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Further Results on the Nonsplit Tree Domination Number in Connected Graphs

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Abstract: Let G = (V, E) be a connected graph. A subset D of V is called a dominating set of G if N[D] = V. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set D of a graph G is called a tree dominating set (ntr - set) if the induced subgraph $\langle D \rangle$ is a tree. The tree domination number $\gamma_{tr}(G)$ of G is the minimum cardinality of a tree dominating set. A tree dominating set D of a graph G is called a nonsplit tree dominating set (nstd - set) if the induced subgraph $\langle V - D \rangle$ is connected. The nonsplit tree domination number $\gamma_{nstd}(G)$ of G is the minimum cardinality of a nonsplit tree dominating set. In this paper, nonsplit tree domination number of unicyclic and cubic and cartesian product of some standard graphs are found.

Keywords: Domination number, connected domination number, tree domination number, nonsplit domination number.

Mathematics Subject Classification: 05C05, 05C69, 05C76.

1. Introduction

Graphs are indispensable tools in every phase of human daily life and in many disciplines where mathematics permeates. Graph theory also has its own invariants. The concept of domination is one of the main parameters in graph theory. The domination is used in the solution of many problems and analysis of events in the historical process of human life. It is also used in solving network problems.

The graphs considered here are nontrivial, finite and undirected. The order and size of G are denoted by n and m respectively. If $D \subseteq V$, then $N(D) = \bigcup_{v \in D} N(v)$ and $N[D] = N(D) \cup D$ where N(v) is the set of vertices of G which are adjacent to v.

The concept of domination in graphs was introduced by Ore in [21]. Cockayne and Hedetniemi explained importance and properties of domination in [4].

Definition 1.1: A subset D of V is called a dominating set of G if N[D] = V. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$.

Xuegang Chen, Liang Sun and Alice McRac [25] introduced the concept of tree domination in graphs.

Definition 1.2: A dominating set D of G is called a tree dominating set, if the induced subgraph $\langle D \rangle$ is a tree. The minimum cardinality of a tree dominating set of G is called the tree domination number of G and is denoted by $\gamma_{tr}(G)$.

Kulli and Janakiram [14, 15] introduced the concept of split and nonsplit domination in graphs.

Definition 1.3: A dominating set D of a graph G is called a nonsplit dominating set if the induced subgraph ⟨V – D⟩ is connected. The nonsplit domination number $\gamma_{nsd}(G)$ of G is the minimum cardinality of a nonsplit dominating set.

Muthammai and Chitiravalli [19, 20] defined the concept of split and nonsplit tree domination in graphs.

Definition 1.4: A tree dominating set D of a graph G is called a nonsplit tree dominating set if the induced subgraph ⟨V – D⟩ is connected. The nonsplit tree domination number $\gamma_{nstd}(G)$ of G is the minimum cardinality of a nonsplit tree dominating set.

The nonsplit tree domination number does not exist for some graphs. If the nonsplit tree domination number does not exist for a given connected graph G, then $\gamma_{nstd}(G)$ is defined to be zero.

Remark 1.1[20]: Since ⟨D⟩ is a tree for any nstd - set D of a connected graph G, $|D| \geq 1$.

Observation 1.1 [20]: For any connected graph G, $\gamma(G) \leq \gamma_{nstd}(G)$, since every nstd - set is a dominating set.

Observation 1.2 [20]: For any spanning subgraph H of G, $\gamma_{nstd}(G) \leq \gamma_{nstd}(H)$.

Remarks 1.2 [20]: For any connected graph G,

(i) $\gamma(G) \leq \gamma_{ns}(G) \leq \gamma_{nstd}(G)$.
(ii) $\gamma(G) \leq \gamma_{tr}(G) \leq \gamma_{nstd}(G)$.
(iii) $\gamma_{nstd}(G) \leq n - 1$.

Theorem 1.1 [20]: For any cycle C_n on n vertices, $\gamma_{nstd}(C_n) = n - 2$, $n \geq 3$.

Theorem 1.2 [20]: For any cycle P_n on n vertices, $\gamma_{nstd}(P_n) = n - 1$, $n \geq 3$.

Theorem 1.3 [20]: $\gamma_{nstd}(K_{r,s}) = 2$, $r, s \geq 2$.

Theorem 1.4 [20]: $\gamma_{nstd}(\overline{C_n}) = 2$, for $n > 5$, where $\overline{C_n}$ is the complement of C_n.

Remark 1.3 [20]: If n = 5, $G \cong \overline{C_5} \cong C_5$, then $\gamma_{nstd}(\overline{C_5}) = 3$.

Remark 1.4 [20]: If $G \cong P_n \circ K_1$, then $\gamma_{nstd}(P_n \circ K_1) = 0$.

Observation 1.3 [20]: For any connected graph G on n vertices with $\gamma_{nstd}(G) > 0$, $1 \leq \gamma_{nstd}(G) \leq n - 1$.

Theorem 1.5 [20]: For any connected graph G with n vertices, $\gamma_{nstd}(G) = 1$ if and only if $G \cong H + K_1$, where H is a connected graph on (n – 1) vertices.

Remarks 1.5 [20]: If G is one of the following graphs, then $\gamma_{nstd}(G) = 1$.

(i) For any wheel W_n ($n \geq 4$), $\gamma_{nstd}(W_n) = 1$.
(ii) For the complete graph K_n with n vertices, $\gamma_{nstd}(K_n) = 1 (n \geq 3)$.
(iii) $\gamma_{nstd}(K_n - e) = 1$, where $n \geq 4$.

2. Nonsplit Tree Domination in Unicyclic and Cubic Graphs

In this paper, we study some bounds for nonsplit tree domination number of Unicyclic graphs and Cubic graphs and characterize graphs for which nonsplit tree domination number $\gamma_{nstd}(G) = 2, 3, n - 1$ and $n - 2$.

Theorem 2.1: Let G be a connected graph with $\gamma_{nstd}(G) > 1$. Then $\gamma_{nstd}(G) = 2$ if and only if there exists an edge cover of G containing exactly one edge e and the subgraph of G induced by vertices of G except the end vertices of e is connected.

Proof: Assume $\gamma_{nstd}(G) = 2$. Then there exists an nstd - set D of G such that $\langle V(G) - D\rangle$ is connected and $\langle D\rangle \cong K_2$ and D is an edge cover of G. Let $e \in \langle D\rangle$ and e = (u, v), where u, $v \in V(G)$. If $\langle V(G) - \{u, v\}\rangle$ is disconnected, then D is not a nstd - set. Therefore $\langle V(G) - \{u, v\}\rangle$ is connected. Conversely, if the condition given in the theorem holds, then $\gamma_{nstd}(G) = 2$.

