# RICCI Yamabe Soliton in three-Dimensional $(\epsilon, \delta) - Trans - Sasakian\ Manifolds$

**Md. Samiul Haque[1], Shivaprasanna G. S.[2], Somashekhara G.[3]**

*1. Department of Mathematics, Acharya institute of technology, Bengaluru-560107, India.*

*2. Department of Mathematics, Dr.Ambedkar institute of technology, Bengaluru-560056, India.*

*3. Department of Mathematics and Statistics, M.S.Ramaiah University of Applied Sciences, Bengaluru-560058, India.*

***Abstract:-*** The objective of the present paper is to carry out Ricci Yamabe soliton in three-dimensional $(\varepsilon, \delta)$-trans-Sasakian manifold. We study partially Ricci Pseudo-symmetric, Weyl Ricci Pseudo-symmetric, projectively flat, Einstein semi-symmetric and ξ-projectively flat $(\varepsilon, \delta)$-trans-Sasakian manifold. Further, we obtained conditions for Ricci Yamabe solitons to be shrinking or expanding or steady.

***Keywords***: *$(\varepsilon, \delta)$-trans-Sasakian manifold, Ricci Yamabe soliton, Einstein manifold.*

## 1. Introduction

J.A.Oubina [6] initiated the notion of almost contact metric manifold called trans-Sasakian manifold in 1985, which contain the classes of cosymplectic, Sasakian and Kenmotsu manifolds which are closely related to the locally conformally Kahler manifolds of type $(0,0), (\alpha, 0)$ and $(0, \beta)$. The Kahler manifolds of type $(0,0), (\alpha, 0)$ and $(0, \beta)$ are known as cosymplectic, $\alpha$-Sasakian and $\beta$-Kenmotsu manifolds respectively. In particular, the trans-Sasakian manifold reduces to a Sasakian if $\alpha = 1,\ \beta = 0$ and Kenmotsu manifolds if $\alpha = 0,\ \beta = 1$.

A.Bejancu and K.L.Duggal [1] initiated the notion of $(\varepsilon)$-Sasakian manifolds and the extended work on this was carried out by X.Xufeng and C.Xiaoli [15] and Rakesh Kumar et al. [9]. U.C.De and A.Sarkar [14] investigated which is conformally flat, Weyl semi-symmetric, $\phi$-recurrent, $(\varepsilon)$-Kenmotsu manifolds. In [1], the researchers obtained Riemannian curvature tensor of $(\varepsilon)$- Sasakian manifolds and have established various relations of curvatures which are in different forms. Authors H.G.Nagaraja et al. [4] introduced and studied $(\varepsilon, \delta)$-trans-Sasakian manifolds which generalizes the manifolds of both $(\varepsilon)$-Sasakian and $(\varepsilon)$-Kenmotsu. Also further investigation was taken up by many scholars Y.B.Maralabhavi et al. [17], G.S.Shivaprasanna et al. [2,3]. In 1988, Hamilton [10] initiated the notion of Ricci flow and Yamabe flow concurrently. The solutions of the Ricci flow and Yamabe flow are known as Ricci soliton and Yamabe soliton respectively. Currently, Guler and Crasmareanu [12] initiated the study of a new geometric flow which is a scalar combination of Ricci and Yamabe flow under the name Ricci-Yamabe map. This is also known as Ricci Yamabe flow of the type $(p, q)$. The Ricci Yamabe flow is an evolution for the metrics on the Riemannian or semi-Riemannian manifolds defined by [12]

$$\frac{\partial}{\partial t} g(t) = -2pRic(t) + q\, R(t)g(t), \quad g_0 = g(0). \tag{1.1}$$

A soliton to the Ricci Yamabe flow is known as Ricci Yamabe solitons if it moves only by one parameter group of diffeomorphism and scaling. To be precise a Ricci Yamabe soliton on Riemannian manifold $(M, g)$ is a data $(g, V, \lambda, p, q)$ satisfying

$$(L_V\, g)(U_1, U_2) + 2pS(U_1, U_2) + (2\lambda - qr)\, g(U_1, U_2) = 0, \tag{1.2}$$

where $r, S$ and $L_V$ is the scalar curvature, the Ricci tensor and the Lie-derivative along the vector field on $M$ respectively and $p,\ q,\ \lambda$ are real constants. The Ricci Yamabe soliton is said to be expanding, shrinking and steady if $\lambda$ is positive, negative and zero respectively. Equation (1.2) is known as Ricci Yamabe soliton of $(p, q)$-type, which is a generalisation of Ricci and Yamabe solitons. The Ricci Yamabe soliton is p-Ricci soliton if q=0 and q-Yamabe soliton if p=0.

