrix equation

$$\begin{pmatrix} (\pi^2+a^2)[C(\pi^2+a^2)+1] & aRa & 0 \\ a & (a^2+\pi^2+H) & -H \\ 0 & \gamma H & -(a^2+\pi^2+\gamma H) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (16)$$

For non-trivial solution, the determinant of the Co-efficient matrix must be zero.

$$Ra = \left(\frac{(\pi^2+a^2)^2}{a^2} + \frac{C(\pi^2+a^2)^3}{a^2} \right) \left(1 + \frac{H}{a^2+\pi^2+\gamma H} \right) . \qquad (17)$$

The value of Rayleigh number Ra given in Equation (9.17) can be minimized with respect to the wavenumber a by setting $\frac{\partial Ra}{\partial a} = 0$ and solving the resulting equation. However, in the present case it is highly impossible to obtain a straight-forward closed form expression for the minimizing value of a. therefore, we use Newton-Raphson iteration scheme to obtain the minimum values of Ra and a as function of H and γ .

Asymptotic Analysis:

Case 1: For very small values of H

When H is very small the critical value of the Rayleigh number Ra is slightly above the critical value for the LTE case. Accordingly we expand Ra given by Eq.(17) in a power series in H as

$$Ra = \left(\frac{(\pi^2+a^2)^2}{a^2} + \frac{C(\pi^2+a^2)^3}{a^2} \right) + \left(\frac{(\pi^2+a^2)}{a^2} + \frac{C(\pi^2+a^2)^2}{a^2} \right) H \\ - \left(\frac{1}{a^2} + \frac{C(\pi^2+a^2)}{a^2} \right) \gamma H^2 + \ldots \quad . \qquad (18)$$

To minimize Ra up to $O(H^2)$ we set $\partial Ra/\partial a = 0$ and obtain an expression of the form

$$(4Ca^6 + (2+6C\pi^2)a^4 - (2\pi^4 + 2C\pi^6) + (2Ca^4 - 2\pi^2 - 2C\pi^4)H \\ +(2+2C\pi^2)H^2\gamma = 0. \qquad (19)$$

We also expand a in power series of H as

$$a = a_0 + a_1 H + a_2 H^2 + \ \ldots \ , \qquad (20)$$

where a_0 is the critical wave number for the LTE case and is given by the expression

$$a_0^{\,2} = \frac{-(C\pi^2+1)+\sqrt{(C\pi^2+1)\ (9C\pi^2+1)}}{4C}. \tag{21}$$

Substituting Eq.(21) into Eq.(20) and rearranging the terms and then equating the coefficients of same powers of *H* will allow us to find a_1 and a_2. Thus we obtain

$$a_1 = \frac{\Delta_1}{\Delta}, \tag{22}$$

$$a_2 = \frac{\Delta_2}{\Delta}, \tag{23}$$

where

$$\Delta_1 = \pi^2 + C\pi^4 - C a_0^{\,4},$$

$$\Delta_2 = (2+2C\pi^2)\gamma + (60C a_0^{\,4} + 12a_0^{\,2} + 36C a_0^{\,2}\pi^2)a_1^{\,2} + 8C a_0^{\,3}a_1,$$

$$\Delta = 12C a_0^{\,5} + 4a_0^{\,3}C + 12C\pi^2 a_0^{\,3}.$$

With these values of a_0, a_1 and a_2, Eq.(20) gives the critical wavenumber and consequently using this in (18) one can obtain the critical Rayleigh number for small *H*.

Case 2: For very large values of H

For large values *H*, the critical Rayleigh number takes the form

$$Ra_c = \left(\frac{(\pi^2+a^2)^2}{a^2} + C\frac{(\pi^2+a^2)^3}{a^2}\right)\left(\frac{1+\gamma}{\gamma} - \frac{(\pi^2+a^2)}{\gamma^2}H^{-1} + \frac{(\pi^2+a^2)^2}{\gamma^3}H^{-2}\right). \tag{24}$$

We minimize this with respect to *a* in a similar way as we did in the small *H* case and obtain the following expression

$$\begin{aligned}&(8Ca^{10} + 6(5C\pi^2+1)a^8 + 4(10C\pi^4+4\pi^2)a^6 + 2(10C\pi^6+6\pi^4)a^4 - 2\pi^8 - 2C\pi^{10})H^{-2}\\ &-(6\gamma C a^8 + 4\gamma(C\pi^2+3C\pi^2\gamma)a^6 + 2\gamma\pi^2(3+2C\pi^2)a^4 - 2\gamma\pi^6(1+C\pi^2))H^{-1}\\ &+4C(\gamma^2+\gamma^3)a^6 + 2(1+3C\pi^2)(\gamma^2+\gamma^3)a^4 - \pi^4 - C\pi^6 = 0.\end{aligned}$$

(25)

Similarly we expand a in the form

$$a = a_o + \frac{a_1}{H} + \frac{a_2}{H^2} + \ldots\ , \tag{26}$$

where a_0 is given by Eq.(9.21) and a_1, a_2 are to be found.