Theorem 2.2: If G is a connected graph with atleast five vertices and $\gamma_{nstd}(G) = 2$, then $diam(G) \leq 3$.

Proof: Let G be a connected graph with $\gamma_{nstd}(G) = 2$. Let D = {u, v} be a nstd - set of G, where u, $v \in V(G)$. Then $\langle V(G) - D\rangle$ is connected.

Let $w \in V(G) - D$. If w is adjacent to u, then d(u, w) = 1. Let w be not adjacent to u. Then w is adjacent to v and d(u, w) = 2, since u v w is a geodesic path in G. Therefore, $d(u, w) \leq 2$. Similarly, $d(v, w) \leq 2$. Therefore, distance between vertices from D to V(G) – D is atmost two.

Let $w_1, w_2 \in V(G) - D$. If w_1, w_2 are adjacent, then $d(w_1, w_2) = 1$. Let w_1 be not adjacent to w_2. If both w_1 and w_2 are adjacent to u or v, then $d(w_1, w_2) = 2$. Let w_1 be adjacent to u and w_2 be adjacent to v. then w_1 u v w_2 is a geodesic path and $d(w_1, w_2) = 3$. Therefore, distance between any two vertices in V(G) – D is atmost 3. Also, d(u, v) = 1.

From the above it can be concluded that the distance between any two vertices in G is atmost three. Therefore $diam(G) \leq 3$.

Theorem 2.3: If G is a connected graph with atleast six vertices and $\gamma_{nstd}(G) = 3$, then $diam(G) \leq 4$.

Proof: Let D = {u, v, w} be a nstd - set of G, where u, v, $w \in V(G)$. Then $\langle V(G) - D\rangle$ is connected. Since $\langle D\rangle$ is a tree, $\langle D\rangle \cong P_3$.

Let $w_1, w_2 \in V(G) - D$. If w_1 is adjacent to all the vertices of D, then $d(w_1, u) = d(w_1, v) = d(w_1, w) = 1$. Let w_1 be adjacent to u and v.

Then $d(w_1, u) = d(w_1, v) = 1$ and $d(w_1, w) = 2$. If w_1 is adjacent to u and w, then $d(w_1, u) = d(w_1, w) = 1$ and $d(w_1, v) = 2$.

Let w_1 be adjacent to exactly one u, v and w, say u. Then $d(w_1, u) = 1$, $d(w_1, v) = 2$ and $d(w_1, w) = 3$. Distance between vertices from D and V(G) – D is atmost three.

Next, distance between any two vertices in V(G) – D is found.

If w_1 is adjacent to w_2, then $d(w_1, w_2) = 1$. If both w_1 and w_2 are adjacent to a vertex of D, then $d(w_1, w_2) = 2$. If w_1 is adjacent to u or w and w_2 is adjacent to v, then $d(w_1, w_2) \leq 3$. If w_1 is adjacent to u and w_2 is adjacent to w, then $d(w_1, w_2) = 4$, since w_1 u v w w_2 is a path in G. Also, distance between any two vertices in $\langle D\rangle$ is 2. Therefore, distance between any two vertices in G is atmost four and hence $diam(G) \leq 4$.

In the following, the connected graphs for which $\gamma_{nstd}(G) = n - 1$ are characterized.

Notation 2.1.1:

Let H_1 be a graph obtained from a cycle C_p on p ($p \geq 5$) vertices by attaching trees at t ($3 \leq t \leq p - 1$) vertices of C_p such that no two adjacent vertices of C_p is of degree 2 in H_1.

Theorem 2.4: Let G be a connected graph on n vertices such that $\gamma_{nstd}(G) > 0$. Then $\gamma_{nstd}(G) = n - 1$ if and only if G is one of the following graphs.

(i) G is a tree on n vertices
(ii) G is obtained from C_3 by attaching trees at any two vertices of C_3.
(iii) G is obtained from C_4 ($K_4 - e$) by attaching trees either at any three vertices of C_4 ($K_4 - e$) or at any two nonadjacent vertices of C_4 ($K_4 - e$)
(iv) G is the graph H_1defined in Notation 3.1.1.

(v) Let u be a vertex of degree 2 of the cycle in H_1. G is a graph obtained from H_1 by joining u to atmost p – 3 vertices of C_p ($p < n$) in H_1, which are nonadjacent to u in C_p.

Proof: Let G be one of the graphs given in the theorem.

Case 1. G is a graph as in (i)

Let u be a pendant vertex in G. Then $V(G) - \{u\}$ is a minimum nstd - set of G.

Case 2. G is a graph as in (ii)

Let $u \in C_3$ be such that $d_G(u) = 2$. Then $V(G) - \{u\}$ is a minimum nstd - set of G.

Case 3.G is a graph as in (iii)

Let $u \in C_4$ be such that $d_G(u) = 2$. Then $V(G) - \{u\}$ is a minimum nstd - set of G.

If $u \in K_4 - e$ be such that $d_G(u) = 3$, then$V(G) - \{u\}$ is a minimum nstd - set of G.

Case 4. G is a graph as in (iv)

If $u \in C_p$ ($p \geq 5$) be such that $d_G(u) = 3$, then $V(G) - \{u\}$ is a minimum nstd - set of G.

Case 5. G is a graph as in (v)

Let $u \in G$ be such that no tree is attached at u. Then $V(G) - \{u\}$ is a minimum nstd - set of G. Therefore, $\gamma_{nstd}(G) = n - 1$.

Conversely, let $\gamma_{nstd}(G) = n - 1$ and $\gamma_{nstd}(G) > 0$. Let D be a γ_{nstd} - set of G with $|D| \geq n - 1$.

Then $\langle D \rangle$ is a tree and $|V(G) - D| = 1$. Let $w \in V(G) - D$. If w is adjacent to exactly one of the vertices of $\langle D \rangle$, then G is a tree. Let w be adjacent to atleast two pendant vertices of $\langle D \rangle$. Let $x \in G$ be a pendant vertex of $\langle D \rangle$. Then the set $V(G) - \{w, x\}$ is a nonsplit tree dominating set of G. Therefore, $\gamma_{nstd}(G) \leq n - 2$ and therefore w is not adjacent to any of the pendant vertices of $\langle D \rangle$. Hence w is adjacent to atleast two nonpendant vertices of $\langle D \rangle$ and G contains cycles.

Let G be unicyclic.If trees are attached at either one or all the vertices of C_3, then $\gamma_{nstd}(G) = n - 2$ (or) 0. That is, $\gamma_{nstd}(G) \neq n - 1$. Then G is a graph as in (i).

