Orthogonal Symmetric Reverse Bi-(σ,τ)-Derivations in Semiprime Rings

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Abstract:- Suppose that $D_1$and D_2 are two symmetric reverse bi-(σ,τ)-derivations of a semi prime ring R. In this paper, we establish some equivalent conditions for the orthogonality between two symmetric reverse bi-(σ,τ)-derivations D_1, D_2 of R. We prove that D_1 and D_2 are orthogonal if and only if any one of the following conditions is satisfied (i) $D_1(u,v)\, D_2(v,w) + D_2(u,v)D_1(v,w) = 0$ (ii) $D_1D_2 = 0$ (iii) $D_1(u,v)D_2(v,w) = 0 = D_2(u,v)D_1(v,w)$ (iv) D_1D_2 is a bi- (σ,τ)-derivation, for all $u,v,w \in R$.

Keywords: *Semiprime ring, Reverse Biderivation, Reverse Bi-(σ,τ)-derivation, Orthogonal Reverse Bi-(σ,τ)-derivation*

1. Introduction

The concept of orthogonality for a pair of derivations of a semiprime ring was introduced by M. Bresar and J.Vukman [7] and they proved some equivalent conditions on derivations of semiprime rings to be orthogonal which are related to classical result of E. Posner [4]. M.N. Daif et al. [9] established some results concerning biderivations. C. Jaya Subba Reddy et al. [1,3] and M.N. Daif et al. [8] proved some results on the orthogonality of derivations, biderivations. K. Kaya et al. [5] and M.Ashraf [6] studied (σ,τ)-derivations in prime rings. The study of orthogonality of symmetric bi-(σ,τ)-derivations in semi prime rings was carried out by C.Jaya Subba Reddy et al. [2] and S.Srinivasulu et al. [10]. In the present paper, we extend the results related to symmetric bi-(σ,τ)-derivation established in [2] to symmetric reverse bi-(σ,τ)-derivations on orthogonality **.**

2. Preliminaries

Throughout this paper, R denotes an associative ring with centre Z(R). A ring R is said to be semiprime if $uRu = \{0\}$ implies $u = 0$. We say that R is 2-torsion-free if $2u = 0,\ \forall\, u \in R$, implies $u = 0$. An additive mapping $d: R \to R$ is said to be a derivation (respectively, reverse derivation) on R if $d(uv) = d(u)v + ud(v)$ (respectively, $d(uv) = d(v)u + vd(u)$ holds $\forall\, u, v \in R$. Suppose that σ and τ are automorphisms of R. An additive mapping $d: R \to R$ is said to be a (σ,τ)-derivation (respectively, reverse (σ,τ)-derivation) on R if $d(uv) = d(u)\sigma(v) + \tau(u)d(v)$ (respectively, $d(uv) = d(v)\sigma(u) + \tau(v)d(u)$, $\forall\, u, v \in R$.
A biadditive mapping $D: RX\,R \to R$ is called symmetric if $D(u,v) = D(v,u), \forall\, u, v \in R$. A symmetric biadditive mapping $D_1: RX\,R \to R$ is said to be a symmetric biderivation on R if $D_1(uv,w) = uD_1(v,w) + D_1(u,w)v$, $\forall\, u, v, w \in R$. A symmetric biadditive mapping $D_1: RX\,R \to R$ is said to be a symmetric bi-(σ,τ)-derivation (respectively, symmetric reverse bi-(σ,τ)-derivation) on R if $D_1(uv,w) = D_1(u,w)\sigma(v) + \tau(u)D_1(v,w)$ (respectively, $D_1(uv,w) = D_1(v,w)\sigma(u) + \tau(v)D_1(u,w)$) holds $\forall\, u, v, w \in R$. Two symmetric reverse bi-(σ,τ)-derivations are said to be orthogonal if $D_1(u,v)RD_2(v,w) = 0 = D_2(v,w)R\,D_1(u,v)$, for all $u, v, w \in R$.