Substituting Eq.(9.26) into Eq.(9.25) and equating the coefficients of like powers of *H* we can find a_1 and a_2 and are given by

$$a_1 = \frac{\Delta_1'}{\Delta'}\ , \tag{27}$$

$$a_2 = \frac{\Delta_2'}{\Delta'}, \tag{28}$$

where

$$\Delta_1' = 6C\gamma a_0^8 + 4\gamma(C\pi^2 + 3C\gamma\pi^2)a_0^6 + 2\gamma(3\pi^2 + 6C\pi^4)a_0^4 - 2\gamma\pi^2(\pi^4 + C\pi^6)\ ,$$

$$\begin{aligned}\Delta_2' &= ((48C\gamma a_0^7 + 24\gamma(C\pi^2 + 3C\gamma\pi^2)a_0^5 + 8\gamma(3\pi^2 + 6C\pi^4)a_0^3)a_1 - ((8Ca_0^{10} + 6(5C\pi^2 + 1)a_0^8 \\ &+ 4(10C\pi^4 + 4\pi^2)a_0^6 + 2(10C\pi^6 + 6\pi^4)a_0^4 - 2\pi^8 - 2C\pi^{10}) - (60C(\gamma^2 + \gamma^3)a_0^4 \\ &+ 12(1 + 3C\pi^2)(\gamma^2 + \gamma^3)a_0^2)a_1^2\ ,\end{aligned}$$

$$\Delta' = 24C(\gamma^2 + \gamma^3)a_0^5 + 8(1 + 3C\pi^2)(\gamma^2 + \gamma^3)a_0^3\ .$$

Again with these values of a_0 , a_1 and a_2 , we compute the critical wavenumber a_c from Eq.(26) and finally using this value of the a_c , one can obtain the critical Rayleigh number Ra_c from Eq.(26) for large H. The expression for the critical Rayleigh number Ra and the critical wavenumber a_c for both small H and large H are evaluated for fixed value of $C = 2$ and comparison of these values with the exact values obtained earlier are given in Table 1 & 2. It is important to note that an excellent agreement between these two results are found when H is small. On the other hand reasonably good agreement is found when H is large.

Table 1. Comparison of Exact and Asymptotic solutions for small values of H

$\log_{10}H$	a_c(E)	a_c(A)	Ra_c(E)	Ra_c(A)
$\gamma = 1$				
-3.5	2.25708	2.25708	701.703	701.703
-3.0	2.25711	2.25711	701.736	701.736
-2.5	2.25719	2.25719	701.837	701.837
-2.0	2.25745	2.25745	702.157	702.157
-1.5	2.25828	2.25829	703.168	703.168
-1.0	2.26085	2.26092	706.344	706.344
-0.5	2.26857	2.26940	716.186	716.180
0.0	2.28935	2.30999	745.444	745.336
0.5	2.33033	3.81790	823.002	825.578
$\gamma = 0.01$				
-3.5	2.25708	2.25708	701.703	701.703
-3.0	2.25711	2.25711	701.736	701.736
-2.5	2.25719	2.25719	701.837	701.837
-2.0	2.25745	2.25745	702.158	702.158
-1.5	2.25829	2.25829	703.171	703.171
-1.0	2.26090	2.26092	706.375	706.375
-0.5	2.26905	2.26925	716.487	716.487
0.0	2.29363	2.29564	748.293	748.300
0.5	2.36144	2.38311	847.380	848.079

Conclusions

The stability of a horizontal couple stress fluid saturated porous layer heated from below when the solid and fluid phases are not in local thermal equilibrium is examined analytically. Darcy model is used for the momentum equation and a two-field model is used for energy equation each representing the solid and fluid phases separately. The condition for the onset of convection is obtained analytically.

The effect of increasing conductivity ratio γ is to decrease the critical Rayleigh number and hence the effect of increasing γ is to destabilize the system. The effect is more pronounced for very small γ. The critical Rayleigh number is independent of γ for very small *H* while for large *H*, it decreases with increasing γ.

It is found that the critical wave number a_c approaches a common limit as $H \to 0$ and $H \to \infty$ in the Newtonian limit while it approaches two different limits, one as $H \to 0$ and another as $H \to \infty$ in the non-Newtonian regime. We found an excellent agreement between the exact solutions and the solutions obtained from asymptotic analysis. We find that the results of the LTE case are recovered in the large *H* limit and the results of Darcy regime are recovered in the small limit C.

References

[1] D.A. Nield and A. Bejan, *Convection in Porous Media*, 3rd ed. (Springer-Verlag, New York, 2006).

[2] *Transport Phenomena in Porous Media*, edited by D.B. Ingham and I. Pop (Pergamon *Emerging Topics in Heat and Mass Transfer in Porous Media*, edited by P.Vadasz (Springer, New York, 2008).

[3] O. M. Phillips, *Geological Fluid Dynamics*, (Cambridge University Press, 2009).

[4] V. Steinberg and H. Brand, "Convective instabilities of binary mixtures with fast chemical reaction in a porous medium," J. Chem. Phys. **78** (5), 2655 (1983).

[5] D. Pritchard and C. N. Richardson, "The effect of temperature-dependent solubility on the onset of thermosolutal convection in a horizontal porous layer," J. Fluid Mech. **571**, 59 (2007).

[6] L. Storesletten, in *Transport Phenomena in Porous Media*, edited by D.B. Ingham, and I. Pop, (Pergamon Press, Oxford, 1998), pp. 261-283.

[7] S.N. Gaikwad, M.S. Malashetty and K.R. Prasad, "An analytical study of linear and nonlinear double diffusive convection in a fluid saturated anisotropic porous layer with Soret effect," Appl. Math. Modell. **33**, 3617 (2009).