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Lattice Identities in the Subgroup Lattices of Groups of 2 × 2 Matrices over Zp

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Abstract: In this article, the properties of the subgroup lattice of the group of 2 × 2 matrices over Z_{17} like modularity, semi modularity, super modularity distributivity, consistency, the General disjointness condition, pseudo complemented and super solvability have been validated.

Keywords: *Lattice, Subgroup lattice, Lattice properties.*

Introduction

Allow L(G) as the Subgroup Lattice of $\mathcal{G}$, where $\mathcal{G}$ is $SL_2(Z_k)$.

If $\mathcal{G}= \{\begin{pmatrix} x & y \\ z & w \end{pmatrix}$: x, y, z, w $\in Z_k$, xw-yz ≠ 0} and

$G = \{\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathcal{G}$: xw-yz = 1}, then G is a subgroup $\mathcal{G}$.

Regarding order of groups, we will show that, $o(\mathcal{G}) = k(k^2-1)(k-1)$ [1] and $o(G) = k(k^2-1)$.[1]

For complete reference we provide the breakup of L(G)while p=17 [2]. Thus, we will investigate regarding to theentire said properties in L(G) of this article.

II Basics

Lattice: Definition 1

A Poset L is said to be a lattice if inf {u, v} and sup {u, v} exists for all u, v∈ L.

Modular Lattice: Definition 2

For a lattice L, **L is modular** if $r \le u$ implies that $u \wedge (v \vee r) = (u \wedge v) \vee r$ for all u, v, r ∈L.

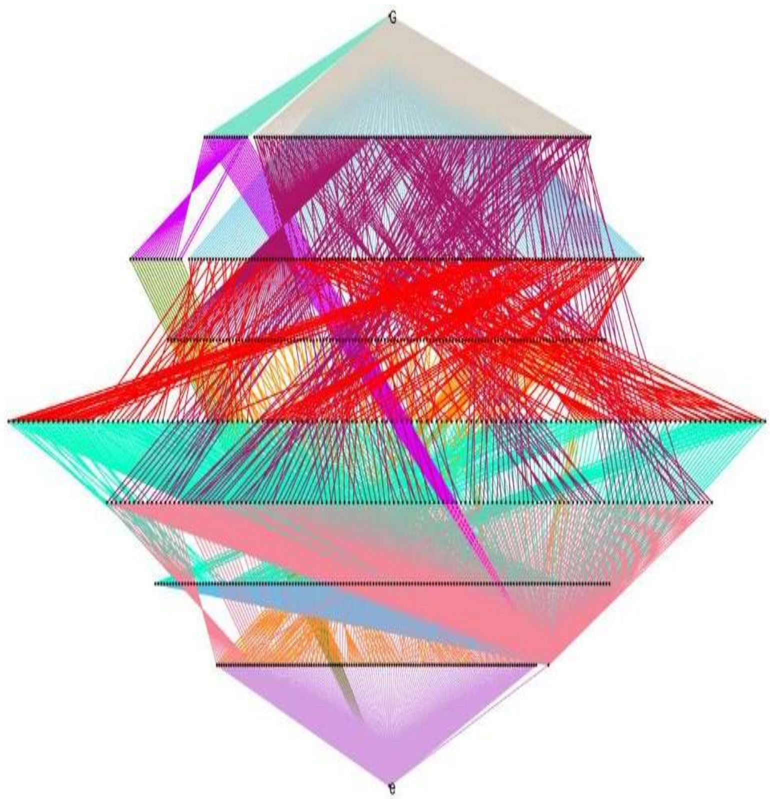
Upper-semi modular: Definition 3

For a lattice L, L is an upper-semi modular if u∨ v covers u and v, u≠ v and u and v cover u ∧ v.

Distributive lattice: Definition 2.5

For a lattice L, L is **distributive** if $u \vee (v \wedge r) = [(u \vee v) \wedge (u \vee r)]$ for all u, v, r∈ L.

Now, we present the drawing of L(G) when p=17 [2] as shown in pic.1.



