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Coefficient Estimates on Janowski Functions

P. Lokesh,

Department of Mathematics

Adhiparasakthi College of Engineering, G.B.Nagar, Kalavai, Ranipet District – 632506

Tamil Nadu, India.

***Abstract*:** In the present paper, a new subclass of bi – pseudo starlike and convex functions related with strongly Janowski functions in the open unit disc is introduced. For functions belonging to the class, estimates on the first two Taylor – Maclaurin coefficients are obtained. Also, to derive new results and known results as special cases of investigation.

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1. Introduction

Let A be the family of functions f which are analytic in the open unit disk $\Delta = \{z \in C : |z| < 1\}$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} d_k z^k \tag{1}$$

Let S mean the subclass of A consisting of univalent functions in Δ. It is well known (refer[1]) that every function $f \in S$ virtually posses an inverse of f, defined by $f^{-1}[f(z)] = z, (z \in \Delta)$ and $f[f^{-1}(w)] = w, (|w| < r_0(f); r_0(f) \geq \frac{1}{4})$, where

$$f^{-1}(w) = w - d_2 w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2 d_3 + d_4)w^4 + \ldots\ldots \tag{2}$$

When the function $f \in A$ is bi univalent, both f and f^{-1} are univalent in Δ. Let $\sum$ be the class of bi univalent functions in Δ given by (1). In fact, Srivastava et al.[12]has revived the study of analytic and bi univalent functions in recent years. Many researchers investigated and propounded various subclasses of bi univalent functions and fixed the initial coefficient coefficients $|d_2|$ and $|d_3|$ (refer [2, 3, 9, 10]).

With a view to recalling the principal of subordination between analytic functions, let f and g be analytic in Δ. We say that the function f is said to be subordination to g, if there exists a Schwarz function ω analytic in Δ with $\omega(0) = 0$ and $|\omega(z)| < 1, (z \in \Delta)$ such that $f(z) = g(\omega(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)(z \in \Delta)$. It is well known that, if the function g is univalent in Δ, then $f \prec g \Leftrightarrow f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Applying the concept of subordination, Janowski [4] investigated the class $\mathrm{P}[A, B]$ of analytic functions p such that $p(z) \prec (1 + Az)/(1 + Bz)$ for $-1 \leq A < B \leq 1, z \in \Delta$.

Let p be analytic function in Δ with $p(0)=1$. Then $p \in \mathrm{P}_{\tau}[A,B]$, if and only if

$$p(z)=\left(\frac{1+Az}{1+Bz}\right)^{\tau}, \tau \in (0,1], -1 \leq A < B \leq 1, z \in \Delta$$

where $p_{1,}, p_2 \in \mathrm{P}[A,B]$. Furthermore, let $p \in \mathrm{P}_{m,\tau}[A,B]$ if and only if

$$p(z)=\left(\frac{m}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{m}{4}-\frac{1}{2}\right)p_2(z).$$

where $p_{1,}, p_2 \in \mathrm{P}_{\tau}[A,B]$ and $m \geq 2$.

Particularly, for $\tau = 1$, the class $\mathrm{P}_{m,\tau}[A,B]$ coincides with the class $\mathrm{P}_{m}[A,B]$ introduced in [5], whereas, for $\tau = 1, A = 1-2\xi$ and $B = -1$, the class $\mathrm{P}_{m,\tau}[A,B]$ reduces to the class $p_m(\xi)$ of analytic univalent functions in p, normalized with $p(0)=1$ and satisfying

$$\int_0^{2\pi}\left|\frac{R(p(z))-\xi}{1-\xi}\right|d\theta \leq m\pi .$$

where $m \geq 2$, $\xi \in [0,1)$ and $z \in \Delta$ refer to [6]. Moreover for $\xi = 0$, the class $p_m(0)= p_m$, introduced by Pinchuk [7]. Furthermore, for $m = 2$ it is well known class p of Caratheodory functions. Also it is noted that, when $m = 2, A = 1$ and $B = -1$, then $p \in P_{2,\tau}[1,-1]$ implies $|\arg p(z)| \leq \frac{\tau\pi}{2}$.