Let (M, g) be 3-dimensional $(\varepsilon, \delta)$-trans-Sasakian manifold . Then the projective curvature tensor P is defined as follows [11]:

$$P(U_1, U_2)U_3 = R(U_1, U_2)U_3 - \frac{1}{2n}[S(U_2, U_3)U_1 - S(U_1, U_3)U_2. \tag{1.3}$$

The present paper is organized as follows: A brief review of Ricci Yamabe soliton and trans-Sasakian manifolds. Section 2 contains preliminaries of almost contact metric manifolds and Weyl conformal curvature tensor. In section 3, 4, 5, 8, it is shown that a Ricci Yamabe soliton in 3-dimensional $(\epsilon, \delta)$-trans-Sasakian manifold with partially Ricci pseudosymmetric is a Einstein manifold if $f(\tau) \neq (\alpha^2 - \beta^2)$, with Weyl Ricci pseudosymmetric is an $\eta-$Einstein manifold and with $\phi-$projectively semi-symmetric condition is an $\eta-$Einstein manifold. In section 6, 7, we study projectively flat and $\xi-$projectively flat Ricci Yamabe soliton.

## 2. Preliminaries

Let $(M, g)$ be an almost contact metric manifold with odd-dimension consisting of a g Riemannian metric, $\eta$ a 1-form, $\zeta$ a vector field, (1, 1) tensor field forming $(\phi, \zeta, \eta, g)$ contact metric structure satisfying

$$\phi^2 U_1 = -U_1 + \eta(U_1)\zeta, \qquad \eta(\zeta) = 1, \qquad \phi\zeta = 0, \qquad \eta \circ \phi = 0. \tag{2.1}$$

Manifold of almost contact metric M is said to be (ε)-almost contact metric manifold when

$$g(\zeta, \zeta) = \varepsilon, \qquad \eta(U_1) = \varepsilon\, g(U_1, \zeta), \tag{2.2}$$

$$g(\phi\, U_1, \phi U_2) = g(U_1, U_2) - \varepsilon\, \eta(U_1)\eta(U_2), \ \forall\, U_1, U_2 \in TM, \tag{2.3}$$

where $\varepsilon = g(\zeta, \zeta) = \pm 1$. Manifold of (ε)-almost contact metric can be called as $(\varepsilon, \delta)$-trans-Sasakian manifold if

$$(\nabla_{U_1} \phi)U_2 = \alpha\, [g(U_1, U_2)\zeta - \varepsilon\, \eta(U_2)U_1] + \beta\, [g(\phi U_1, U_2)\zeta - \delta\, \eta(U_2)\phi U_1], \tag{2.4}$$

holds good for some smooth functions $\alpha$ *and* $\beta$ on $M$ and $\delta = \pm 1, \varepsilon = \pm 1$. For $\alpha = 1$, $\beta = 0$, an $(\varepsilon, \delta)$-trans-Sasakian manifold gets reduced to an (ε)-Sasakian and for $\alpha = 0$, $\beta = 1$ it reduces to a manifold of (δ)-Kenmotsu. Let $(M, g)$ be a (ε, δ)- trans-Sasakian manifold. Then from (2.4), it can be conveniently seen that

$$\nabla_{U_1} \zeta = -\varepsilon\alpha\phi U_1 - \beta\delta\phi^2\, U_1, \tag{2.5}$$

$$(\nabla_{U_1} \eta)U_2 = -\alpha\, g(U_2, \phi U_1) + \beta\, g(\phi U_1, \phi U_2), \tag{2.6}$$

$$\zeta\alpha + 2\alpha\beta\, \delta = 0. \tag{2.7}$$

In a 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold, R is the curvature tensor and S is Ricci tensor will be represented by [5]

$$R(U_1, U_2)U_3 = \left(2A - \frac{r}{2}\right) [g(U_2, U_3)U_1 - g(U_1, U_3)U_2] + B[(g(U_2, U_3)\eta(U_1)\zeta$$
$$- g(U_1, U_3)\eta(U_2)\zeta) + \eta(U_3)(\eta(U_2)U_1 - \eta(U_1)U_2)], \tag{2.8}$$

$$R(U_1, U_2)\zeta = (\beta^2 - \alpha^2)\, [\eta(U_1)U_2 - \eta(U_2)U_1], \tag{2.9}$$

$$R(\zeta, U_1)U_2 = (\beta^2 - \alpha^2)[\eta(U_2)U_1 - \varepsilon\, g(U_1, U_2)\zeta], \tag{2.10}$$

$$S(U_1, U_2) = Ag(U_1, U_2) + B\, \eta(U_1)\eta(U_2), \tag{2.11}$$

where $A = \frac{r}{2} - (\alpha^2 - \beta^2)$, $B = 3(\alpha^2 - \beta^2) - \varepsilon \frac{r}{2}$ $and$ $\varepsilon\delta = 1$.