Pic.1:L(G) when p = 17

Row I : (Left to Right**)** R_1 to R_{18} and Q_1 to Q_{136}

Row II : (Left to Right**)** P_1 to P_{18} and O_1 to O_{153}

Row III : (Left to Right**)** N_1 to N_{136}

Row IV : (Left to Right**)** M_1 to M_{153}

Row V : (Left to Right**)** L_1 to L_{136}

Row VI : (Left to Right**)**and K_1 to K_{153}

Row VII : (Left to Right**)** J_1 to J_{136} and $\mathcal{K}_1$

3. Main Properties

Property 3.1

If p =17, then L(G) is not modular.

Proof:

From Pic.1, we take three subgroups, K_{55}, R_1, $O_{93} \in L(G)$.

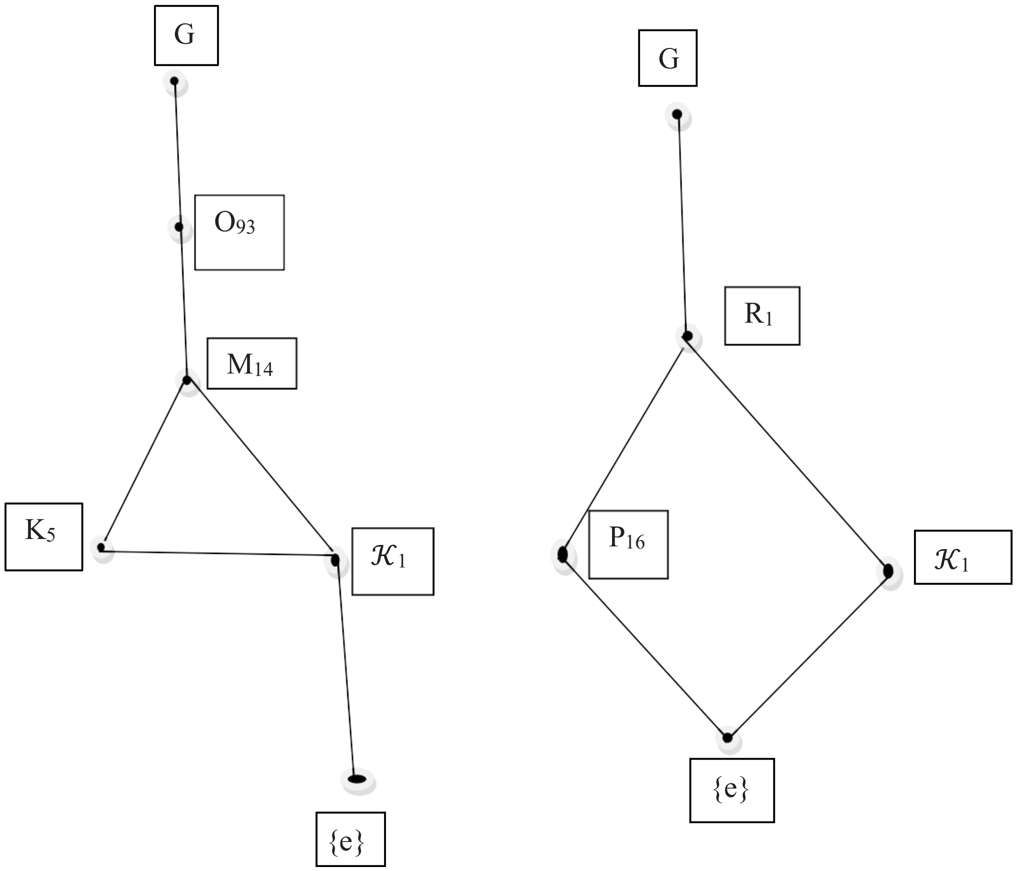


Fig 3.1.1

$K_{55} \vee (R_1 \wedge O_{93}) = K_{55} \vee \mathcal{K}_1 = K_{55}$

But, $(K_{55} \vee R_1) \wedge O_{93} = G \wedge O_{93} = O_{93}$

Hence $K_{55} \vee (R_1 \wedge O_{93}) \neq (K_{55} \vee R_1) \wedge O_{93}$

Otherwise, $(K_{55} \wedge O_{93}) \vee (R_1 \wedge O_{93}) = K_{55} \vee \mathcal{K}_1 = K_{55}$.