The aim of this work, introduce a new subclass $\sum J_{[A,B]}^{\lambda,\tau}(m,\delta)$ and determine the bounds for the initial Taylor – Maclaurin coefficients of $|d_2|$ and $|d_3|$ for $f \in \sum J_{[A,B]}^{\lambda,\tau}(m,\delta)$.

Definition 1.1 A function $f \in \Sigma$ is said to be the class $f \in \sum J_{[A,B]}^{\lambda,\tau}(m,\delta)$ if the following conditions are satisfied:

$$(1-\delta)\frac{z(f'(z))^{\lambda}}{f(z)}+\delta\frac{\left[z(f'(z))'\right]^{\lambda}}{f'(z)} \in \mathrm{P}_{m,\tau}[A,B], (z \in \Delta) \tag{3}$$

and

$$(1-\delta)\frac{w(g'(w))^{\lambda}}{g(w)}+\delta\frac{\left[w(g'(w))'\right]^{\lambda}}{g'(w)} \in \mathrm{P}_{m,\tau}[A,B], (w \in \Delta). \tag{4}$$

where $-1 \leq A < B \leq 1, m \geq 2, \lambda \geq 1, \tau \in (0,1]$ and $\delta \in [0,1]$, and $g(w)$ is given by (2).

Special cases:

(i) For $\delta = 0$, a new a class $f \in \sum J_{[A,B]}^{\lambda,\tau}(m)$ of the functions $f \in \Sigma$ such that

$$\frac{z\left(f'(z)\right)^{\lambda}}{f(z)} \in \mathrm{P}_{m,\tau}[A,B],(z \in \Delta)$$

$$\frac{w\left(g'(w)\right)^{\lambda}}{g(w)} \in \mathrm{P}_{m,\tau}[A,B],(w \in \Delta)$$

where $-1 \le A < B \le 1, m \ge 2, \lambda \ge 1, \tau \in (0,1]$ and $g(w)$ is given by (2).

(ii) For $\delta = 1$, a new a class $f \in \sum J_{[A,B]}^{\lambda,\tau}(m)$ of the functions $f \in \Sigma$ such that

$$\frac{\left[z\left(f'(z)\right)'\right]^{\lambda}}{f'(z)} \in \mathrm{P}_{m,\tau}[A,B],(z \in \Delta)$$

$$\frac{\left[w\left(g'(w)\right)'\right]^{\lambda}}{g'(w)} \in \mathrm{P}_{m,\tau}[A,B],(w \in \Delta)$$

where $-1 \le A < B \le 1, m \ge 2, \lambda \ge 1, \tau \in (0,1]$ and $g(w)$ is given by (2).

2. Main Results

The following lemmas are required to prove our investigations.

Lemma 2.1. [8] Let $q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n$ be subordinate to $Q(z) = \sum_{n=1}^{\infty} Q_n z^n$. If $Q(z)$ is univalent in Δ and $Q(\Delta)$ is convex, then $|q(n)| \le |Q(1)|, n \ge 1$.

The following lemma can be easily proved by using Lemma 2.1along with the definition of $\mathrm{P}_{\tau}[A,B]$.

Lemma 2.2. For $-1 \le A < B \le 1, m \ge 2, \tau \in (0,1]$ and let $p \in \mathrm{P}_{\tau}[A,B]$ with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, then $|p_n| \le \tau(A-B), n \ge 1$.

Lemma 2.3. Let $-1 \le A < B \le 1, m \ge 2, \tau \in (0,1]$ and let $p \in \mathrm{P}_{m,\tau}[A,B]$ with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Then

$$|p_n| \le \frac{m\tau}{2}(A-B), n \ge 1$$

Proof. This proof is straight forward by using Lemma 2.2 along with the definition of $\mathrm{P}_{m,\tau}[A,B]$.