If trees are attached at one (or) two adjacent vertices of C_4, then $\gamma_{nstd}(G) = n - 2$. If trees are attached at all the vertices of C_4, Then $\gamma_{nstd}(G) = 0$

Then G is the graph as in (iii)

If trees are attached at all the vertices of C_p ($5 \leq p < n$), then $\gamma_{nstd}(G) = 0$. Therefore, trees can be attached at atmost p – 1 vertices of C_p.

If any two adjacent vertices of C_p is of degree 2 in G, then $\gamma_{nstd}(G) \leq n - 2$. Then G is the graph as in (iv).

Let G be not unicyclic. The G has atleast two cycles. Let w_1, w_2 be two adjacent vertices of the cycle in G. If trees are not attached at both w_1 and w_2, then $\gamma_{nstd}(G) \leq n - 2$. Then G is the graph as in (v) (or) G is the graph obtained from $K_4 - e$ by attaching trees either at any three vertices of $K_4 - e$ (or) at any two nonadjacent vertices of $K_4 - e$.

Hence the Theorem follows.

Notation 2.1.2:

Let C_p be a cycle on p ($p \geq 5$) vertices. Let u and v be any two adjacent vertices of the cycle C_p. Let H_2 be a graph obtained from C_p by joining the vertices u, v or both to atmost p – 3 vertices of C_p, which are not adjacent to atleast one of u and v.

Theorem 2.5: Let G be a connected graph on n vertices such that $\gamma_{nstd}(G) > 0$. Then $\gamma_{nstd}(G) = n - 2$ if and only if G is one of the following graphs.

(i) G is a cycle on n vertices.

(ii) G is obtained from C_3 (or) C_4 by attaching trees at one vertex of C_3 (or) C_4.

(iii) G is obtained fromC_4 (or) K_4 by attaching trees at any two adjacent vertices of C_4 (or) K_4.

(iv) G is obtained from $K_4 - e$ by attaching trees at two adjacent vertices of $K_4 - e$ in which one vertex is of degree two in $K_4 - e$.

(v) G is agraph obtained from the graph H_2 by attaching trees at two vertices w_1 and w_2, where w_1 is adjacent to u and w_2 is adjacent to v in C_p ($p = 5$ or 6) in H_2 and attaching trees at atmost one (or) two remaining vertices of C_p($p = 5$ or 6), where $p < n$, where H_2 is a graph in notation 3.1.2.

(vi) G is obtained from the graph H_2 by attaching trees at two vertices w_1 and w_2, where w_1 is adjacent to u and w_2 is adjacent to v in C_p ($p \geq 7$) in H_2 and attaching trees at the vertices of $V(C_p) - \{u, v, w_1, w_2\}$ such that atleast one of the vertices of induced P_3 in the cycle C_p get trees attached.

Proof:

Let G be one of the graphs mentioned in the Theorem.

Case 1.

Let G be a cycle on n vertices and let u, v be any two adjacent vertices of C_3. Then $V(G) - \{u, v\}$ is a γ_{nstd}- set of G and $\gamma_{nstd}(G) = n - 2$.

Case 2.

Let G be a graph as in (ii) and let D be a set containing any two adjacent vertices of C_3 (or C_4) of degree 2. Then $V(G) - D$ is a γ_{nstd} - set and $\gamma_{nstd}(G) = n - 2$.

Case 3.G is a graph as in (iii)

Let u, v be two adjacent vertices of C_4 (or K_4). Then the set $V(G) - \{u, v\}$ is a minimum nstd - set of G and $\gamma_{nstd}(G) = n - 2$.

Case 4. G is a graph as in (iv)

Let $u, v \in K_4 - e$ be such that $d_G(u) = 3$ and $d_G(v) = 2$ in $K_4 - e$. Then $V(G) - \{u, v\}$ is a γ_{nstd} - set of G and $\gamma_{nstd}(G) = n - 2$.

Case 5. G is a graph as in (v) or (vi)

In this case, the set $V(G) - \{u, v\}$ is a γ_{nstd} - set of G and $\gamma_{nstd}(G) = n - 2$.

Conversely, assume $\gamma_{nstd}(G) = n - 2$ and $\gamma_{nstd}(G) > 0$. Let D be a γ_{nstd} - set of G with $|D| \geq n - 2$. Then $\langle D \rangle$ is a tree and $|V(G) - D| = 2$. Let $u, v \in V - D$. If each of u and v is adjacent to a distinct pendant vertex of $\langle D \rangle$, then G is a cycle. If atleast one of u and v is adjacent to two verticesof $\langle D \rangle$, one of them is a pendant vertex w of $\langle D \rangle$, then $V(G) - \{u, v, w\}$ is a nstd - set of G and $\gamma_{nstd}(G) \leq n - 3$. Therefore, none of the vertices u and v is adjacent to a pendant vertex in $\langle D \rangle$ and u and v are adjacent to vertices of $\langle D \rangle$which are not the pendant vertices of $\langle D \rangle$. Therefore, $\langle D \rangle$ contains cycles.

Let G be unicyclic. If trees are attached at two or all the vertices of C_3, then $\gamma_{nstd}(G) = n - 1$ (or) $\gamma_{nstd}(G) = 0$. Then G is the graph as in (ii)

If trees are attached at two nonadjacent vertices of C_4, then $\gamma_{nstd}(G) = n - 1$. If trees are attached at all the vertices of C_4, Then $\gamma_{nstd}(G) = 0$. Then G is the graph as in (iii).

If trees are attached at all the vertices of C_p ($5 \leq p < n$), then $\gamma_{nstd}(G) = 0$ and if trees can be attached at $p - 1$ vertices of C_p, $\gamma_{nstd}(G) = n - 1$.Therefore, trees can be attached at atmost $p - 2$ vertices of C_p.

Let G be not unicyclic. If trees are attached at two nonadjacent vertices (or) three vertices K_4, then $\gamma_{nstd}(G) = n - 1$. If trees are attached at all the vertices of K_4, Then $\gamma_{nstd}(G) = 0$. Then G is the graph as in (iii).

If trees are attached at three vertices or two adjacent vertices of degree3, then $\gamma_{nstd}(G) = n - 1$. Then G is the graph as in (iv).

Let H_2 be the graph as given in Notation 3.1.2, in which the cycle is C_p, where p = 5 or 6.

If trees are not attached at one of w_1 and w_2 of H_2, then $\gamma_{nstd}(G) \leq n - 3$. Then G is the graph as in (v). If the graph G is not as in (vi), then $\gamma_{nstd}(G) \leq n - 3$.Therefore, G is one of the graphs given in the Theorem.