But, $[(K_{55} \wedge O_{93}) \vee R_1] \wedge O_{93} = (K_{55} \vee R_1) \wedge O_{93} = G \wedge O_{93} = O_{93}$.

Therefore, $(K_{55} \wedge O_{93}) \vee (R_1 \wedge O_{93}) \neq [(K_{55} \wedge O_{93}) \vee R_1] \wedge O_{93}$.

Consequently, L(G) is non modular when p = 17.

Property 3.2

L(G) is not upper semi modular if p=17.

Proof:

From Pic.1, we take two subgroups, K_{14}, $M_1 \in$ L(G).

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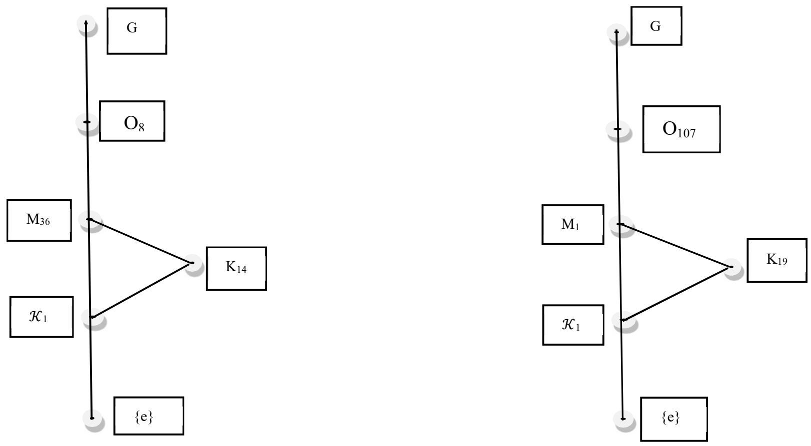

$K_{14} \wedge M_1 = \mathcal{K}_1$.which is covered by K_{14} while $K_{14} \vee M_1 = G$. which does not cover $M_{1.}$

Therefore L(G) is not upper semi modular when p = 17.

Property 3.3

If p =17, then L(G) is not super modular.

Proof:

From Pic.1, we choose four subgroups, $J_1, K_{19}, N_{104}, P_1 \in L(G)$.