In view of (1.2) and (2.5) we obtain

$$S(U_1, U_2) = \left(\frac{\{qr-2\lambda - 2\beta\,\delta\}}{\{2p\}}\right) g(U_1, U_2) + \left(\frac{\{\beta\}}{\{p\}}\right) \eta(U_1)\eta(U_2), \tag{2.12}$$

$$S(U_1, \zeta) = \varepsilon \left(\frac{\{qr-2\lambda\}}{\{2p\}}\right) \eta(U_1), \tag{2.13}$$

$$Q\zeta = \left(\frac{\{qr-2\lambda\}}{\{2p\}}\right) \zeta. \tag{2.14}$$

For a three-dimensional almost contact metric manifold M, the Weyl conformal curvature tensor $\widehat{W}$ is defined as

$$\widehat{W}(U_1, U_2)U_3 = R(U_1, U_2)U_3 - [S(U_2, U_3)U_1 - S(U_1, U_3)U_2 + g(U_2, U_3)QU_1$$
$$- g(U_1, U_3)QU_2] + (\frac{r}{2}) [g(U_2, U_3)U_1 - g(U_1, U_3)U_2]. \tag{2.15}$$

In view of (2.1), (2.3), (2.10), (2.13), (2.14) and (2.15) we get

$$\widehat{W}(\zeta, U_2)U_3 = \left[(\beta^2 - \alpha^2) + \varepsilon \left(\frac{\{(p+q)-2\lambda\}}{\{2p\}}\right)\right] [\eta(U_3)U_2 - \varepsilon\, g(U_2, U_3)\zeta] - [S(U_2, U_3)\zeta - \varepsilon\, \eta(U_3)QU_2], \tag{2.16}$$

$$\widehat{W}(\zeta, U_2)\zeta = \left[(\beta^2 - \alpha^2) + \varepsilon \left(\frac{\{(p+q)-2\lambda\}}{\{2p\}}\right)\right] [U_2 - \eta(U_2)\zeta] - \left[\varepsilon \left(\left(\frac{\{qr-2\lambda\}}{\{2p\}}\right) \eta(U_2)\zeta - QU_2\right)\right], \tag{2.17}$$

$$\widehat{W}(\zeta, \zeta)U_3 = 0. \tag{2.16}$$

A 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold (M, g) is known as Einstein semi-symmetric if and only if $R.\hat{E} = 0$ [18], where $\hat{E}$ is the Einstein tensor given by

$$\hat{E}(U_2, U_3) = S(U_2, U_3) - \frac{r}{3}\, g(U_2, U_3). \tag{2.19}$$

A 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold (M, g) is known as partially Ricci pseudosymmetric if and only if the relation given by [13]

$$R.S = f(\tau)Q(g, S), \tag{2.20}$$

holds on the set $\cup = (u \in M: Q(g, S) \neq 0 \;\; at \;\; u)$, where $f \in C^{\infty}(M)$ for $\tau \in \cup$. $R.S$, $Q(g, S)$ and $(U_1 \wedge_g U_2)$ are defined as follow

$$(R(U_1, U_2).S)(V_1, V_2) = -S(R(U_1, U_2)V_1, V_2) - S(V_1, R(U_1, U_2)V_2), \tag{2.21}$$

$$Q(g, S) = \left((U_1 \wedge_g U_2).S\right)(V_1, V_2), \tag{2.22}$$

$$(U_1 \wedge_g U_2)U_3 = g(U_2, U_3)U_1 - g(U_1, U_3)U_2, \tag{2.23}$$

for all $U_1, U_2, U_3, V_1$ $and$ $V_2$ on M.

A 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold (M, g) is known as Weyl Ricci pseudosymmetric if the tensor $\widehat{W}.S$ and $Q(g, S)$ are linearly dependent [7,8,16] that is given by

$$\widehat{W}.S = L_S Q(g, S), \tag{2.24}$$

and holds on the set $\cup_S = (u \in M: \widehat{W} \neq 0 \;\; at \;\; u)$, where $L_S$ is some function on $\cup_S$.