In the following, the connected unicycilc graphs for which $\gamma_{nstd}(G) = 2$, 3 and 4 are obtained.

Theorem 2.6: If G is a connected unicyclic graph with $\gamma_{nstd}(G) > 0$, then $\gamma_{nstd}(G) = 2$ if and only if either G is C_4 or G is obtained from C_3 by attaching a pendant edge at a vertex of C_3.

Proof: If G is C_4, then $\gamma_{nstd}(G) = 2$. If G is a graph obtained from C_3 by attaching a pendant edge at a vertex of C_3, then the set containing the pendant vertex and its support is a minimum nstd - set of G. Therefore, $\gamma_{nstd}(G) = 2$.

Conversely, assume $\gamma_{nstd}(G) = 2$. Let D be a minimum nstd - set of G such that |D| = 2. If the cycle in G is C_p, $p \geq 5$, then $\gamma_{nstd}(G) \geq 3$. Therefore, the cycle in G is either C_3 or C_4. If a pendant edge is attached at a vertex of C_4, then $\gamma_{nstd}(G) \geq 3$. Similarly, if a path of length atleast two is attached at a vertex of C_3 or a pendant edge is attached at atleast two vertices of C_3, Then also $\gamma_{nstd}(G) \geq 3$.

Hence, either G is C_4 or G is obtained from C_3 by attaching a pendant edge at a vertex of C_3.

Theorem 2.7: If G is a connected unicyclic graph with $\gamma_{nstd}(G) > 0$, then $\gamma_{nstd}(G) = 3$ if and only if G is one of the following graphs.

(a) G is isomorphic to C_5.

(b) G is obtained from C_4 by attaching a pendant edge at a vertex of C_4.

(c) G is obtained from C_3 by attaching a path of length two at a vertex of C_3.

Proof: If G is one of the graphs given in the Theorem, $\gamma_{nstd}(G) = 3$.

Conversely, assume $\gamma_{nstd}(G) = 3$.

Let D be a minimum nstd - set of G such that |D| = 3. If the cycle in G is C_p, $p \geq 6$, then $\gamma_{nstd}(G) \geq 4$. Therefore, the cycle in G is C_3, C_4 or C_5. If a pendant edge is attached at a vertex of C_5, then $\gamma_{nstd}(G) \geq 4$.If either a path of length atleast two is attached at a vertex or a pendant edge is attached at any two vertices of C_4, then $\gamma_{nstd}(G) \geq 4$. If either a path of length atleast three is attached at a vertex or a pendant edge is attached at any two vertices of C_3, then $\gamma_{nstd}(G) \geq 4$.

Therefore, G is one of the graphs given in the Theorem.

Similarly, the following theorem can be proved.

Theorem 2.8: If G is a connected unicyclic graph with $\gamma_{nstd}(G) > 0$, then $\gamma_{nstd}(G) = 4$ if and only if G is one of the following graphs.

(a) G is isomorphic to C_6.

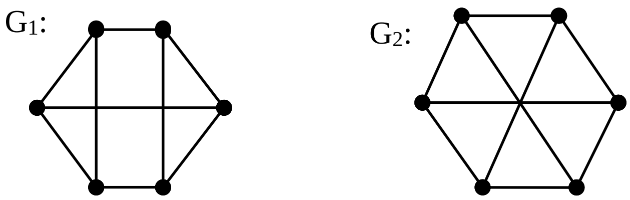
(d) G is obtained from C_p (p = 3, 4) by attaching a pendant edge each at two adjacent vertices of C_p.

(e) G is obtained from C_4 by attaching a path of length two at a vertex of C_3.

(f) G is obtained from C_3 by attaching a path of length three at a vertex of C_3.

In the following, connected cubic graphs for which $\gamma_{nstd}(G) = 2$ and 3 are obtained.

Theorem 2.9: Let G be a connected cubic graph. Then $\gamma_{nstd}(G) = 2$ if and only if G is one of the following graphs.

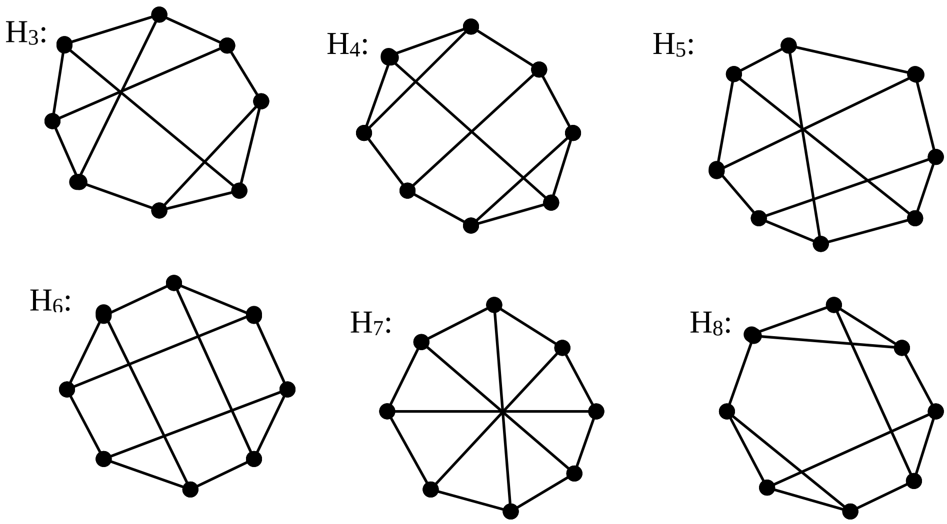


Proof: If G is one of the graphs G_1 and G_2, then $\gamma_{nstd}(G) = 2$. Conversely, let D be a $\gamma_{nstd}(G)$ - set of G. Then |D| = 2 and $\langle D\rangle \cong K_2$. Since G is cubic, each vertex of K_2 is adjacent to two vertices of $\langle V(G) - D\rangle$and $\langle V(G) - D\rangle$contains atmost four vertices. Since |V(G)| is even, |V(G) – D| = 2 (or) 4.

If |V(G) – D| = 2, then $G \cong K_4$ and $\gamma_{nstd}(K_4) = 1$. Therefore, |V(G) – D| = 4 and $\langle V(G) - D\rangle$ is either a path or a cycle on four vertices. If $\langle V(G) - D\rangle \cong P_4$, then there exists no cubic graph.

Hence $G \cong C_4$ and G is isomorphic to one of the graphs G_1 and G_2.

Theorem 2.10: Let G be a connected cubic graph. Then $\gamma_{nstd}(G) = 3$ if and only if G is one of the following graphs. H_3, H_4, H_5, H_6, H_7 and H_8.