Theorem 2.4. For $-1 \le A < B \le 1, m \ge 2, \lambda \ge 1, \tau \in (0,1]$, $\delta \in [0,1]$ and let $f \in \sum J_{[A,B]}^{\lambda,\tau}(m,\delta)$ be given by (1). Then

$$|d_2| \le \min\left\{\sqrt{\frac{m\tau(A-B)}{2\left((1+2\delta)(3\lambda-1)+(1+3\delta)\left(2\lambda^2-4\lambda+1\right)\right)}}, \frac{m\tau}{2(1+\delta)(2\lambda-1)}\right\},$$

$$|d_3| \leq \left\{ \begin{array}{c} \dfrac{m\tau(A-B)}{2(1+2\delta)(3\lambda-1)} + \dfrac{m\tau(A-B)}{2((1+2\delta)(3\lambda-1)+(1+3\delta)(2\lambda^2-4\lambda+1))}; \\ \dfrac{m\tau(A-B)}{2(1+2\delta)(3\lambda-1)}\left(1+\dfrac{2m\tau(A-B)(1+3\delta)(2\lambda^2-4\lambda+1)}{4(1+\delta)^2(2\lambda-1)^2}\right); \\ \dfrac{m\tau(A-B)}{2(1+2\delta)(3\lambda-1)}\left(1+\dfrac{2m\tau(A-B)(2(1+2\delta)(3\lambda-1)+(1+3\delta)(2\lambda^2-4\lambda+1))}{4(1+\delta)^2(2\lambda-1)^2}\right) \end{array} \right\}$$

and

$$\left|d_3-\eta d_2^2\right| \leq \frac{m\tau(A-B)}{2(1+2\delta)(3\lambda-1)}.$$

where $\eta = \dfrac{2(1+2\delta)(3\lambda-1)+(1+3\delta)(2\lambda^2-4\lambda+1)}{(1+2\delta)(3\lambda-1)}$.

Proof. Let $f \in \sum J_{[A,B]}^{\lambda,\tau}(m,\delta)$ be given by (1). Then there exists two analytic functions $u,v \in P_{m,\tau}[A,B]$ with

$$u(z)=1+u_1z+u_2z^2+u_3z^3+\ldots(z\in\Delta) \tag{5}$$

and

$$v(w)=1+v_1w+v_2w^2+v_3w^3+\ldots(w\in\Delta) \tag{6}$$

with $u(0)=v(0)=0, |u(z)|<1, |v(z)|<1, z,w\in\Delta$, such that

$$(1-\delta)\frac{z(f'(z))^\lambda}{f(z)}+\delta\frac{\left[z(f'(z))'\right]^\lambda}{f'(z)}=u(z) \tag{7}$$

and

$$(1-\delta)\frac{w(g'(w))^\lambda}{g(w)}+\delta\frac{\left[w(g'(w))'\right]^\lambda}{g'(w)}=v(w) \tag{8}$$

From (7) and (8), comparing the coefficients of z, w, z^2 and w^2, it is obtained that

$$(1+\delta)(2\lambda-1)d_2=u_1, \tag{9}$$

$$(1+2\delta)(3\lambda-1)d_3+(1+3\delta)(2\lambda^2-4\lambda+1)d_2^2=u_2, \tag{10}$$

$$-(1+\delta)(2\lambda-1)d_2=v_1, \tag{11}$$

$$(1+2\delta)(3\lambda-1)(2d_2^2-d_3)+(1+3\delta)(2\lambda^2-4\lambda+1)d_2^2=v_2. \tag{12}$$

From (9) and (11), it can be written as

$$d_2 = \frac{u_1}{(1+\delta)(2\lambda-1)} = -\frac{v_1}{(1+\delta)(2\lambda-1)}. \tag{13}$$