## 3. Ricci Yamabe soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold with Partially Ricci Pseudosymmetric condition

Suppose that $(M, g)$ be a partially Ricci pseudosymmetric 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold, then by (2.20), (2.22), we get

$$(R(U_1, U_2).S)(U_3, V_1) = f(\tau)[(U_1 \wedge_g U_2).S)(U_3, V_1)]. \tag{3.1}$$

By using (2.21) in (3.1), we get

$$S(R(U_1, U_2)U_3, V_1) + S(U_3, R(U_1, U_2)V_1) = f(\tau)\left[S\left((U_1 \wedge_g U_2)U_3, V_1\right) + S(U_3, (U_1 \wedge_g U_2)V_1)\right]. \tag{3.2}$$

Adopting $U_1 = V_1 = \zeta$ in (3.2), we obtain

$$S(R(\zeta, U_2)U_3, \zeta) + S(U_3, R(\zeta, U_2)\zeta) = f(\tau)[S((\zeta \wedge_g U_2)U_3, \zeta) + S(U_3, (\zeta \wedge_g U_2)\zeta)]. \tag{3.3}$$

In view of (2.1), (2.2), (2.3), (2.8), (2.13) and (2.23), the above equation reduces to

$$[f(\tau) + (\beta^2 - \alpha^2)]\left[S(U_2, U_3) - \left(\frac{qr-2\lambda}{2p}\right) g(U_2, U_3)\right] = 0. \tag{3.4}$$

Which implies either $f(\tau) = -(\beta^2 - \alpha^2)$ or $S(U_2, U_3) = \left(\frac{\{qr-2\lambda\}}{2p}\right) g(U_2, U_3)$.

Thus, we can state the result as:

**Theorem 3.1.** A Ricci Yamabe soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold with partially Ricci pseudosymmetric is an Einstein manifold if $f(\tau) \neq (\alpha^2 - \beta^2)$.

Comparing (3.4) with (2.11) we yield

$$\lambda = \left(\frac{q-p}{2}\right) r - p(\beta^2 - \alpha^2). \tag{3.5}$$

**Theorem 3.2.** Ricci Yamabe soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold with partially Ricci pseudosymmetric is shrinking or expanding or steady accordingly as $\left(\frac{q-p}{2}\right) r < p(\beta^2 - \alpha^2)$ or $\left(\frac{q-p}{2}\right) r > p(\beta^2 - \alpha^2)$ or $\left(\frac{q-p}{2}\right) r = p(\beta^2 - \alpha^2)$.

If $q = 0$ then (3.5) becomes

$$\lambda = -\frac{pr}{2} + p(\alpha^2 - \beta^2).$$

**Corollary 3.1.** p-Ricci soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold with partially Ricci pseudosymmetric is shrinking or expanding or steady accordingly as $\frac{r}{2} > (\alpha^2 - \beta^2)$ or $\frac{r}{2} < (\alpha^2 - \beta^2)$ or $\frac{r}{2} = (\alpha^2 - \beta^2)$.

If p=0 then (3.5) implies

$$\lambda = \frac{qr}{2}.$$

**Corollary 3.2.** q-Yamabe soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold with partially Ricci pseudosymmetric is shrinking or expanding or steady accordingly as $r < 0$ or $r > 0$ or $r = 0$.

## 4. Ricci Yamabe soliton in 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold with Weyl Ricci Pseudosymmetric condition

Let $(M,g)$ be a Weyl Ricci pseudosymmetric 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold. Then by (2.24) we obtain

$$\big(\widehat{W}(U_1,U_2).S\big)(V_1,V_2) = L_S\, Q(g.S)(V_1,V_2;\ U_1,U_2)]. \tag{4.1}$$

Above equation also can be written as

$$S\big(\widehat{W}(U_1,U_2)V_1,V_2\big) + S\big(V_1,\widehat{W}(U_1,U_2)V_2\big) \ = L_S\left[S\left(\big(U_1\wedge_g U_2\big)V_1,V_2\right) \ +\ S\big(V_1,\big(U_1\wedge_g U_2\big)V_2\big)\right]. \tag{4.2}$$

Putting $U_1 = V_2 = \zeta$ in (4.2), we obtain

$$S\big(\widehat{W}(\zeta,U_2)V_1,\zeta\big) + S\big(V_1,\widehat{W}(\zeta,U_2)\zeta\big) = L_S\left[S\left(\big(\zeta\wedge_g U_2\big)V_1,\zeta\right) + S\big(V_1,\big(\zeta\wedge_g U_2\big)\zeta\big)\right]. \tag{4.3}$$

By virtue of (2.13), (2.16), (2.17) and (2.23), equation (4.3) implies

$$\left[(\beta^2-\alpha^2) + L_S + \varepsilon\left(\tfrac{(p+q)r-2\lambda}{2p}\right)\right]\left[S(U_2,V_1) - \left(\tfrac{qr-2\lambda}{2p}\right)g(U_2,V_1)\right]$$

$$-\left[\varepsilon\left(\tfrac{qr-2\lambda}{2p}\right)S(V_1,U_2) - S(V_1,QU_2)\right] = 0. \tag{4.4}$$

