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Note on Studying Multistep Multiderivative Methods and it's Application to Solve Ordinary Differential Equations

Vagif Ibrahimov [1], Mehriban Imanova [2]

[1] *Computational Mathematics, Baku State University,*
Institute of Control Systems Named after Academician

[2] *Science Development Foundation under the President of the Republic of Azerbaijan*

Abstract:- As is known Multistep Multiderivative Methods are investigated for a very long time. Recently some modifications of these methods have been constructed. These methods are successfully applied to solve initial-value problems for the Ordinary Differential and Volterra integro-differential Equations of any order and also to solving Volterra integral equations. Here, have investigated application of the Multistep Third Derivative Method to solve initial value problems for the ODEs of first, second and third orders. Shown that the Multistep Methods with the constant coefficients can be presented in different forms, depending on the object of research. By study these problems have found some necessary conditions for the stability of the above noted Multistep Multiderivative Methods. It is known that one and the same numerical methods can be applied to solve initial-value problems for ODEs of the different orders. Here, it has been proven that in the application of one and the same Multistep Multi Derivative Method to solving initial-value problems for the ODEs of the same orders, the maximal value for its accuracy depends from the values of some coefficients can be receive different values. This idea is illustrated by using the initial-value problem for Ordinary Differential Equations with special structure. A classic representative of such problems is the initial value problem for the ODEs of the second order. Here, fully studied this problem, and have investigated the extended version of the named problem. And also have considered investigating some relation between different types of Multistep Multiderivative Methods with constant coefficients. By using these relations, have defined some connection for the degree and order of stable methods.

Keywords: *Initial-Value Problem (IVP), Ordinary differential Equations (ODEs) of the 1st -3rd orders, Stability and Degree, Multistep Multiderivative Methods, Bilateral Method.*

1. Introduction

One of the most popular Mathematical Models is the model that uses the initial-value problem for Ordinary Differential Equations with the different orders, which are used in all areas of Natural science. Therefore, the result obtained in this direction is of interest for the many scientists who work in the application of Mathematical Methods in solving ODEs different areas of natural sciences. For the illustration of this, let us to consider the following problem:

$$y' = f(x, y), \;\; y(x_0) = y_0, \;\; x_0 \le x \le X. \tag{1}$$

For the investigation the numerical solution of this problem, let us suppose that the solution $y(x)$ of the problem (1), is continuous and has been defined at the interval $[x_0, X]$. But the continuous to totality of arguments function of $f(x, y)$ is defined in some closed set in which has the continuous partial derivatives up to p, inclusively. For the finding the approximately values of the solution of the problem (1), let us define that by the

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y_i and the corresponding exact value by the $y(x_i)\ (i=0,1,...,N)$. Here the mesh points x_{i+1} are defined as the $x_{i+1}=x_i+h\ \ (i=0,1,..,N-1)$, where $0<h$ is the step-size. One of the popular Multistep Method for solving problem (1), can be presented as follows (see for example [1]-[11]):

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=h\sum_{i=0}^{k}\beta_i f_{n+i},\ \ n=0,1,..,N-k, \tag{2}$$

here $f_m=f(x_m,y_m)\ (m=0,1,2,..,N)$ and $\alpha_k\neq 0$.

The condition of $\alpha_k\neq 0$ related with the finding the solution of the equation (2) (so as y_{n+k} is the solution of the equation(2)). It is ese to understand that to find the value y_{n+k}, one can use the following method (see for example [12]-[18]):

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=h\sum_{i=0}^{k}\beta_i f_{n+i}+h^2\sum_{i=0}^{k}\gamma_i g_{n+i};$$
$$n=0,1,2,..,N-k,\ \alpha_k\neq 0. \tag{3}$$

Here the function $g(x,y)=f'_x(x,y)+f'_y(x,y)f(x,y)$.

By above described way, receive that, methods (2) and (3) can be applied to solving problem (1). In this case, arises question about that, which these methods is better. To answer this question for the comparison of the method (2) and (3), let us begin from the value of Computational work. By using the calculation of the values of function $g(x,y)$,receive that the value of Computational work in using of method (3) is increases twice. Noted that the order of accuracy for the methods receiving from the class of (3), also, is increases. For the comparison of the methods (2) and (3), let us to consider the following section.