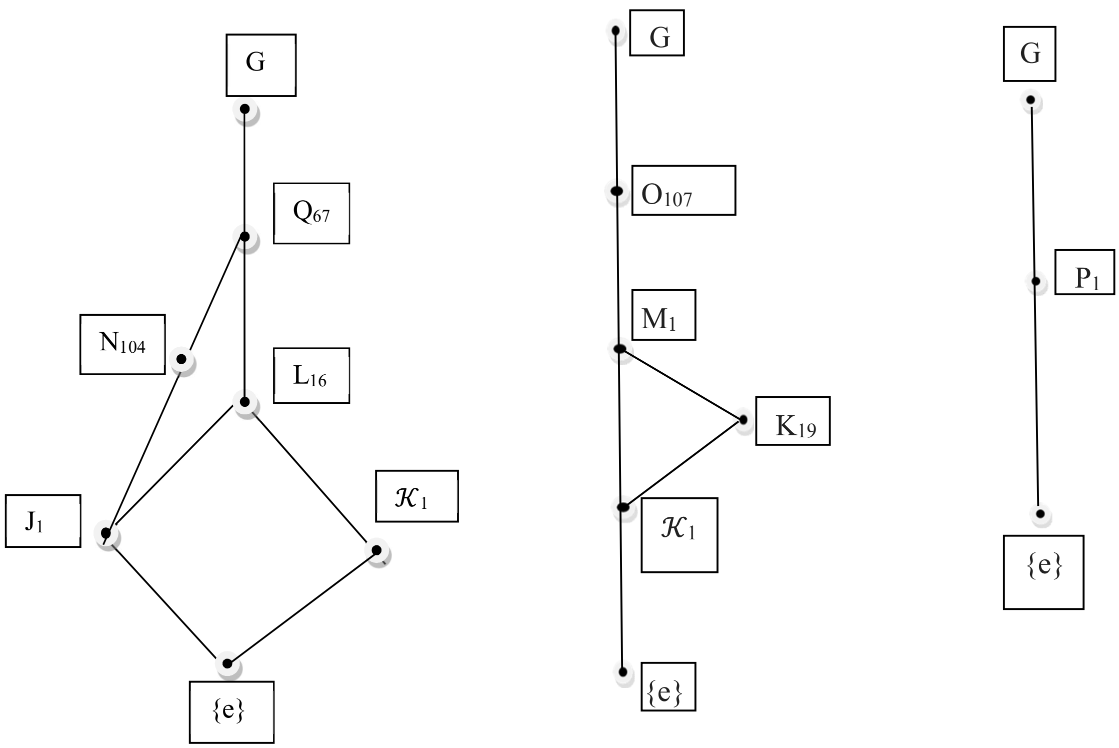

Fig.3.3.1

$(J_1 \vee K_{19}) \wedge (J_1 \vee N_{104}) \wedge (J_1 \vee P_1) = G \wedge N_{104} \wedge G = N_{104}$.

But, $J_1 \vee [K_{19} \wedge N_{104} \wedge (J_1 \vee P_1)] \vee [N_{104} \wedge P_1 \wedge (J_1 \vee K_{19})] \vee [K_{19} \wedge P_1 \wedge (J_1 \vee N_{104})]$

$= J_1 \vee [K_{19} \wedge N_{104} \wedge G] \vee [N_{104} \wedge P_1 \wedge G] \vee [K_{19} \wedge P_1 \wedge N_{104}]$

$= J_1 \vee \{e\} \vee \{e\} \vee \{e\}$

$= J_1$.

Therefore, $(J_1 \vee K_{19}) \wedge (J_1 \vee N_{104}) \wedge (J_1 \vee P_1) \neq J_1 \vee [K_{19} \wedge N_{104} \wedge (J_1 \vee P_1)] \vee [N_{104} \wedge P_1 \wedge (J_1 \vee K_{19})] \vee [K_{19} \wedge P_1 \wedge (J_1 \vee N_{104})]$

Consequently, L(G) is not super modular when p = 17.

Property 3.4

If p =17, then L(G) is not distributive.

Proof:

From Pic.1, We take three subgroups K_{14}, M_{36}, $L_{16} \in L(G)$.

$K_{14} \vee (M_{36} \wedge L_{16}) = K_{14} \vee \mathcal{K}_1 = K_{14}$.

But, $(K_{14} \vee M_{36}) \wedge (K_{14} \vee L_{16}) = M_{36} \wedge G = M_{36}$.

Therefore, $K_{14} \vee (M_{36} \wedge L_{16}) \neq (K_{14} \vee M_{36}) \wedge (K_{14} \vee L_{16})$.

Consequently, L(G) is not distributive when p = 17.