We assume throughout the paper that R is a 2-torsion-free semi prime ring while σ and τ are automorphisms of R and d is a reverse (σ,τ)-derivation of R such that dτ = τd, σd = dσ. We also assume two symmetric reverse bi-(σ,τ)-derivations $D_1: RX\,R \to R$ and $D_2: RX\,R \to R$ such that $D_1\tau = \tau D_1$, $D_2\tau = \tau D_2$, $\sigma D_1 = D_1\sigma$. $\sigma D_2 = D_2\sigma$.

Lemma:1 [[7], Lemma 1]

If R is a 2-torsion free semi prime ring and $u, v \in R$, then the following conditions are equivalent:

1. $urv = 0$, for all $r \in R$. 2. $vru = 0$, for all $r \in R$. 3. $urv + vru = 0$, for all $r \in R$

If anyone of the above conditions is fulfilled, then $uv = vu = 0$

Lemma 2: [[3], Lemma 2]

Let R be a semiprime ring. Suppose that two bi-additive mappings $D_1: R\ X\ R \to R$ and $D_2: R\ X\ R \to R$ satisfies $D_1(u,v)RD_2(v,u) = 0$, for all $u, v \in R$, then $D_1(u,v)R\,D_2(v,w) = 0$, for all $u, v, w \in R$.

3. Main Results

Theorem 1:

Let R be a 2 torsion free semi prime ring. Then the following conditions are equivalent:

1.Two symmetric reverse bi-(σ,τ)-derivations D_1 and D_2 are orthogonal.

2.$D_1(u,v)D_2(v,w) + D_2(u,v)D_1(v,w) = 0, \ \forall\, u,v,w \in R$.

Proof: (1) ⟹ (2)

Suppose that D_1 and D_2 are orthogonal reverse bi-(σ,τ)-derivations

Then, $D_1(u,v)RD_2(v,w) = 0$ and also $D_2(u,v)RD_1(v,w) = 0, \ \forall\, u,v,w \in R$

which means $D_1(u,v)D_2(v,w) = 0$ and $D_2(u,v)D_1(v,w) = 0$ (Since $1 \in R$)

and so $D_1(u,v)D_2(v,w) + D_2(u,v)D_1(v,w) = 0, \forall\, u,v,w \in R$

(2) ⟹ (1)

Suppose that $D_1(u,v)D_2(v,w) + D_2(u,v)D_1(v,w) = 0, \ \forall\, u,v,w \in R$ (3.1)

Replace w by uw in the equation (3.1)

$D_1(u,v)D_2(v,uw) + D_2(u,v)D_1(v,uw) = 0$

$(D_1(u,v)D_2(v,w) + D_2(u,v)D_1(v,w))\sigma(u) + D_1(u,v)\,\tau(w)D_2(v,u) + D_2(u,v)\tau(w)D_1(v,u) = 0$

Using (3.1) and the fact that τ is an automorphism, we deduce that

$D_1(u,v)w_1D_2(v,u) + D_2(u,v)w_1D_1(v,u) = 0, \forall\, u,v,w_1 \in R$

By Lemma 1, we get $D_1(u,v)RD_2(v,u) = 0$

By Lemma 2, we get $D_1(u,v)RD_2(v,w) = 0$

Therefore D_1 and D_2 are orthogonal reverse bi-(σ,τ)-derivations. Hence Proved.

Theorem 2:

Let R be a 2 torsion free semi prime ring. Then the following conditions are equivalent:

1.Two symmetric reverse bi-(σ,τ)-derivations D_1 and D_2 are orthogonal. 2. $D_1D_2 = 0$.