2. §1. On some connection between the methods (2) and (3).

As is known one of the conception for the comparison of multistep methods is the degree, which can be defined as the following form:

Definition 1. Integer value p is called as the degree for the method of (3), if the following holds (see for example []):

$$\sum_{i=0}^{k}\left(\alpha_i y(x+ih)-h\beta_i y'(x+ih)-h^2\gamma_i y''(x+ih)\right)=$$
$$=O(h^{p+1}),h\to 0. \tag{4}$$

It is not difficult to define the value of p by using asymptotic equality of (4) for the method (2). For this aim, it's enough to put $\gamma_i=0\ \ (i=0,1,..,k)$ in the equality of (4).

And now, let us to comprise methods (2) and (3) by the using the maximum value of the degree of these methods. Noted that for the method (2) the maximum value of the order of accuracy $p_{\max}=2k$, but for the method (3), $p_{\max}=3k+1$. By simple comparison, receive that method (3) is more exact than the method (2). However, let us note that, not all methods with maximum accuracy are convergent. Therefore, scientists have explored convergent Multistep Methods and determined that for the convergence of these methods the necessary conditions are their stability. And now let us to consider the definition of the conception of stability (see for example [19]-[31]).

Definition 2. The method (2) is called as stable if the roots of the polynomial $\rho(\lambda)=\alpha_k\lambda^k+\alpha_{k-1}\lambda^{k-1}+...+\alpha_1\lambda+\alpha_0$ are located in the unit circle and on the boundary of which, there are not multiple roots.

As was noted above, method (2) can be received from the method (3) as a special case. But definition of stability for the method (3) doesn't match the definition of 2 and can be presented as following.

Definition 3. Method (3) is called as stable, if the roots of the polynomial $\rho(\lambda)$ in the case $|\beta_k|+|\beta_{k-1}|+...+|\beta_0|\neq 0$, located in the unit circle and on the boundary of which, there are not multiple roots, if $|\beta_k|+|\beta_{k-1}|+...+|\beta_1|+|\beta_0|=0$, then method (3) is called as the stable if the roots of the polynomial $\rho(\lambda)$ lie in the unite circle on the boundary of which, there are not multiple roots, except double root $\lambda=1$.Noted that in this case, the definition of degree can be given as following.

Definition 4. Integer values p, in the case $\beta_i=0\ (i=0,1,..,k)$ is called as the degree for the method (3), if the following holds:

$$\sum_{i=0}^{k}\left(\alpha_i y(x+ih)-h^2\beta_i y''(x+ih)\right)=O(h^{p+2}),\ h\to 0. \tag{5}$$

As was it noted above, method (2) is derived from the method (3) as a special case. But the properties of these methods are not completely the same (compares the asymptotic equalities (4) and (5)).

Comment. Methods (2) and (3) have been investigated by many authors and have received some connection between degree p and order k. Noted that if method (2) is stable and has degree of p, then $p\leq 2[k/2]+2$ and for all the k, there is method with the degree $p_{max}=2[k/2]+2$. If the method (2) is explicit, that is $\beta_k=0$, then $p\leq k$. But if the method (3) is stable, then $p\leq 2k+2$ for the case $|\beta_k|+|\beta_{k-1}|+...+|\beta_1|+|\beta_0|\neq 0$ and $p\leq 2[k/2]+2$ for the case $\beta_i=0\ (i=0,1,..,k)$. It follows from here that the method (3) can be taken as the better, so as the stable methods of type (3) are more exact.

Let us for the construction more exact Multistep Multi derivative Methods, to consider Multistep third derivative Methods.

3. §2. Multistep Third derivative Methods with constant coefficients.

It is known that the Multistep Third derivative Method in one version can be presented as following (see for example [39]-[]):

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=\delta_1 h\sum_{i=0}^{k}\beta_i y'_{n+i}+\delta_2 h^2\sum_{i=0}^{k}\gamma_i y''_{n+i}+h^3\sum_{i=0}^{k}l_i y'''_{n+i}. \tag{6}$$

It is not verify that this method can be applied to solve the problem (1) and the following:

$$y'''=F(x,y,\delta_3 y',\delta_4 y''),\ \ y(x_0)=y_0;$$
$$y'(x_0)=y'_0;\ \ y''(x_0)=y''_0,\ x_0\leq x\leq X. \tag{7}$$

How it follows from here, dependents from the values $\delta_j (i = 1,2,3,4)$, one can receive special case of the problem (7). And also, there are special methods for solving stated problem.