If

$$S(U_2,V_1) = \left(\tfrac{qr-2\lambda}{2p}\right)g(U_2,V_1) \ \text{ and } \ (\beta^2-\alpha^2) + L_S + \varepsilon\left(\tfrac{(p+q)r-2\lambda}{2p}\right) \neq 0,$$

then using (2.12), (4.4) implies

$$S(U_2,V_1) = \left(\tfrac{qr-2\lambda}{qr-2\lambda-2\beta\,\delta}\right)\left[\varepsilon\left(\tfrac{qr-2\lambda}{2p}\right)g(U_2,V_1) \ - \tfrac{\beta}{p}\eta(U_2)\eta(V_1)\right]. \tag{4.5}$$

Thus, we can state the following:

**Theorem 4.3.** Ricci Yamabe soliton in 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold with Weyl Ricci Pseudosymmetric is an η-Einstein manifold.

Contracting (4.5) and (2.11) we have

$$\lambda \ = \frac{q\beta r + (pqr-2p\beta\delta)\left(3(\alpha^2-\beta^2)-\frac{\varepsilon r}{2}\right)}{2\beta + 2p\left(3(\alpha^2-\beta^2)-\frac{\varepsilon r}{2}\right)}. \tag{4.6}$$

Thus, we can state the following theorem:

**Theorem 4.4.** Ricci Yamabe soliton 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold with Weyl Ricci Pseudosymmetric is shrinking or expanding or steady accordingly as $\frac{q\beta r + (pqr-2p\beta\delta)\left(3(\alpha^2-\beta^2)-\frac{\varepsilon r}{2}\right)}{2\beta + 2p\left(3(\alpha^2-\beta^2)-\frac{\varepsilon r}{2}\right)} < 0$ or $\frac{q\beta r + (pqr-2p\beta\delta)\left(3(\alpha^2-\beta^2)-\frac{\varepsilon r}{2}\right)}{2\beta + 2p\left(3(\alpha^2-\beta^2)-\frac{\varepsilon r}{2}\right)} > 0$ or $\left\{q\beta r + (pqr-2p\beta\delta)\left(3(\alpha^2-\beta^2)-\frac{\varepsilon r}{2}\right)\right\} \ = 0$

provided $\left\{2\beta + 2p\left(3(\alpha^2-\beta^2)-\frac{\varepsilon r}{2}\right)\right\} \neq 0.$

If q = 0 then (4.6) can be shown as

$$\lambda = \frac{p\beta\delta\left(3(\beta^2-\alpha^2)+\frac{\varepsilon r}{2}\right)}{\beta + p\left(3(\alpha^2-\beta^2)-\frac{\varepsilon r}{2}\right)}.$$

**Corollary 4.3.** p-Ricci soliton in 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold with Weyl Ricci Pseudosymmetric is shrinking or expanding or steady accordingly as $\frac{p\beta\delta\left(3(\beta^2-\alpha^2)+\frac{\varepsilon r}{2}\right)}{\beta+p\left(3(\alpha^2-\beta^2)-\frac{\varepsilon r}{2}\right)}<0$ or $\frac{p\beta\delta\left(3(\beta^2-\alpha^2)+\frac{\varepsilon r}{2}\right)}{\beta+p\left(3(\alpha^2-\beta^2)-\frac{\varepsilon r}{2}\right)}>0$ or $\left(p\beta\delta\left(3(\beta^2-\alpha^2)+\frac{\varepsilon r}{2}\right)\right)=0$ provided $\left(\beta+p\left(3(\alpha^2-\beta^2)-\frac{\varepsilon r}{2}\right)\right)\neq 0$.

If p = 0 then (4.6) yield

$\lambda=\frac{qr}{2}$.

**Corollary 4.4.** q-Yamabe soliton in 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold with Weyl Ricci Pseudosymmetric is shrinking or expanding or steady accordingly as $r<0$ or $r>0$$ or $r=0$.

## 5. Ricci Yamabe solitons in 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold with Einstein Semi Symmetric

Condition Let (M, g) be the Einstein semi symmetric 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold, using (2.19) we have

$$\left(R(U_1,U_2).\hat{E}\right)(V_1,V_2)=0, \tag{5.1}$$

$$\hat{E}(R(U_1,U_2)V_1,V_2)+\hat{E}(V_1,R(U_1,U_2)V_2)=0. \tag{5.2}$$

By virtue of (2.19) and (5.2) we get

$$S(R(U_1,U_2)V_1,V_2)+S(V_1,R(U_1,U_2)V_2)=\frac{r}{3}[g(R(U_1,U_2)V_1,V_2)+g(V_1,R(U_1,U_2)V_2)]. \tag{5.3}$$

Adopting $U_1=\zeta$ in (5.3) and then taking $V_1=\zeta$ and using (2.1), (2.2), (2.3), (2.10) and (2.13) in (5.3), we obtain

$$S(U_2,V_1)=\left[\frac{qr-2\lambda}{2p}+\frac{r(1-\varepsilon)}{3}\right]g(U_2,V_1). \tag{5.4}$$

Thus, we state the theorem as:

**Theorem 5.5.** Ricci Yamabe soliton in 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold with Einstein Semi Symmetric is an Einstein manifold.