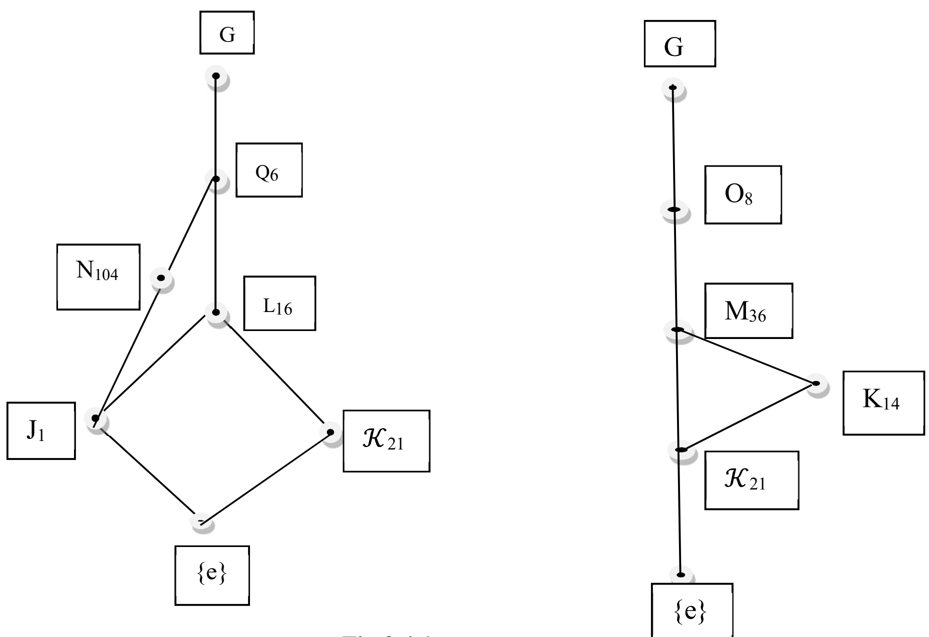

Fig 3.4.1

Property 3.5

If p =17, then L(G) is not consistent.

Proof:

We choose the join irreducible element $P_1 \in L(G)$ for the case p = 17, we find that

when p = 17 , $\mathcal{K}_1 \vee P_1 = G = O_1 \vee O_2$ in the upper interval $[\mathcal{K}_1, G]$

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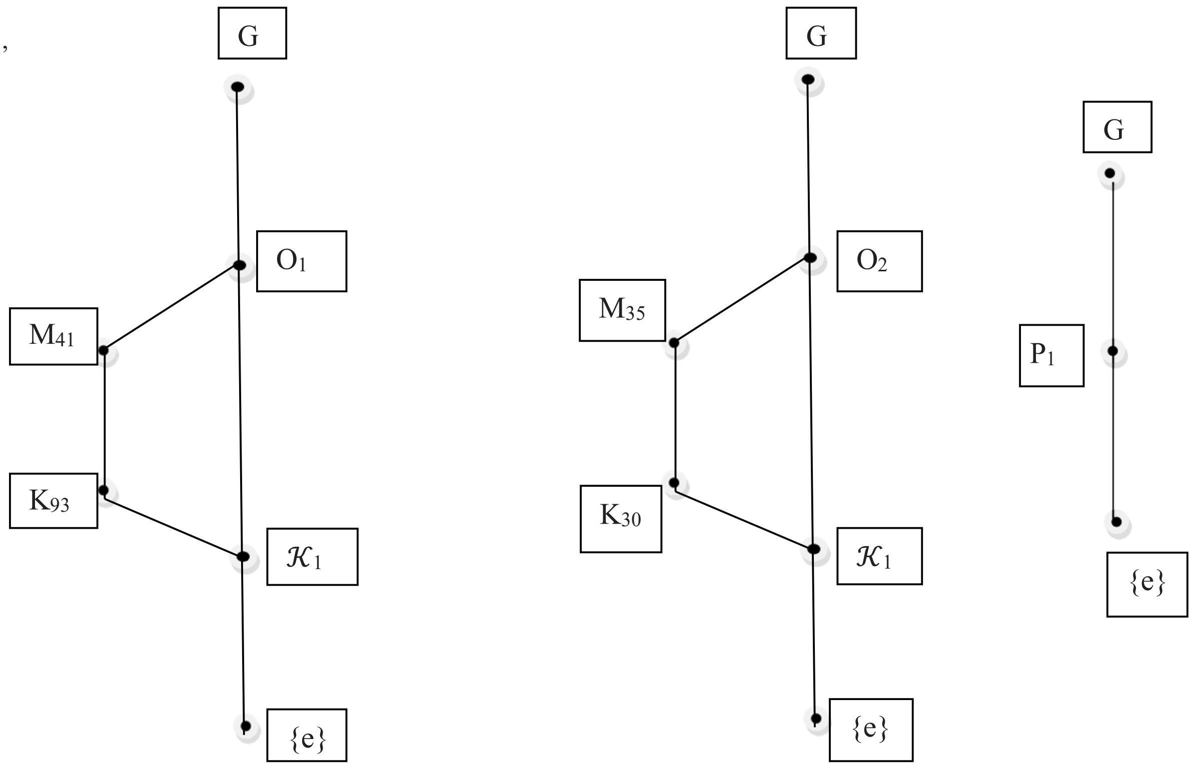