Proof: (1)⟹ (2)

Suppose that D_1 and D_2 are orthogonal symmetric reverse bi-(σ,τ)-derivations

we prove that $D_1D_2 = 0$

Since D_1, D_2 are orthogonal, we can have $D_1(u,v)RD_2(v,w) = 0$,

and so $D_1(u,v)\,rD_2(v,w) = 0, \forall\, u, v, w, r \in R$

Therefore, $D_1(D_1(u,v)\,rD_2(v,w), m) = 0, \forall\, u,v,w,r,m \in R$

$D_1(rD_2(v,w),m)\ \sigma(D_1(u,v)) + \tau(rD_2(v,w))D_1(D_1(u,v),m) = 0, \forall\, u,v,w,r,m \in R$

$(D_1(D_2(v,w),m)\sigma(r) + \tau(D_2(v,w)D_1(r,m))\sigma(D_1(u,v)) + \tau(rD_2(v,w))D_1(D_1(u,v),m) = 0,$

$\forall\, u,v,w,r,m \in R$

$D_1(D_2(v,w),m)\sigma(r)\sigma(D_1(u,v)) + \tau(D_2(v,w))D_1(r,m))\sigma(D_1(u,v)) + \tau(rD_2(v,w))D_1(D_1(u,v),m) = 0$

Using $\tau D_2 = D_2\tau$; $\sigma D_1 = D_1\sigma$ and σ and τ are automorphisms of R, we obtain

$D_1(D_2(v,w),m)r_1D_1(u_1,v_1) + D_2(v_1,w_1)D_1(r,m)\ D_1(u_1,v_1) + r_1D_2(v_1,w_1)D_1(D_1(u,v),m) = 0,$

$\forall\ u_1,v_1,w_1,r_1,u\ ,v\ ,w,m \in R$

$D_1(D_2(v,w),m)r_1D_1(u_1,v_1) = 0$ (By using the condition of orthogonality of $D_{1\,,}D_2$)

In Particular if we put $u_1 = D_2(v,w)$ in the above equation, we get

$D_1D_2(v,w)\ r_1D_1(D_2(v,w),v_1) = 0, \forall\ v_1,r_1,v\ ,w \in R$

$D_1D_2(v,w)\ r_1\ D_1D_2(v,w) = 0,$

$D_1D_2(v,w) = 0$ (Using the Semiprimeness of R.) and so $D_1D_2 = 0$

(2) $\Longrightarrow$ (1)

Let D_1 and D_2 be two reverse bi-(σ,τ)-derivations such that $D_1D_2 = 0$

we prove that D_1 and D_2 are orthogonal.

$D_1D_2(uv,w) = D_1(D_2(uv,w),m)$

= $D_1((D_2(v,w)\sigma(u) + \tau(v)D_2(u,w)$, m)

= $D_1(D_2(v,w)\sigma(u),m\ ) + D_1(\tau(v)D_2(u,w)$, m)

= $D_1(\sigma(u),m)\sigma(D_2(v,w)) + \tau(\sigma(u))D_1(D_2(v,w),m)$ + $D_1(D_2(u,w),m)\sigma(\tau(v)) + \tau(D_2(u,w)D_1(\tau(v)$, m)

Therefore.$D_1D_2(uv,w) = D_1(\sigma(u),m)\sigma(D_2(v,w)) + \tau(\sigma(u))D_1(D_2(v,w),m)$ +

$D_1(D_2(u,w),m)\sigma(\tau(v)) + \tau(D_2(u,w))D_1(\tau(v)$, m)

= $D_1(\sigma(u),m)\sigma(D_2(v,w)) + \tau(\sigma(u))D_1D_2(v,w) + D_1D_2(u,w)\sigma(\tau(v)) + \tau(D_2(u,w)D_1(\tau(v)$, m)

= $D_1(\sigma(u),m)\sigma(D_2(v,w)) + \tau(D_2(u,w)D_1(\tau(v)$, m) (Since $D_1D_2 = 0$)

Using $D_2\sigma = \sigma D_2$; $\tau D_2 = D_2\tau$; and σ and τ are automorphisms of R, we get

= $D_1(u_1,m)D_2(v_1,w_1) + D_2\ (u_1,w_1)D_1(v_1$, m)

Taking $u_1 = u,\ v_1 = v,\ m = w_1 = w$, we get $D_1(u,w)\ D_2(\,v,w)$ + $D_2(u,w)D_1(v,w)$ = 0

$D_1(u,w)\ D_2(\,w,v)$ + $D_2(u,w)D_1(w,v)$ = 0, $\forall\, u,v,w \in R$ (Since $D_{1,}D_2$ are symmetric)

By Theorem 1, we can conclude that D_1 and D_2 are orthogonal reverse bi-(σ,τ)-derivations.