Contrasting (5.4) with (2.11) we obtain

$$\lambda=\frac{r}{2}\left(q-p+\frac{2p(1-\varepsilon)}{3}\right)-p(\beta^2-\alpha^2). \tag{5.5}$$

Thereby, we state the following:

**Theorem 5.6.** Ricci Yamabe solitons in 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold with Einstein Semi Symmetric is shrinking or expanding or steady accordingly as $\frac{r}{2}\left(q-p+\left(\frac{2p(1-\varepsilon)}{3}\right)\right)<p(\beta^2-\alpha^2)$ or $\frac{r}{2}\left(q-p+\left(\frac{2p(1-\varepsilon)}{3}\right)\right)>p(\beta^2-\alpha^2)$ or $\frac{r}{2}\left(q-p+\left(\frac{2p(1-\varepsilon)}{3}\right)\right)=p(\beta^2-\alpha^2)$.

If q = 0 then (5.5) implies

$\lambda=-\frac{pr}{2}\left(\frac{1+\varepsilon}{3}\right)+p(\alpha^2-\beta^2)$.

**Corollary 5.5.** p-Ricci solitons in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold with Einstein Semi Symmetric is shrinking or expanding or steady accordingly as $\frac{r}{2}\left(\frac{1+\varepsilon}{3}\right) > (\alpha^2 - \beta^2)$ or $\frac{r}{2}\left(\frac{1+\varepsilon}{3}\right) < (\alpha^2 - \beta^2)$ or $\frac{r}{2}\left(\frac{1+\varepsilon}{3}\right) = (\alpha^2 - \beta^2)$.

If p = 0 then (5.5) becomes

$\lambda = \frac{qr}{2}$.

**Corollary 5.6.** q-Yamabe solitons in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold with Einstein Semi Symmetric is shrinking or expanding or steady accordingly as $r < 0$ or $r > 0$ or $r = 0$.

## 6. Projectively Flat Ricci Yamabe soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold

If Ricci Yamabe soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold is projectively flat, then

$$g(P(U_1, U_2)U_3, \ \phi V_1) = 0. \tag{6.1}$$

Making use of (1.3) we have

$$g(R(U_1, U_2)U_3, \phi V_1) - \frac{1}{6}\left[S(U_2, U_3)g(U_1, \phi V_1) - S(U_1, U_3)g(U_2, \phi V_1)\right] = 0. \tag{6.2}$$

Assuming $U_2 = U_3 = \zeta$ in (6.2)

$$g(R(U_1, \zeta)\zeta, \phi V_1) - \frac{1}{6}\left[S(\zeta, \zeta)g(U_1, \phi V_1) - S(U_1, \zeta)g(\zeta, \phi V_1)\right] = 0. \tag{6.3}$$

By virtue of (2.1), (2.2), (2.9), (2.13) and (6.3) we yield

$$g(U_1, \phi V_1)\left[(\beta^2 - \alpha^2) + \varepsilon\left(\frac{qr - 2\lambda}{12p}\right)\right] = 0. \tag{6.4}$$

Which implies

$$\lambda = \frac{\varepsilon qr}{2} - 6p(\alpha^2 - \beta^2). \tag{6.5}$$

Or

$$r = \left(\frac{2\lambda - 12p(\beta^2 - \alpha^2)}{\varepsilon q}\right). \tag{6.6}$$

Thereby, we state the result as:

**Theorem 6.7.** Projectively flat Ricci Yamabe soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold is shrinking or expanding or steady according as $\frac{\varepsilon qr}{2} < 6p(\alpha^2 - \beta^2)$ or $\frac{\varepsilon qr}{2} > 6p(\alpha^2 - \beta^2)$ or $\frac{\varepsilon qr}{2} = 6p(\alpha^2 - \beta^2)$.

Above theorem leads to the following corollary:

**Corollary 6.7.** Projectively flat Ricci Yamabe soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold is of constant scalar curvature.

If q = 0 then (6.5) can be written as

$\lambda = 6p(\beta^2 - \alpha^2)$.