Variables $\delta_j (i = 1,2,3,4)$ can only have values $\delta_i = 0,1 \ (i = 1,2,..,4)$. Obviously, the method (6) can be successfully applied to solve problem (7). Noted that by the selection of the variable δ_3 *and* δ_4 in the problem (7), one can receive different problem for solving which, one can use the method (6). Noted that in the method (6) by selected δ_1 *and* δ_2 one can receive the different methods with the different properties. Thus, we find that there is a wide choice for the method and for the tasks. For example, let us consider the case, when $\delta_4 = 0$. In this case, receive that there is no need for calculation of y''_m. Therefore, one can use method (6) in following form:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h^3 \sum_{i=0}^{k} l_i F_{n+i}; \ n = 0,1,.., N-k, \tag{8}$$

here $F_m = F(x_m, y_m, y'_m); \ m = 0,1,2,.., N-k.$

And in other case, if $\delta_3 = 0$, then one can take $\delta_1 = 0$. Noted that in these cases the degree for the receiving method decreases. The conception of stability for the method (6) defined in different form. For example, if $\beta_i = 0 \ (i = 0,1,.., k)$ then for the definition of stability one can use the definition 3 if $|\gamma_k| + |\gamma_{k-1}| + ... + |\gamma_1| + |\gamma_0| \neq 0$. If $\beta_i = \gamma_i = 0 \ \ (i = 0,1,2,.., k)$, then method receiving from the (6), will be stable if the roots of the polynomial $\rho(\lambda)$ are located in the unit circle on the boundary of which there is not multiple root, except the threefold root $\lambda = 1$. Noted that method (6) can be applied to solve problem (1). Some specialists suggested applied Multistep Method of type (6) to solve problem (1). In this case, the values y''_m and $y'''_m \ (m = 0,1,.., N)$ are calculated by the following way :

$$y''(x) = f'_x(x, y) + f'_y(x, y) f(x, y); \ \ y''(x) = g(x, y)$$
$$y'''(x) = \varphi(x, y); \ \ \varphi(x, y) = dg(x, y) / dx.$$

By this way, receive that finding the numerical solution of the problem (1) by using method (6) is difficult, which relation with the calculation of the functions $g(x, y)$ *and* $\varphi(x, y)$. Because in the calculation of these function, have to calculate values of the functions:

$$f'_x(x, y), \ f'_y(x, y), \ \varphi'_x(x, y) \ \varphi'_y(x, y).$$

which are not the same with the following functions:

$f(x, y)$ and $\varphi(x, y)$

For showing the advantages of the method (6), let us consider application of the method (6) to solve problem (7) in the case $\delta_i = 0,1 \ (i = 1,2,..,4)$. In this case arises necessity for the calculation of the values y'_i, y''_i and $y'''_i \ (i = 0,1,2,.., k)$. Noted that if by some methods have been calculated the values y'_i *and* y''_i, then by using problem (7) one can calculate the value y'''_i. For the showing of these problems let us consider the problem (1), and assume that by some method an exact solution to problem (1) has been found, taking into account which can differentiate the resulting equality, then receive the following problems:

$$y''(x) = g(x, y) \ y(x_0) = y_0,$$
$$y'(x_0) = f(x_0, y_0), \ x_0 \le x \le X, \tag{9}$$

$$y'''(x) = \varphi(x, y) \ y(x_0) = y_0, \ y'(x_0) = f(x_0, y_0),$$
$$y''(x_0) = \varphi(x_0, y_0), \ x_0 \le x \le X. \tag{10}$$

By using this sequences of problem, one can solve the problem (1), with help of the method (6). It is easy to understand that for finding values one can use the numerical solution of the problems (9) and (10). Noted that for the finding values of the numerical solution of problem (9), one can used the method (3). By the continue what was described, ways receive that to solve problem (10) one can used method (2). By using above comparison for the exactness above mentioned multistep methods, receive that in this $k+4$ case the order of accuracy will equal. To increase the order of accuracy (the degree) of results obtained by above described way one can use the methods of hybrid types. One of simple class of hybrid methods can presented as:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i+\nu_i} \ (|\nu_i| < 1; \ i = 0,1,2,.., k) \tag{11}$$