Proof: If G is one of the graphs H_3, H_4, H_5, H_6, H_7 and H_8, then $\gamma_{nstd}(G) = 3$. Conversely, let D be a $\gamma_{nstd}(G)$ - set of G. Then $|D| = 3$. Since $\langle D\rangle$ is a tree, $\langle D\rangle \cong P_3$. G is cubic implies G has atmost eight vertices and $\langle V(G) – D\rangle$ has atmost five vertices. Since $|V(G)|$ is even, $V(G) – D$ has 3 or 5 vertices. If $|V(G) – D| = 3$, then there is no cubic graph on six vertices, since for these graphs $\gamma_{nstd}(G) = 2$.
Then $|V(G) – D| = 5$ and $\langle V(G) – D\rangle \cong P_5$ or C_5. If $\langle V(G) – D\rangle \cong P_5$, then the number of edges between $V – D$ and D is eight.
Therefore, $\langle V(G) – D\rangle \cong C_5$ and G is one of the graphs H_3, H_4, H_5, H_6, H_7 and H_8.

Theorem 2.11: For any connected graph G with n vertices and m edges and $\gamma_{nstd}(G) > 0$, $\left\lceil \frac{n}{(\Delta+1)} \right\rceil \leq \gamma_{nstd}(G) \leq 2m – n$.

Proof:

$$\text{Since } \left\lceil \frac{n}{(\Delta+1)} \right\rceil \leq \gamma(G) \leq \gamma_{nst}(G), \left\lceil \frac{n}{(\Delta+1)} \right\rceil \leq \gamma_{nst}(G).$$

Also $\gamma_{nst}(G) \leq n –1 = 2(n –1) – n +1$. Since G is connected, $m \geq n – 1$. Therefore, $\gamma_{nst}(G) \leq 2m – n +1$.

If $\gamma_{nst}(G) = 2m – n + 1$, then $2m – n + 1 \leq n – 1$.

That is, $m \leq n – 1$. If $\gamma_{nst}(G) = 2m – n +1$, then $m \leq n – 1$. Therefore, G is a tree. But for a tree, $\gamma_{nst}(G) = 0$ and hence $\gamma_{nst}(G) < 2m – n + 1$. That is, $\gamma_{nstd}(G) \leq 2m – n$.

Theorem 2.12: Let G be a connected graph with $diam(G) = 2$ and $\gamma_{nst}(G) > 0$ and let $v \in V(G)$ be such that $d(v) = k$. If $N(v)$ is an independent set of G and $\langle V(G) –N[v]\rangle$ is connected, then $\gamma_{nst}(G) \leq k + 1$.

Proof: Let $v \in V(G)$ be such that $d(v) = k$. Assume $N(v)$ is independent and $\langle V(G) –N[v]\rangle$ is connected. Let $D = N[v]$. Then $\langle D\rangle$ is a tree and $\langle V(G) – D\rangle$ is connected. Since $diam(G) = 2$, each vertex in $V(G)$ is adjacent to atleast one vertex in D. Therefore, D is nonsplit tree dominating set of G and hence $\gamma_{nst}(G) \leq |D| = k +1$.

3. Nonsplit Tree Domination Number of Cartesian Product of Graphs

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In this section, nonsplit tree domination numbers of $P_2 \times C_n$, $P_3 \times C_n$, $P_4 \times C_n$, $P_5 \times C_n$, $P_2 \times P_n$, $P_3 \times P_n$, $P_4 \times P_n$, $P_5 \times P_n$, $P_6 \times P_n$, $C_3 \times C_n$, $C_4 \times C_n$ are found.

Theorem 3.1: For the graph $P_2 \times C_n$, $\gamma_{nstd}(P_2 \times C_n) = n$, $n \geq 4$.

Proof: Let $G \cong P_2 \times C_n$ and let $V(G) = \bigcup_{i=1}^{n}\{v_{i1}, v_{i2}\}$ where $\langle\{v_{i1}, v_{i2}\}\rangle \cong P_2^i$, i = 1, 2 and $\langle\{v_{1j}, v_{2j}, \ldots, v_{nj}\}\rangle \cong C_n^j$, j = 1, 2, ... ,n and P_2^i is the i^{th} copy of P_2 and C_n^j is the j^{th} copy of C_n in G.

Let $D = \{v_{12}\} \cup (\bigcup_{i=1}^{n-1}\{v_{i1}\})$.Then $D \subseteq V(G)$. Here v_{n1} is adjacent to v_{11} and for i = 2, 3,… ,n, v_{i2} is adjacent to v_{i1} in D. Therefore, D is a dominating set of G and $\langle D\rangle \cong P_n$ and $\langle V(G) - D\rangle \cong P_n$. Since $\langle D\rangle$ is a tree and $\langle V(G) - D\rangle$ is connected, D is a nstd - set of G and $\gamma_{nstd}(G) \leq |D| = n$.

Let D′ be a minimum nstd - set of C_n. Then $|D'| = \gamma_{ns}(C_n) = n - 2$. To obtain a nstd - set of $P_2 \times C_n$, atleast two vertices of $P_2 \times C_n$ are to be added with D′. Therefore, any nstd - set of $P_2 \times C_n$ contains atleast n vertices. Thus, $\gamma_{nstd}(P_2 \times C_n) \geq n$.

Therefore, $\gamma_{nstd}(G) = \gamma_{nstd}(P_2 \times C_n) = n$.

Remark 3.1: $\gamma_{nstd}(P_2 \times C_3) = 2$, since the set $\{v_{11}, v_{12}\}$ is a minimum nonsplit tree dominating set of $P_2 \times C_3$.

Example 3.1:

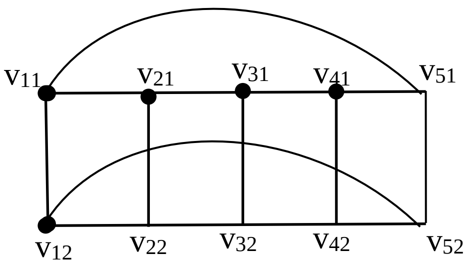

Figure 3.1

In the graph $P_2 \times C_5$ given in Figure 3.1, minimum nstd - set is

$D = \{v_{11}, v_{21}, v_{31}, v_{41}, v_{12}\}$ and $\langle V(P_2 \times C_4) - D\rangle \cong P_5$, and $\gamma_{nstd}(P_2 \times C_4) = 5$.

Theorem 3.2: For the graph $P_3 \times C_n$, $\gamma_{nstd}(P_3 \times C_n) = n + 1$, $n \geq 4$.