**Corollary 6.8.** Projectively Flat p-Ricci soliton in in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold is shrinking or expanding or steady accordingly as $\beta^2 < \alpha^2$ or $\beta^2 > \alpha^2$ or $\beta^2 = \alpha^2$.

If p = 0 then (6.5) can be shown as

$\lambda = \frac{\varepsilon q r}{2}$ .

**Corollary 6.9.** Projectively flat q-Yamabe soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold. If 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold is shrinking or expanding or steady accordingly as $\varepsilon r < 0$ or $\varepsilon r > 0$ or $\varepsilon r = 0$.

**7. ζ-Projectively flat Ricci Yamabe soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold**

If Ricci Yamabe soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold is ζ-projectively flat, then

$$P(U_1, U_2)\zeta = 0. \quad (7.1)$$

From (1.3) we obtain

$$R(U_1, U_2)\zeta - \frac{1}{6}[S(U_2, \zeta)U_1 - S(U_1, \zeta)U_2] = 0. \quad (7.2)$$

Taking inner product with $V_1$ in the above equation, then we have

$$g(R(U_1, U_2)\zeta, V_1) - \frac{1}{6}[S(U_2, \zeta)g(U_1, V_1) - S(U_1, \zeta)g(U_2, V_1)] = 0. \quad (7.3)$$

Substitute $U_2 = \zeta$ in the above equation to get

$$g(R(U_1, \zeta)\zeta, V_1) - \frac{1}{6}[S(\zeta, \zeta)g(U_1, V_1) - S(U_1, \zeta)g(\zeta, V_1)] = 0. \quad (7.4)$$

In view of (2.2), (2.9), (2.13) and (7.4) to yield

$$g(U_1, V_1) = -\varepsilon\, \eta(U_1)\eta(V_1). \quad (7.5)$$

Put (7.5) in (7.3), we have

$$R(U_1, U_2)\zeta = 0. \quad 7.6)$$

By applying (7.6) in (7.2), we conclude the following

$$\lambda = \frac{qr}{2}. \quad (7.7)$$

Or

$r = \frac{2\lambda}{q}$.

Which leads the following theorem:

**Theorem 7.8.** $\zeta$-Projectively flat Ricci Yamabe soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold is shrinking or expanding or steady according as $r < 0$ or $r > 0$ or $r = 0$.

If q = 0 then (7.7) yield

$\lambda = 0$.

Above theorem leads to the following corollary:

**Corollary 7.10.** ζ-Projectively flat p-Ricci soliton in 3-dimensional $(\varepsilon, \delta)$- trans-Sasakian manifold is steady.

**8. Ricci Yamabe soliton in 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold with ϕ-Projectively Semi-Symmetric Condition**

**Definition 8.1.** A Riemannian manifold $(M,g)$ is said to be $\phi$-projectively semi-symmetric with respect to semi-symmetric metric connection if $P(U_1,U_2).\phi=0$ holds on M.

Let $M$ be a $\phi$-Projectively semi-symmetric 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold with respect to semi-symmetric metric connection. The condition $P(U_1,U_2).\phi=0$ turns into

$$(P(U_1,U_2).\phi)U_3 = P(U_1,U_2).\phi\, U_3 - \phi\, P(U_1,U_2)U_3. \tag{8.1}$$

Using (1.3) we obtain

$$R(U_1,U_2)\phi\, U_3 - \phi\, R(U_1,U_2)U_3 - \frac{1}{6}[S(U_2,\phi\, U_3)U_1 - S(U_1,\phi\, U_3)U_2 - S(U_2,U_3)\phi\, U_1 + S(U_1,U_3)\phi\, U_2] = 0. \tag{8.2}$$

Applying (2.8) and (2.12) in (8.2), we have

$$l_1[g(U_2,\phi\, U_3)U_1 - g(U_1,\phi\, U_3)U_2 - g(U_2,U_3)\phi\, U_1 + g(U_1,U_3)\phi\, U_2]$$
$$+l_2[g(U_2,\phi\, U_3)\eta(U_1)\zeta - g(U_1,\phi\, U_3)\eta(U_2)\zeta - \eta(U_2)\eta(U_3)\phi\, U_1 + \eta(U_1)\eta(U_3)\phi\, U_2]$$
$$-\frac{1}{6}[S(U_2,\phi\, U_3)U_1 - S(U_1,\phi\, U_3)U_2 - S(U_2,U_3)\phi\, U_1 + S(U_1,U_3)\phi\, U_2] = 0. \tag{8.3}$$