One can prove that, if method (11) is stable, then there exist hybrid method of (11) type with the maximum degree $p_{\max} = 2k+2$. In this case $k=1$, the method with the degree $p=4$ can presented as:

$$y_{n+1} = y_n + h(f_{n+1/2+\alpha} + f_{n+1/2-\alpha})/2, \ \alpha = \sqrt{3}/6. \tag{12}$$

As was noted above, if method (3) is stable and has the degree of p, then $p_{\max=2k+2}$. Thus, receive that one can use hybrid methods to calculate the values of the functions $y'''(x)$, participated in the problem (10). Considering that the construction of numerical methods for calculating the values of the function $y'''(x)$, are investigated, which are not enough. So, here, some methods have been constructed for calculation of the values for above named function.

4. Construction of Some Numerical Methods

By using the way of unknown methods, one can receive the following system of algebraic equations for finding the values of the coefficients in the method of (7):

$$\sum_{i=0}^{k} \alpha_i = 0, \ \sum_{i=0}^{k} \beta_i = \sum_{i=0}^{k} i\alpha_i; \ \sum_{i=0}^{k} \gamma_i + \sum_{i=0}^{k} i\beta_i = \sum_{i=0}^{k} \frac{i^2}{2!}\alpha_i; ...;$$
$$\sum_{i=0}^{k} \frac{i^{l-1}}{(l-1)!}\beta_i + \sum_{i=0}^{k} \frac{i^{l-2}}{(l-2)!}\gamma_i + \sum_{i=0}^{k} \frac{i^{l-3}}{(l-3)!}l_i; \ l = 3,4,.., p. \ (0! = 1) \tag{13}$$

By using the solution of this system one can be constructed explicit and implicit multistep third derivative methods with the different properties. Let us constructed some simple methods of the third derivaties type. It is not difficult to understand that the explicit methods are simple, therefor, let us construct some explicit methods. For this aim, let us put $k=2$. In this case, one constructed the following explicit multistep methods of the third derivative types:

$$y_{n+2} = y_{n+1} + h(15y'_{n+1} - 13y'_n) \backslash 2 - h^2(31y''_{n+1} + 29y''_n)/10 +$$
$$+ h^3(111y'''_{n+1} - 49y'''_n)/120, \tag{14}$$

$$y_{n+2} = (y_{n+1} + y_n)/2 + h(31y'_{n+1} - 25y'_n)\backslash 4 -$$
$$- h^2(63y''_{n+1} + 57y''_n)/20 + h^3(233y'''_{n+1} - 97y'''_n)/240 \qquad (15)$$

These methods are stable and have the degree $p = 6 \ or \ p = 3k$. For the comparison explicit and implicit method, let us consider the following implicit method:

$$y_{n+1} = y_n + h(y'_n + y'_{n+1})\backslash 2 + h^2(y''_n + y''_{n+1})/10 +$$
$$+ h^3(y'''_{n+1} + y'''_n)/120 \qquad (16)$$

Noted that methods (14)-(16) have same degree, method (16) is one step, but the methods (14) and (15) are the two step methods. All the methods have its own disadvantages and advantages. In our case, considering any method as the primary is difficult.

5. Conclusion

As is known in the study and application of the Multistep Multiderivative Methods one of the basic question is the define the maximal value of its degree. In usually specialists for this aim are used the Dahlquist's law. As was shown above the Dahlquist's results are extended by Ibrahimov. For the illustration of this, here have considered investigation of Multistep Third derivative Methods. For the application of these methods, one of the important question in the dependence of the choosing of the stable method, which can be applied to solving different problems. For example as the Multistep Second derivative Methods and the Multistep Methods with the special structure (as the Stormer method). By using some relation between of Multistep Multiderivative Methods for the ODEs of the first, second and third orders, here have given comparison these Multistep Multiderivative Methods. For the illustration of resaving results have used some known and unknown methods. And also have recommending some ways for the application of the methods with high derivatives to solve initial-value problem by using implicit and explicit methods. Noted here have comprised the concrete stable methods with maximum degree. The way described here is new and hope that, this direction will be find its follows.

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