Proof: Let $G \cong P_3 \times C_n$, $n \geq 4$ and let $V(G) = \bigcup_{i=1}^{n}\{v_{i1}, v_{i2}, v_{i3}\}$ such that $\langle\{v_{i1}, v_{i2}, v_{i3}\}\rangle \cong P_3^i$, i = 1, 2, 3 and $\langle\{v_{1j}, v_{2j}, \ldots, v_{nj}\}\rangle \cong C_n^j$, j = 1, 2, ... ,n, where P_3^i is the i^{th} copy of P_3 and C_n^j is the j^{th} copy of C_n in G.

Let $D = (\{v_{11}, v_{13}\}) \cup (\bigcup_{i=1}^{n-1}\{v_{i2}\})$ and then $D \subseteq V(G)$ and $|D| = n - 1 + 2 = n + 1$. Here v_{n2} is adjacent to v_{12} and for i = 2, 3, … ,n – 1, v_{i1} and v_{i3} are adjacent to v_{i2} in D. Therefore, D is a dominating set of G and $\langle D\rangle$ is a tree obtained by attaching a pendant edge at one support of P_n and $\langle V(G) - D\rangle \cong P_{2n-1}$. Therefore, D is a nstd - set of G and $\gamma_{nstd}(G) \leq |D| = n + 1$.

The maximum length of a path obtained from C_n is n – 1. Let D′ be a nstd - set of G. Then D′ contains (n – 1) vertices of C_n and atleast two vertices of $P_3 \times C_n$, for adjacency. Therefore, D′ contains atleast (n + 1) vertices and $\gamma_{nstd}(P_3 \times C_n) \geq n + 1$.

Thus, $\gamma_{nstd}(P_3 \times C_n) = n + 1$.

Remark 3.2: $\gamma_{nstd}(P_3\times C_3)$ =3, since the set $\{v_{11}, v_{12}, v_{13}\}$ is a minimum nstd - set of $P_3\times C_3$.

Theorem 3.3: For the graph $P_4\times C_n$, $\gamma_{nstd}(P_4\times C_n) = 2n - 1, n \geq 4$.

Proof: Let $G \cong P_4\times C_n$, n ≥ 4 and let $V(G) = \bigcup_{i=1}^{n}\{v_{i1}, v_{i2}, v_{i3}, v_{i4}\}$ such that $\langle\{v_{i1}, v_{i2}, v_{i3}, v_{i4}\}\rangle \cong P_4^i$, i = 1, 2, 3, 4 and $\langle\{v_{1j}, v_{2j}, \dots, v_{nj}\}\rangle \cong C_n^j$, j = 1, 2, ... ,n, where P_4^i is the i^{th} copy of P_4 and C_n^j is the j^{th} copy of C_n in G.

Case1. n ≥ 6

Let $D = \left(\bigcup_{i=1}^{4}\{v_{1i}\}\right) \cup \left(\bigcup_{i=1}^{n-3}\{v_{i+3,1}\}\right) \cup \left(\bigcup_{i=1}^{n-3}\{v_{i+2,3}\}\right) \cup \{v_{n-2,2}\}$. Then $D \subseteq V(G)$ and $|D| = 2n - 1$. Here v_{31} is adjacent to v_{41} and for i = 1, 2, 3, 4, v_{2i} and v_{ni} is adjacent to v_{1i} and for i = 3, 4, …,n – 1, v_{i4} and v_{i2} are adjacent to v_{i3}. Therefore, D is a dominating set of G and ⟨D⟩ is a tree obtained from P_{2n-3} by attaching a pendant edge each at $v_{n-2,1}$ and $v_{n-2,3}$ and ⟨V(G) – D⟩ is a connected graph obtained from C_4 by attaching a path of length (n + 2) and (n – 6) at v_{22} and v_{32} respectively. Therefore, D is a nstd - set of G and $\gamma_{nstr}(G) \leq |D| = 2n - 1$.Thus, $\gamma_{nstd}(G) = \gamma_{nstd}(P_4\times C_n) = 2n - 1$.

Case 2. n = 4

The set D = $\{v_{11}, v_{12}, v_{13}, v_{14}, v_{22}, v_{23}, v_{33}\}$ is a minimum nstd - set of G and $\gamma_{nstd}(P_4\times C_4) = 7$.

Case 3. n = 5

The set D = $\{v_{11}, v_{12}, v_{13}, v_{14}, v_{41}, v_{51}, v_{52}, v_{53}, v_{43}\}$ is a minimum nstd - set of G and $\gamma_{nstd}(P_4\times C_5) = 9$.

Theorem 3.4: For the graph $P_2 \times P_n$, $\gamma_{nstd}(P_2 \times P_n) = n, n \geq 2$.

Proof: Let $G \cong P_2 \times P_n$ and let $V(G) = \bigcup_{i=1}^{n}\{v_{i1}, v_{i2}\}$ where $\langle\{v_{i1}, v_{i2}\}\rangle \cong P_2^i$, i = 1, 2 and $\langle\{v_{1j}, v_{2j}, \dots, v_{nj}\}\rangle \cong P_n^j$, j = 1, 2, ... ,n and P_2^i is the i^{th} copy of P_2 and P_n^j is the j^{th} copy of P_n in G.

Let $D = \bigcup_{i=1}^{n}\{v_{i1}\}$. Then $D \subseteq V(G)$. Here v_{i2}, for i = 1, 2, 3, …, n, is adjacent to v_{i1} in D. Therefore, D is a dominating set of G and $\langle D\rangle \cong P_n$ and $\langle V(G) - D\rangle \cong P_n$. Since ⟨D⟩ is a tree and⟨V(G) – D⟩ is connected, D is a nstd - set of G and $\gamma_{nstr}(G) \leq |D| = n$.

Let D′ be a minimum tree dominating set of G. Then $|D'| = \gamma_{tr}(P_n) = n - 2$. To obtain anstd - set of $P_2 \times P_n$, atleast two vertices of $P_2 \times P_n$ are to be added with D′ and therefore, any nstd - set of $P_2 \times P_n$ contains atleast n vertices. Thus, $\gamma_{nstd}(P_2 \times P_n) \geq n$. Hence, $\gamma_{nstd}(P_2 \times P_n) = n$.

Theorem 3.5: For the graph $P_3 \times P_n$, $\gamma_{nstd}(P_3 \times P_n) = 2n, n \geq 3$.

Proof: Let $G \cong P_3 \times P_n$ and let $V(G) = \bigcup_{i=1}^{n}\{v_{i1}, v_{i2}, v_{i3}\}$ where $\langle\{v_{i1}, v_{i2}, v_{i3}\}\rangle \cong P_3^i$, i = 1, 2, 3 and $\langle\{v_{1j}, v_{2j}, \dots, v_{nj}\}\rangle \cong P_n^j$, j = 1, 2, ... ,n and P_3^i is the i^{th} copy of P_3 and P_n^j is the j^{th} copy of P_n in G.