Fig.3.5.1

Therefore L(G) is not consistent when $p = 17$.

Property 3.6.

If $p = 17$, then the General disjointness condition is not true in L(G).

Proof:

From Pic.1, we take three subgroups $\mathcal{K}_1, J_1, J_2 \in L(G)$.

Now let $\mathcal{K}_1 \wedge J_1 = 0$ and $(\mathcal{K}_1 \vee J_1) \wedge J_2 = L_{16} \wedge J_2 = 0$.

Then, $\mathcal{K}_1 \wedge (J_1 \vee J_2) = \mathcal{K}_1 \wedge G = \mathcal{K}_1 \neq 0$

$\mathcal{K}_1 \wedge (J_1 \vee J_2) \neq 0$.

Hence the General disjointness condition is not true in L(G) when $p = 17$.

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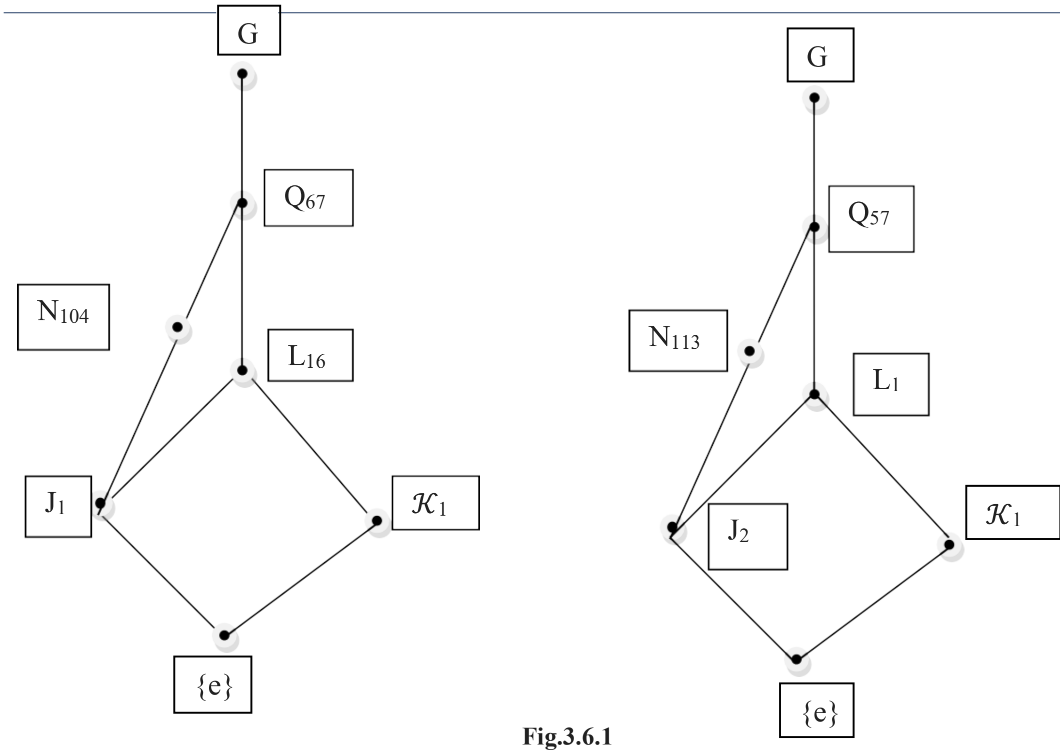

Fig.3.6.1

Property 3.7

If p =17, then L(G) is not pseudo complemented.

