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Recent Applications of Multiple Hypergeometric Transformations in Function Spaces and Associated Reduction Formulas

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Abstract

In this research paper we have obtained recent applications of multiple hypergeometric transformations in function spaces and associated reductions formulas of Kampé de Ferlet's, Exton's double series, and Srivastava triple series. Further, we have presented some interesting integrals involving the Lauricella's F_A, Lauricella-Saran's F_E , F_G etc., Exton's $G_{F:G,H}^{A:B,C}$, Horn's H_2 and Jain's $3\varphi_K^{(1)}, 3\varphi_N^{(1)}$ and generalized hypergeometric series pFq arise as special cases of our main results. Also, some special cases were considered as an application of presented integral functions in function spaces.

Keywords: Generalized hypergeometric function, Horn's function, Jain's function, Exton's function.

MSC: 33B15, 33C20, 33C05, 33C65.

Introduction:

The Horn's function (see [1]-[3]) are defined as follows:

$$H_2\left[\alpha,\beta,\gamma,r;s;a,b\right]=$$

$$\sum_{f,g=0}^{\infty}\frac{(\alpha)_{f+g}(\beta)_f(\gamma)_g}{(s)_f}.\frac{a^f}{f!}.\frac{b^g}{g!},$$

$$(j+1)k=1 \qquad (1)$$

$$H_4\left[\alpha,\beta;\gamma;r;a,b\right]=$$

$$\sum_{f,g=0}^{\infty}\frac{(\alpha)_{2f+g}(\beta)_f}{(\gamma)_f(r)_g}.\frac{a^f}{f!}.\frac{b^g}{g!},$$

$$4j=(k-1)^2 \tag{2}$$

$$H_7[\alpha;\beta,\gamma;a,b]=$$

$$\sum_{f,g=0}^{\infty}\frac{(\alpha)_{2f+g}}{(\beta)_f(\gamma)_g}.\frac{a^f}{f!}.\frac{b^g}{g!},$$

$$4|a|<1 \tag{3}$$

the positive quantities j and k are the associated radii of absolute convergence of the double Fourier series $\sum_{f,g=0}^{\infty}\beta_{f,g}a^f.\mathrm{b}^g, |a|<j, |b|<k$

In this paper we define a general double hypergeometric function due to the author [4].

$$F_{F:G;H}^{A:B;D}\left[\begin{matrix}(\alpha):(\beta);(r);\\(w):(u);(v);\end{matrix}a,b\right]=$$

$$\sum_{f,g=0}^{\infty}\frac{(\alpha)_{f+g}(\beta)_f(r)_g}{(w)_{f+g}(u)_g(v)_g}.\frac{a^f}{f!}.\frac{b^g}{g!} \tag{4}$$

In 1964, Srivastava (see [5]-[7]) defined H_A by triple series

$$H_A[\alpha,\beta,\gamma;r,s;\mathrm{a,b,c}]=$$

$$\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_{f+\ell}(\beta)_{f+g}(\gamma)_{g+\ell}}{(r)_f(s)_{g+\ell}}.\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!} \tag{5}$$

whose region of convergence is

$$|a|<d, |b|<e, |c|<j,$$

$d+e+j=1+\ell j$ which is the generalization of the result of author [8]. The general triple hypergeometric functions $F^{(3)}$, defined as follows:

$$F^{(3)}\left[\begin{matrix}(\alpha)::(\beta);(\beta^1);(\beta^{11}):(\gamma):(\gamma^1):(\gamma^{11});\\(r)::(s);\ (s^1);\ (s^{11}):\ (t):\ (t^1):\ (t^{11});\end{matrix}x,y,z\right]$$

$$=\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_{f+g+\ell}(\beta)_{f+g}(\beta^1)_{g+\ell}(\beta^{11})_{f+\ell}}{(r)_{f+g+\ell}(s)_{f+g}(s^1)_{g+\ell}(s^{11})_{f+\ell}}\times$$

$$\frac{(r)_f\left(r^1\right)_g\left(r^{11}\right)_\ell}{(t)_f\left(t^1\right)_g\left(t^{11}\right)_\ell}.\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!}. \tag{6}$$

It is clear that

$$(\alpha)_f=\prod_{\xi=1}^{A}\left(\alpha_\xi\right)_f$$

$$=\prod_{\xi=1}^{A}\frac{\Gamma\left(\alpha_{\xi+f}\right)}{\Gamma\left(\alpha_\xi\right)} \tag{7}$$

$$(\beta)_g=\prod_{\in=1}^{B}\frac{\Gamma\left(\beta_\in+g\right)}{\Gamma\left(\beta_\in\right)} \tag{8}$$

and $(\gamma)_\ell=\prod_{\delta=1}^{C}\frac{\Gamma\left(\gamma_\delta+\ell\right)}{\Gamma\left(\gamma_\delta\right)}$ (9)

where A of the (α) parameters, B of the (β) parameters, and C of the (γ) parameters.