Theorem 3:

Let R be a 2 torsion free semiprime ring. Two symmetric reverse bi- (σ,τ)-derivations D_1 and D_2 are orthogonal if and only if $D_1(u,v)D_2\,(v,w) = D_2\,(u,v)\ D_1(v,w) = 0$, for all $u,v,w \in R$.

Proof: (1) $\Longrightarrow$ (2)

Case (i) :

D_1 and D_2 are orthogonal reverse bi-(σ,τ)-derivations $\Longleftrightarrow D_1(u,v)D_2\,(v,w) = 0$

Let D_1 and D_2 be orthogonal.

Then we prove $D_1(u,v)RD_2(v,w)$= 0 = $D_2(v,w)RD_1(u,v),\ \forall\, u,v,w \in R.$

Since $1 \in R$, we have $D_1(u,v)D_2(v,w) = 0 = D_2(v,w)D_1(u,v)$, $\forall\, u,v,w \in R$ (3.2)

and hence we can conclude that $D_1(u,v)D_2(v,w) = 0$

Conversely, suppose that $D_1(u,v)D_2(v,w) = 0$, $\forall\, u,v,w \in R$. (3.3)

Replacing w = uw in the equation (3.3), we get $D_1(u,v)D_2(v,w)\sigma(u) + D_1(u,v)\,\tau(w)D_2(v,u) = 0$, $\forall\, u,v,w \in R$

By using (3.3) in the above equation, we obtain ,

$D_1(u,v)\,\tau(w)D_2(v,u) = 0$, $\forall\, u,v,w \in R$.

$D_1(u,v)\,w_1D_2(v,u) = 0$, $\forall\, u,v,w_1 \in R$. (Since σ is an automorphism of R)

$D_1(u,v)RD_2(v,u) = 0$

$D_1(u,v)RD_2(v,u) = 0$ (By Lemma 2)

Hence, D_1 and D_2 are orthogonal reverse bi-(σ,τ)-derivations.

Case (ii) : D_1 and D_2 are orthogonal reverse bi-(σ,τ)-derivations $\Leftrightarrow D_2(u,v)D_1(v,w) = 0$

Let D_1 and D_2 be orthogonal, then $D_1(u,v)RD_2(v,w) = 0 = D_2(v,w)R\,D_1(u,v)$, $\forall\, u,v,w \in R$.

Since $1 \in R$, we have $D_1(u,v)D_2(v,w) = 0 = D_2(v,w)\,D_1(u,v)$, $\forall\, u,v,w \in R$.

By the symmetricity of D_1 and D_2, we can have

$D_2(v,w)\,D_1(u,v) = 0$ implies $D_2(w,v)D_1(v,u) = 0$

which is same as $D_2(u,v)D_1(v,w) = 0$, $\forall\, u,v,w \in R$. Hence, proved .

Conversely, Suppose that $D_2(u,v)D_1(v,w) = 0$, $\forall\, u,v,w \in R$. (3.4)

Replacing w = uw in the above equation (3.4),

$D_2(u,v)D_1(v,w)\sigma(u) + D_2(u,v)\tau(w)D_1(v,u) = 0$

By using the equation (3.4) in the above equation and τ is an automorphism of R, we obtain

$D_2(u,v)RD_1(v,u) = 0$

$D_2(u,v)RD_1(v,w) = 0$, $\forall\, u,v,w \in R$ (By Lemma 2)

Hence, D_1 and D_2 are orthogonal reverse bi-(σ,τ)-derivation.

Theorem 4:

Let R be a 2 torsion free semiprime ring. Two symmetric reverse bi- (σ,τ)-derivations D_1 and D_2 are orthogonal if and only if D_1D_2 is a bi- (σ,τ)-derivation .

Proof: D_1 and D_2 are orthogonal $\Leftrightarrow$ D_1D_2 is a bi- (σ,τ)-derivation

Suppose that D_1 and D_2 are orthogonal reverse bi-(σ,τ)-derivations.