Replacing $U_1$ by $\phi U_1$ and taking the inner product with $V_1$ and using (2.1), we have

$$l_1[g(U_2,\phi\, U_3)g(\phi\, U_1,V_1) - g(U_1,U_3)g(U_2,V_1) + \varepsilon\, \eta(U_2)\eta(U_3)g(U_2,V_1)$$
$$+ g(U_2,U_3)g(U_1,V_1) - \varepsilon\, g(U_2,U_3)\eta(U_1)\eta(V_1) + g(\phi\, U_1,U_3)g(\phi\, U_2,V_1)]$$
$$+l_2[-\varepsilon\, \eta(U_2)\eta(V_1)g(U_1,U_3) + \eta(U_2)\eta(U_3)g(U_1,V_1) + (1-\varepsilon)\eta(U_1)\eta(U_2)\eta(U_3)\eta(V_1)]$$
$$-\frac{1}{6}[S(U_2,\phi\, U_3)U_1 - S(U_1,\phi\, U_3)U_2 - S(U_2,U_3)\phi\, U_1 + S(U_1,U_3)\phi\, U_2] = 0. \tag{8.4}$$

Substituting $U_1 = V_1 = e_i$ in (8.4) and using (2.1), we yield

$$S(U_2,U_3) = a_1\, g(U_2,U_3) + a_2\eta(U_2)\eta(U_3). \tag{8.5}$$

Where

$l_1 = \frac{r}{2} + 2(\beta^2-\alpha^2)$, $l_2 = 3(\alpha^2-\beta^2) - \frac{\varepsilon r}{2}$, $l_3 = \frac{qr-2\lambda-2\beta\delta}{2p}$, $a_1 = \frac{6}{3-\varepsilon}\left[(1-\varepsilon)l_1 + \frac{l_3}{3}\right]$, $a_2 = \frac{6}{3-\varepsilon}\left[2\varepsilon\, l_1 + 2(2-\varepsilon)l_2 - \frac{\varepsilon l_3}{3}\right]$.

Which leads to the following theorem:

**Theorem 8.9.** Let $M$ be a $\phi$-Projectively semi-symmetric Ricci Yamabe soliton in 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold. Then $M$ is an $\eta$-Einstein manifold.

Contracting (8.5) and (2.11) we get

$$\lambda = r\left[\left(\frac{3-5\varepsilon}{4}\right)p + q\right] - \left[\beta\delta + \left(\frac{9-11\varepsilon}{2}\right)(\alpha^2-\beta^2)p\right]. \tag{8.6}$$

Thus, we can state the theorem as:

**Theorem 8.10.** Ricci Yamabe soliton in 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold with $\phi$-Projectively Semi-Symmetric is shrinking or expanding or steady according as

$r\left[\left(\frac{3-5\varepsilon}{4}\right)p+q\right]<\left[\beta\delta+\left(\frac{9-11\varepsilon}{2}\right)(\alpha^2-\beta^2)p\right]$ or $r\left[\left(\frac{3-5\varepsilon}{4}\right)p+q\right]>\left[\beta\delta+\left(\frac{9-11\varepsilon}{2}\right)(\alpha^2-\beta^2)p\right]$ or

$r\left[\left(\frac{3-5\varepsilon}{4}\right)p+q\right]=\left[\beta\delta+\left(\frac{9-11\varepsilon}{2}\right)(\alpha^2-\beta^2)p\right].$

If $q = 0$ then (8.6) implies

$\lambda=p\left[r\left(\frac{3-5\varepsilon}{4}\right)+\left(\frac{9-11\varepsilon}{2}\right)(\beta^2-\alpha^2)\right]-\beta\delta.$

**Corollary 8.11.** p-Ricci soliton in 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold with $\phi$-Projectively Semi-Symmetric is shrinking or expanding or steady accordingly as$p\left[r\left(\frac{3-5\varepsilon}{4}\right)+\left(\frac{9-11\varepsilon}{2}\right)(\beta^2-\alpha^2)\right]<\beta\delta$ or $p\left[r\left(\frac{3-5\varepsilon}{4}\right)+\left(\frac{9-11\varepsilon}{2}\right)(\beta^2-\alpha^2)\right]>\beta\delta$ or $p\left[r\left(\frac{3-5\varepsilon}{4}\right)+\left(\frac{9-11\varepsilon}{2}\right)(\beta^2-\alpha^2)\right]=\beta\delta.$

If $p = 0$ then (8.6) becomes

$\lambda = qr - \beta\delta.$

**Corollary 8.12.** q- Yamabe soliton in 3-dimensional $(\varepsilon,\delta)$- trans-Sasakian manifold with $\phi$-Projectively Semi-Symmetric is shrinking or expanding or steady accordingly as $qr<\beta\delta$ or $qr>\beta\delta$ or $qr=\beta$.

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