Let $D = \left(\bigcup_{i=1}^{n}\{v_{i1}\}\right) \cup \left(\bigcup_{i=1}^{n-3}\{v_{i+3,3}\}\right) \cup \{v_{12}, v_{n2}, v_{13}\}$ .Then $D \subseteq V(G)$. Here v_{23} is adjacent to v_{13} and v_{33} is adjacent to v_{43} and for v_{i2} in V – D is adjacent to v_{i1} in D for i = 2, 3, … ,n – 2, v_{i2} in V(G) – D is adjacent to v_{i1} in D. Therefore, D is a dominating set of G and $\langle D\rangle \cong P_{2n}$, $\langle V(G) - D\rangle$ is a connected graph obtained from C_4 by attaching path of length n – 4 at v_{32}. Since $\langle D\rangle$ is a tree and $\langle V(G) - D\rangle$ is connected, D is a nstd - set of G and $\gamma_{nstd}(G) \le |D| = 2n$.

The graph $P_3 \times P_n$ can be divided into two blocks $P_2 \times P_n$ and P_n.

$\gamma_{nstd}(P_2 \times P_n) = n$ and $\gamma_{nstd}(G) = n - 2$. Let D be a minimum nstd - set of $P_3 \times P_n$. Then D contains 2n – 2 vertices from the blocks P_n and $P_2 \times P_n$. Since $\langle D\rangle$is a tree, atleast 2 vertices (from the block $P_2 \times P_n$) can be added with 2n – 2 vertices of D.

Therefore, D contains atleast 2n vertices, $\gamma_{nstd}(P_3 \times P_n) \ge 2n$.

Hence, $\gamma_{nstd}(P_3 \times P_n) = 2n$.

Theorem 3.6: For the graph $P_4 \times P_n$, $\gamma_{nstd}(P_4 \times P_n) = 2(n + 1)$, $n \ge 4$.

Proof: Let $G \cong P_4 \times P_n$, $n \ge 4$ and let $V(G) = \bigcup_{i=1}^{n}\{v_{i1}, v_{i2}, v_{i3}, v_{i4}\}$ such that $\langle\{v_{i1}, v_{i2}, v_{i3}, v_{i4}\}\rangle \cong P_4^i$, i = 1, 2, 3, 4 and $\langle\{v_{1j}, v_{2j}, \dots, v_{nj}\}\rangle \cong P_n^j$, j = 1, 2, ... ,n where P_4^i is the i^th^ copy of P_4 and P_n^j is the j^th^ copy of P_n in G.

Let $D = \left(\bigcup_{i=1}^{n}\{v_{i1}\}\right) \cup \left(\bigcup_{i=1}^{n}\{v_{i4}\}\right) \cup \{v_{12}, v_{13}\}$. Then $D \subseteq V(G)$. Here for i = 1, 2, …,n, v_{i2} and v_{i3} in V(G) – D are adjacent to v_{i1} and v_{i4} respectively in D. Therefore, D is a dominating set of G and $\langle D\rangle \cong P_{2(n+1)}$, $\langle V(G) - D\rangle \cong P_2 \times P_{n-1}$. Since $\langle D\rangle$ is a tree and $\langle V(G) - D\rangle$ is connected, D is a nstd - set of G and $\gamma_{nstd}(G) \le |D| = n + n + 2 = 2(n + 1)$.

The graph $P_4 \times P_n$ can be divided into two blocks $P_2 \times P_n$ and $P_2 \times P_n$. $\gamma_{nstd}(P_2 \times P_n) = n$. Let D be a minimum nstd - set of $P_4 \times P_n$. Then D contains n vertices from each block $P_2 \times P_n$. Since $\langle D\rangle$ is a tree, atleast 2 vertices (atleast one vertex from each block) can be added with 2n vertices of D. Therefore, D contains atleast 2n + 2 vertices, $\gamma_{nstd}(P_4 \times P_n) \ge 2n + 2$. Hence, $\gamma_{nstd}(P_4 \times P_n) = 2n + 2$.

Theorem 3.7: For the graph $P_5 \times P_n$, $\gamma_{nstd}(P_5 \times P_n) = 3n$, $n \ge 5$.

Proof: Let $G \cong P_5 \times P_n$, $n \ge 5$ and let $V(G) = \bigcup_{i=1}^{n}\{v_{i1}, v_{i2}, v_{i3}, v_{i4}, v_{i5}\}$ such that $\langle\{v_{i1}, v_{i2}, v_{i3}, v_{i4}, v_{i5}\}\rangle \cong P_5^i$ and $\langle\{v_{1j}, v_{2j}, \dots, v_{nj}\}\rangle \cong P_n^j$, for $1 \le j \le n$ where P_5^i is the i^th^ copy of P_5 and P_n^j is the j^th^ copy of P_n in G and $|V(G)| = 5n$.

$$\text{Let } D = \left(\bigcup_{i=1}^{n}\{v_{i1}\}\right) \cup \left(\bigcup_{i=1}^{4}\{v_{1,i+1}\}\right) \cup \left(\bigcup_{i=1}^{n-2}\{v_{i+1,5}\}\right) \cup \left(\bigcup_{i=1}^{n-3}\{v_{i+3,3}\}\right) \cup \{v_{n2}\}.$$

Then $D \subseteq V(G)$. Here v_{n4} is adjacent to v_{n3} and v_{23} is adjacent to v_{13} and v_{33} is adjacent to v_{43} and v_{n5} is adjacent to $v_{n-1,5}$ and for i = 2, 3, 4, …, n – 1, v_{i2} and v_{i4} in V(G) – D are adjacent to v_{i1} and v_{i5} respectively in D.

Therefore, D is a dominating set of G and $\langle D\rangle \cong P_{3n}$, $\langle V(G) - D\rangle$ is a connected graph obtained from $P_2 \times P_3$ by attaching a path of length n – 2 and n – 4 at v_{34} and v_{32} respectively. Since $\langle D\rangle$ is a tree and $\langle V(G) - D\rangle$ is connected, D is a nstd - set of G and $\gamma_{nstd}(G) \le |D| = n + 4 + n - 2 + n - 3 + 1 = 3n$.