By the definition of orthogonality of D_1 and D_2,

we can write $D_1(u,v)RD_2(v,w) = 0 = D_2(u,v)RD_1(v,w)$, $\forall\, u,v,w \in R$ (3.5)

We have to prove that D_1D_2 is a bi- (σ,τ)-derivation

Using the same process we followed in Theorem 2, we obtain

$$D_1D_2(uv,w) = D_1(\sigma(u),m)\sigma\big(D_2(v,w)\big) + \tau\big(\sigma(u)\big)D_1(D_2(v,w),m) + D_1(D_2(u,w),m)\sigma\big(\tau(v)\big) + \tau(D_2\,(u,w))D_1(\tau(v),\mathrm{m})$$

Using $\sigma D_2 = D_2\sigma$; $\tau D_2 = D_2\tau$ and σ and τ are automorphisms of R, we obtain

$=D_1(u_1,m)D_2(v_1,w_1)+\tau(u_1)D_1(D_2(v,w),m)+D_1(D_2(u,w),m)\sigma(v_1)+\ D_2(u_1,w_1)D_1(v_1,\mathrm{m})$,

$\forall\, u,v,w,u_1,v_1,w_1,m\in R$

$=D_1(u_1,m)D_2(v_1,w_1)+\ \tau(u_1)D_1D_2\,(v,w)+D_1D_2(u,w)\sigma(v_1)+D_2\,(u_1,w_1)D_1(v_1,\mathrm{m})$,

$\forall\, u,v,w,u_1,v_1,w_1,m\in R$

Replacing m = w_1 in the above equation,

$=D_1(u_1,w_1)D_2(v_1,w_1)\ +\tau(u_1)D_1D_2\,(v,w)+D_1D_2\,(u,w)\sigma(v_1)\ +D_2(u_1,w_1)D_1(v_1,w_1)$

Using the orthogonality of D_1 and D_2 , the above equation becomes

$D_1D_2(uv,w)=\tau(u_1)D_1D_2(v,w)+D_1D_2\,(u,w)\sigma(v_1)$, $\forall\, u,v,w,u_1,v_1\in R$

Taking $u_1=u$ and $v_1=v$, the above equation becomes

$D_1D_2(uv,w)=\tau(u)D_1D_2(v,w)+D_1D_2(u,w)\sigma(v)$ and hence D_1D_2 is a bi- (σ,τ)-derivation.

Conversely, Suppose that D_1D_2 is a bi- (σ,τ)-derivation .

Since D_1D_2 is a bi- (σ,τ)-derivation , we can have

$$D_1D_2(uv,w)=\ D_1D_2(u,w)\sigma(v)+\ \tau(u)D_1D_2(v,w) \tag{3.6}$$

By the earlier discussion, we can write

$D_1D_2(uv,w)=D_1(u_1,w_1)D_2(v_1,w_1)+\ \tau(u_1)D_1D_2(v,w)+D_1D_2\,(u,w)\sigma(v_1)+D_2\,(u_1,w_1)D_1(v_1,w_1)$,

$\forall\, u,v,w,u_1,v_1,w_1\in R$

In particular, taking $u_1=u$, $v_1=v$, $w_1=$ w, we get

$$D_1D_2(uv,w)=D_1(u,w)D_2(v,w)+\ \tau(u)D_1D_2(v,w)+D_1D_2\,(u,w)\sigma(v)+\ D_2\,(u,w)D_1(v,w),\quad \forall\, u,v,w\in R \tag{3.7}$$

Combining the equations (3.6) and (3.7), we obtain

$D_1D_2(u,w)\sigma(v)+\ \tau(u)D_1D_2(v,w)=\ D_1(u,w)D_2(v,w)+\ \tau(u)D_1D_2(v,w)+D_1D_2\,(u,w)\sigma(v)+$
$D_2(u,w)D_1(v,w)$

$0=D_1(u,w)D_2(v,w)+D_2(u,w)D_1(v,w)$

Hence, we can conclude that D_1 , D_2 are orthogonal.

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