Applying Lemma 2.3. it follows that

$$|d_2| \le \frac{m\tau(A-B)}{2(1+\delta)(2\lambda-1)} \tag{15}$$

Adding (10) and (12), it is clear that

$$2\left((1+2\delta)(3\lambda-1)+(1+3\delta)\left(2\lambda^2-4\lambda+1\right)\right)d_2^2 = u_2+v_2 \tag{16}$$

By applying Lemma 2.3 and simple computations yields

$$|d_2| \le \sqrt{\frac{m\tau(A-B)}{2\left((1+2\delta)(3\lambda-1)+(1+3\delta)\left(2\lambda^2-4\lambda+1\right)\right)}}. \tag{17}$$

Subtracting (10) and (12),

$$d_3 = \frac{u_2-v_2}{2(1+2\delta)(3\lambda-1)} + d_2^2 \tag{18}$$

Now, employing Lemma 2.3 and (15) , it is found that

$$|d_3| \le \frac{m\tau(A-B)}{2(1+2\delta)(3\lambda-1)} + \frac{m\tau(A-B)}{2\left((1+2\delta)(3\lambda-1)+(1+3\delta)\left(2\lambda^2-4\lambda+1\right)\right)}. \tag{19}$$

On making use of (9) and (10), it can be easily obtained that

$$|d_3| \le \frac{m\tau(A-B)}{2(1+2\delta)(3\lambda-1)}\left(1+\frac{2m\tau(A-B)(1+3\delta)\left(2\lambda^2-4\lambda+1\right)}{4(1+\delta)^2(2\lambda-1)^2}\right). \tag{20}$$

Again, using (9) and (10), finally found

$$|d_3| \le \frac{m\tau(A-B)}{2(1+2\delta)(3\lambda-1)}\left(1+\frac{2m\tau(A-B)\left(2(1+2\delta)(3\lambda-1)+(1+3\delta)\left(2\lambda^2-4\lambda+1\right)\right)}{4(1+\delta)^2(2\lambda-1)^2}\right).$$

(21)

From (12), it can be write

$$\frac{2(1+2\delta)(3\lambda-1)+(1+3\delta)\left(2\lambda^2-4\lambda+1\right)}{(1+2\delta)(3\lambda-1)}d_2^2 - d_3 = \frac{v_2}{(1+2\delta)(3\lambda-1)} \tag{22}$$

By applying Lemma 2.3. it implies

$$\left|d_3-\eta d_2^2\right| \le \left|\frac{v_2}{(1+2\delta)(3\lambda-1)}\right| \le \frac{m\tau(A-B)}{2(1+2\delta)(3\lambda-1)}. \tag{23}$$

where $\eta = \dfrac{2(1+2\delta)(3\lambda-1)+(1+3\delta)\left(2\lambda^2-4\lambda+1\right)}{(1+2\delta)(3\lambda-1)}$.

It is noted that for specializing the parameters, as mentioned in special cases of (i) and (ii) in Definition 1.1, the following new results.

Taking the parameter $\delta = 0$, the following corollary 2.5 is obtained.

Corollary 2.5 Let $f \in \sum J_{[A,B]}^{\lambda,\tau}(m)$. Then

$$|d_2| \le \min\left\{\sqrt{\frac{m\tau(A-B)}{2(2\lambda^2-\lambda)}}, \frac{m\tau}{2(2\lambda-1)}\right\}$$

and

$$|d_3| \le \left\{\begin{array}{c} \dfrac{m\tau(A-B)}{2(3\lambda-1)} + \dfrac{m\tau(A-B)}{2(2\lambda^2-\lambda)}; \\ \dfrac{m\tau(A-B)}{2(3\lambda-1)}\left(1+\dfrac{2m\tau(A-B)(2\lambda^2-4\lambda+1)}{4(2\lambda-1)^2}\right); \\ \dfrac{m\tau(A-B)}{2(3\lambda-1)}\left(1+\dfrac{2m\tau(A-B)(2\lambda^2+2\lambda-1)}{4(2\lambda-1)^2}\right) \end{array}\right\}$$

with $-1 \le A < B \le 1, m \ge 2, \lambda \ge 1, \tau \in (0,1]$. Moreover

$$|d_3 - \eta d_2^2| \le \frac{m\tau(A-B)}{2(3\lambda-1)}$$

where $\eta = \dfrac{2\lambda^2+2\lambda-1}{(3\lambda-1)}$.