The graph $P_5 \times P_n$ can be divided into two blocks $P_3 \times P_n$ and $P_2 \times P_n$. $\gamma_{nstd}(P_2 \times P_n) = n$ and $\gamma_{nstd}(P_3 \times P_n) = 2n$ Let D be a minimum nstd - set of $P_5 \times P_n$. Then D contains 3n vertices from above the blocks $P_2 \times P_n$ and $P_3 \times P_n$. Therefore, D contains atleast 3n vertices, $\gamma_{nstd}(P_3 \times P_n) \ge 3n$.

Theorem 3.8: For the graph $P_6 \times P_n$, $\gamma_{nstd}(P_6 \times P_n) = 3(n + 1)$, $n \ge 6$.

Proof: Let $G \cong P_6 \times P_n$ and let $V(G) = \bigcup_{i=1}^{n}\{v_{i1}, v_{i2}, v_{i3}, v_{i4}, v_{i5}, v_{i6}\}$ where

$\langle\{v_{i1}, v_{i2}, v_{i3}, v_{i4}, v_{i5}, v_{i6}\}\rangle \cong P_6^i$, i = 1, 2, 3, 4, 5, 6 and $\langle\{v_{1j}, v_{2j}, \dots, v_{nj}\}\rangle \cong P_n^j$, j = 1, 2, ... ,n and P_6^i is the i^{th} copy of P_6 and P_n^j is the j^{th} copy of P_n in G.

$$\text{Let } D = \left(\bigcup_{i=1}^{n}\{v_{i1}\}\right) \cup \left(\bigcup_{i=1}^{5}\{v_{1,i+1}\}\right) \cup \left(\bigcup_{i=1}^{n-1}\{v_{i+1,6}\}\right) \cup \left(\bigcup_{i=1}^{n-2}\{v_{i+2,3}\}\right) \cup \{v_{n2}\}.$$

Then $D \subseteq V(G)$. Here, for i = 3,4, ... ,n, v_{i2} and v_{i4} in $V(G) - D$ are adjacent to v_{i3} in D and for i = 3, 4, 5, … ,n,v_{i5} in $V(G) - D$ is adjacent to v_{i6} in D and for$1 \le i \le 6$, v_{2i} in $V(G) - D$ is adjacent to v_{1i} in D. Therefore, D is a dominating set of G and $\langle D\rangle \cong P_{3n+3}$ and $\langle V(G) - D\rangle$ is a connected graph obtained from $P_2 \times P_6$ by attaching a path of length n – 1 at v_{24}. Since $\langle D\rangle$ is a tree and $\langle V(G) - D\rangle$ is connected, D is a nstd - set of G and

$\gamma_{nstd}(G) \le |D| = n + 5 + n - 1 + n - 2 + 1 = 3(n + 1)$.

The graph $P_6 \times P_n$ can be divided into two blocks $P_5 \times P_n$ and P_n.

$\gamma_{nstd}(P_5 \times P_n) = 3n$ and $\gamma_{nstd}(P_n) = n - 2$. Let D be a minimum nstd - set of $P_6 \times P_n$. Then D contains 3n – 2 vertices from the blocks P_n and $P_5 \times P_n$. Since $\langle D\rangle$ is a tree, atleast five vertices (from the blocks) can be added with 3n – 2 vertices of D.

Therefore, D contains atleast 3n + 3 vertices, $\gamma_{nstd}(P_6 \times P_n) \ge 3n + 3$.

Hence, $\gamma_{nstd}(P_6 \times P_n) = 3n + 3$.

Theorem 3.9: For the graph $C_3 \times C_n$, $\gamma_{nstd}(C_3 \times C_n) = n + 1$, $n \ge 3$.

Proof: Let $G \cong C_3 \times C_n$ and let $V(G) = \bigcup_{i=1}^{n}\{v_{1i}, v_{2i}, v_{3i}\}$ where $\langle\{v_{i1}, v_{i2}, v_{i3}\}\rangle \cong C_3^i$, i = 1, 2, 3 and $\langle\{v_{1j}, v_{2j}, \dots, v_{nj}\}\rangle \cong C_n^j$, j = 1, 2, ... ,n and C_3^i is the i^{th} copy of C_3 and C_n^j is the j^{th} copy of C_n in G.

Let $D = \left(\bigcup_{i=2}^{n}\{v_{i2}\}\right) \cup \{v_{13}, v_{23}\}$. Then $D \subseteq V(G)$. Here v_{11} and v_{12} are adjacent to v_{13} and for i = 2, 3, … ,n, v_{i1} and v_{i3} in $V(G) - D$ are adjacent to v_{i2} in D. Therefore, D is a dominating set of G and $\langle D\rangle \cong P_{n+1}$ and $\langle V(G) - D\rangle$ is a connected graph obtained from C_n by attaching path of length n – 2 at v_{n1} and an edge at v_{12}.

Since $\langle D\rangle$ is a tree and $\langle V(G) - D\rangle$ is connected, D is a minimum nstd - set of G and $\gamma_{nstd}(G) = |D| = 2 + n - 1 = n + 1$.

Theorem 3.10: For the graph $C_4 \times C_n$, $\gamma_{nstd}(C_4 \times C_n) = 2n - 2$, $n \ge 4$.

Proof: Let $G \cong C_4 \times C_n$ and let $V(G) = \bigcup_{i=1}^{n}\{v_{1i}, v_{2i}, v_{3i}, v_{4i}\}$ where $\langle\{v_{i1}, v_{i2}, v_{i3}, v_{i4}\}\rangle \cong C_4^i$, i = 1, 2, 3 and $\langle\{v_{1j}, v_{2j}, \dots, v_{nj}\}\rangle \cong C_n^j$, j = 1, 2, ... ,n and C_4^i is the i^{th} copy of C_3 and C_n^j is the j^{th} copy of C_n in G.

Let $D = \left(\bigcup_{i=1}^{n-1}\{v_{i1}\}\right) \cup \left(\bigcup_{i=1}^{n-2}\{v_{i+2,3}\}\right) \cup \{v_{n-1,2}\}$.Then D ⊆ V(G). Here v_{13} and v_{23} in V(G) – D are adjacent to v_{n3} and v_{33} in D and v_{n4} and v_{n2} in V(G) – D are adjacent to v_{n3} in D and v_{n1} is adjacent to v_{11} in D and for i = 1, 2, 3, … ,n – 1, v_{i2} and v_{i4} in V(G) – D are adjacent to v_{i1} in D. Therefore, D is a dominating set of G and ⟨D⟩ is a tree obtained from P_{2n-3} by attaching a path of length 1 at $v_{n-1,3}$. Since ⟨D⟩ is a tree and ⟨V(G) – D⟩ is connected, D is a nstd - set of G and is also minimum.

Hence $\gamma_{nstd}(G) = |D| = 1 + n - 1 + n - 2 = 2n - 2$.

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