Proof:

From Pic.1, we take one subgroup $K_{14} \in L(G)$

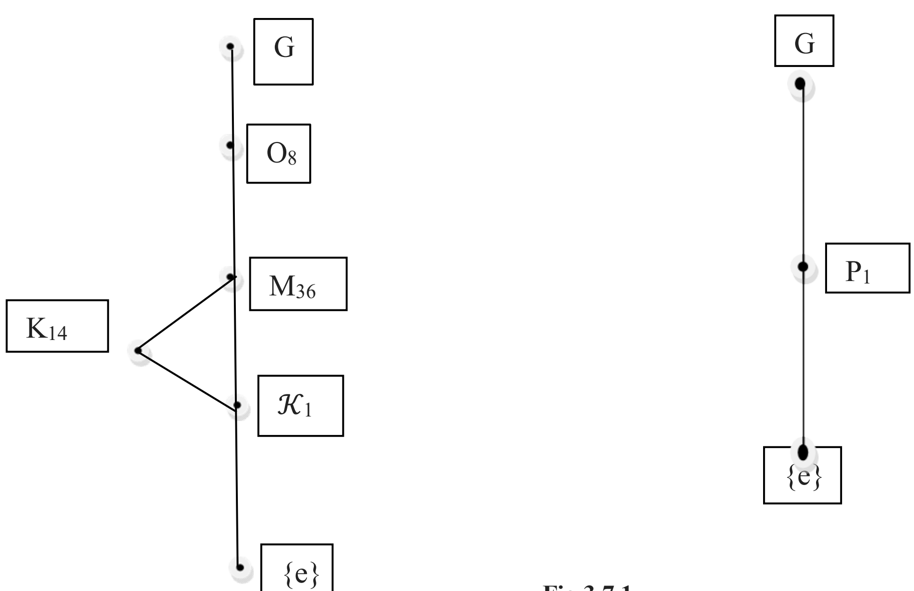

Fig.3.7.1

Then, $K_{14} \wedge P_1 = 0$ and if for any $\mathcal{K}_1 \in L(G)$ such that $\mathcal{K}_1 \subset P_1$.

But, $\mathcal{K}_1 \wedge K_{14} = \mathcal{K}_1 \neq 0$.

Therefore, $\mathcal{K}_1 \wedge K_{14} \neq 0$.

Consequently an element $P_1 \in L(G)$ is not pseudo complement of $K_{14} \in L(G)$.

Hence L(G) is not pseudo complemented when p = 17.

Property 3.8

Every atom is non – modular if p =17.

Proof:

Consider an atom among the atoms J_2, J_3, J_4 say J_2.

We have $P_{16} \subset R_1$

Now, $P_{16} \vee (J_2 \wedge R_1) = P_{16} \vee \{e\} = P_{16}$

But, $(P_{16} \vee J_2) \wedge R_1 = G \wedge R_1 = R_1$

Therefore, $P_{16} \vee (J_2 \wedge R_1) \neq (P_{16} \vee J_2) \wedge R_1$

Therefore J_2 is not modular in L(G) when p = 17.

Similarly we can prove that J_3 and J_4 are not modular.

By Similar argument, we can prove that all the other atoms in L(G) when p = 17 are not modular.

Hence there is no atom in L(G) when p= 17, which is modular.

Property 3.9

If p =17, then L(G) is not super solvable.

Proof:

By Property 3.8, we have no atom in L(G) is modular So, there is no maximal chain in L(G) with modular element

Therefore, L(G) is not super solvable when p = 17.

4. Conclusion

In this article, the properties of L(G) over Z_{17} like modularity, semi modularity, super modularity distributivity, consistency, the GD condition, pseudo complemented and super solvability have been proved and validated.

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