Setting the parameter $\delta = 1$, the following corollary 2.6. becomes

Corollary 2.6. Let $f \in \sum J_{[A,B]}^{\lambda,\tau}(m)$. Then

$$|d_2| \le \min\left\{\sqrt{\frac{m\tau(A-B)}{2(8\lambda^2-7\lambda+1)}}, \frac{m\tau}{4(2\lambda-1)}\right\}$$

And

$$|d_3| \le \left\{\begin{array}{c} \dfrac{m\tau(A-B)}{6(3\lambda-1)} + \dfrac{m\tau(A-B)}{2(8\lambda^2-7\lambda+1)}; \\ \dfrac{m\tau(A-B)}{6(3\lambda-1)}\left(1+\dfrac{2m\tau(A-B)(2\lambda^2-4\lambda+1)}{4(2\lambda-1)^2}\right); \\ \dfrac{m\tau(A-B)}{6(3\lambda-1)}\left(1+\dfrac{2m\tau(A-B)(4\lambda^2+\lambda-1)}{8(2\lambda-1)^2}\right) \end{array}\right\}$$

with $-1 \le A < B \le 1, m \ge 2, \lambda \ge 1, \tau \in (0,1]$. Moreover

$$\left|d_3 - \eta d_2^2\right| \le \frac{m\tau(A-B)}{6(3\lambda-1)}$$

where $\eta = \dfrac{2(4\lambda^2+\lambda-1)}{3(3\lambda-1)}$.

References

[1] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[2] B. A. Frasin and M. K. Aouf, *New subclasses of bi-univalent functions,* Appl. Math. Lett., 24 (2011), 1569–1573.

[3] S. P. Goyal and P. Goswami, *Estimate for initial Maclaurin coefficients of bi-univalent functions for a class*

[4] *defined by fractional derivatives*, J. Egyptian Math. Soc., 20 (2012), 179-182.

[5] W. Janowski, *Some extremal problems for certain families of analytic functions*, Ann.Polon. Math. **28** (1973), 297-326.

[6] K.I. Noor, *Some radius of convexity problems for analytic functions of bounded boundary rotation*, Punjab Univ. J. Math. **21** (1988), 71-81.

[7] K. Padmanabhan and R. Parvatham, *Properties of a class of functions with bounded boundary rotation,* Ann. Polon. Math. **31** (1975), 311-323.

[8] B. Pinchuk, *Functions with bounded boundary rotation*, Isr. J. Math. **10** (1971), 7-16.

[9] W. Rogosinski, *On the coefficients of subordinate functions*, Proc. Lond. Math. Soc.51(1975), 109-116.

[10] H. M. Srivastava, S. S. Eker and R. M. Ali, *Coefficient bounds for a certain class of analytic and bi-univalent functions,* Filomat, 29(2015), 1839–1845.

[11] H. M. Srivastava, S. S. Eker, S. G. Hamidi, J. M. Jahangiri, *Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator*, Bull. Iranian Math. Soc.,44(1)(2018), 149-157.

[12] H. M. Srivastava, S. Gaboury and F. Ghanim, *Coefficient estimates for some general subclasses of analytic and bi-univalent functions*, Africa Math., 28(2017), 693-706.

[13] H. M. Srivastava, A. K. Mishra and P. Gochhayat, *Certain subclasses of analytic and Bi univalent functions*, Appl. Math. Lett., 23(2010), 1188–1192.