Exton's modified series are defined as follows:

$$G_{F:G;H}^{A:B;C}\left[\begin{matrix}(\alpha):(\beta);(\gamma);\\(w):(u);(v);\end{matrix}a,b\right]$$

$$=\sum_{f,g=0}^{\infty}\frac{(\alpha)_{f-g}(\beta)_f(r)_g}{(w)_{f-g}(u)_f(v)_g}.\frac{a^f}{f!}.\frac{b^g}{g!} \tag{10}$$

The Saran functions (see [9]) are defined as follows:

$$F_4:F_E\left[\alpha,\alpha,\alpha,\beta,\gamma,\gamma;r,s,w;a,b,c\right]=$$

$$\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_{f+g+\ell}(\beta)_f(\gamma)_{g+\ell}}{(r)_f(s)_g(w)_\ell}.\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!},$$

where $j+\left(k^{\frac{1}{2}}+x^{\frac{1}{2}}\right)=1$ (11)

$$F_8:F_G\left[\alpha,\alpha,\alpha;\beta,\gamma,r;s,w,w;a,b,c\right]=$$

$$\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_{f+g+\ell}(\beta)_f(\gamma)_g(r)_\ell}{(s)_f(w)_{g+\ell}}.\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!},$$

where $j+k=1=j+x$ (12)

$$F_3 : \mathrm{F}_K\left[\alpha,\beta,\beta,\gamma,r,\gamma;s,w,u;a,b,c\right]=$$

$$\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_f(\beta)_{g+\ell}(\gamma)_{f+\ell}(r)_g}{(s)_g(w)_g(u)_\ell}.\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!},$$

where $(1-j)(1-k)=x$ (13)

$$F_{11} : \mathrm{F}_M\left[\alpha,\beta,\beta,\gamma,r,\gamma;s,w,w;a,b,c\right]=$$

$$\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_f(\beta)_{g+\ell}(\gamma)_{f+\ell}(r)_g}{(s)_f(w)_{g+\ell}}.\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!},$$

where $j+x=1=k$ (14)

and

$$F_6 : F_N\left[\alpha,\beta,\gamma,r,s,r;w,u,u;a,b,c\right]=$$

$$\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_f(\beta)_g(\gamma)_\ell(r)_{f+g}(s)_g}{(w)_f(u)_{g+\ell}}.\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!},$$

where $(1-j)k+(1-k)x=0$ (15)

In 1963, Pandey (see [10]) transformed the integral representations of F_A and F_G into two interesting hypergeometric series which is similar to Horn's type (see [11]-[13]).

$$G_A\left[\alpha,\beta,\gamma;r;a,b,c\right]=$$

$$\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_{g+\ell-f}(\beta)_{f+\ell}(\gamma)_g}{(r)_{g+\ell-f}}.\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!} \quad (16)$$

it provides a generalization of Appell's function F_1 and Horn's function G_1 and G_2; and

$$G_B\left[\alpha,\beta_1,\beta_2,\beta_3;r;a,b,c\right]=$$

$$\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_{g+\ell-f}(\beta_1)_f(\beta_2)_f(\beta_3)_\ell}{(r)_{g+\ell-f}}.\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!} \quad (17)$$

it provides the generalizations of Appell's function F_1 and Horn function G_2.

By the means of the linearization method, Li and Chu [14] established analytical formulae for a class of non-terminating ${}_3F_2$- series with unit argument. Several closed formulae are presented as applications. Shpot and Srivastava [15] derived three term summation formulas for Clausenian hypergeometric series ${}_3F_2(1)$ with negative parameter differences. Tremblay [16] give a list of twenty-four presumed new transformations involving the Gauss hypergeometric functions with quadratic rational arguments. They are obtained from known transformation formulas of the hypergeometric function, for the most part in Goursat's thesis [17]. Trembly [18] add twelve new transformations formulas for the Gauss hypergeometric function having higher order rational arguments. Choi and Rathie [19] established 176 interesting summation formulas for the Kampe de Feriet function in the form of 16 general summation formulas based on the transformation formulas due to Liu and Wang. The results are derived with the help of generalizations of Kummer's summation theorem, Gauss second summation theorem and Bailey's summation theorem established earlier by Lavoie et al. Motivated by the work done by [20-28], we derive some applications of multiple hypergeometric transformations in function spaces and associated reduction formulas of Kampe de Feriet's function. Exton's double series, and Srivastava triple series.

In this research note we prove the following theorem:

Theorem: Let $\mathrm{Re}(\alpha+s-1)>0$ and $\mathrm{Re}(\alpha+\beta+r+s-1)$, and $\mathrm{Re}(\gamma-1)>0$

then

$$\sigma = F^{2:2;2}_{2:1;1}\left[\begin{matrix}\alpha+\beta-\gamma+r+s,\alpha+\beta+r+s-1: \alpha,\beta; & r,s & ; \\ \alpha+r \quad , \quad a+s \quad : \quad c & ;\alpha+\beta-\gamma+r+s; \end{matrix} a,b\right]=$$

$$(1-a)^{1-\alpha-\beta-r-s}\,{}_4F_3\left[\begin{matrix}\frac{\alpha+\beta+r+s-1}{2},\alpha+\beta+r+s,\alpha,y-\beta; \\ \alpha+r \quad , \quad \alpha+s \quad , \quad y \quad ;\end{matrix} \frac{-4a}{(1-a)^2}\right] \qquad (18)$$

for the proof of our equation (18), the left hand side of equation (18) can be expressed in the following power series

$$\sigma=\sum_{f,g=0}^{\infty}\frac{(\alpha+\beta-\gamma+r+s)_{f+g}(\alpha+\beta+r+s-1)_{f+g}}{(\alpha+r)_{f+g}(\alpha+s)_{f+g}}\cdot\frac{(\alpha)_f(\beta)_f(r)_g(s)_g}{(\gamma)_f(\alpha+\beta-\gamma+r+s)_g}\cdot\frac{a^{f+g}}{f!.g!} \qquad (19)$$

with the help of Rainville result (see [29], p. 56 [1])

$$\sum_{f,g=0}^{\infty}B(f,g)=\sum_{f=0}^{\infty}\sum_{g=0}^{\infty}B(f-g,g)$$

and Pochhammer's identities (see [30-31])

(i) $(\alpha)_{f+g}=(\alpha)_f(\alpha+f)_g$

(ii) $(\alpha)_f=(-1)^f(1-\alpha-f)_f$, $m\in z$

(iii) $(\alpha)_{f-g} = \dfrac{(\alpha)_m (-1)^g}{(1-\alpha-f)_g}$,

and (iv) $(\alpha)_{2g} = 2^{2g}\left(\dfrac{\alpha}{2}\right)_g\left(\dfrac{\alpha+1}{2}\right)_g$

then we can easily obtain

$$\sigma = \sum_{f=0}^{\infty}\frac{(\alpha)_f(\beta)_f(\alpha+\beta-\gamma+r+s)_f}{f!(\gamma)_f(\alpha+r)_f(\alpha+s)_f}.(\alpha+\beta+r+s-1)_f.\mathrm{a}^f$$

$${}_4F_3\left[\begin{matrix}-f, & s, & 1-\gamma-f, & r; \\ \alpha+\beta-\gamma+r+s, & 1-\alpha-f, & 1-\alpha-f, & 1-\beta-f;\end{matrix}\right]$$

$${}_4F_3\left[\begin{matrix}-f,\alpha,b,c; \\ \eta_1,\eta_2,\eta_3\ ;\end{matrix} 1\right] = \frac{(\eta_2-c)_f(\eta_3-c)_f}{(\eta_2)_f(\eta_3)_f}\times$$

$${}_4F_3\left[\begin{matrix}-f, & \eta_1-a, & \eta_1-b,c; \\ 1-\eta_2-f+c, & 1-\eta_3-f+c, & \eta_1\ ;\end{matrix} 1\right] \tag{20}$$

thus

$$\sigma = \sum_{f=0}^{\infty}\sum_{g=0}^{f}\frac{(\alpha+\beta+r+s-1)_{f+g}(\alpha+\beta-\gamma+r+s)_f}{(\alpha+s)_f(\gamma)_f}\frac{1}{(\alpha+\beta-\gamma+r+s)_g} \tag{21}$$

with the help of Rainville see [29], page 57 (2) and Whipple's theorem (see [29]; p. 90, Th. 32 (1));

$${}_3F_2\left[\begin{matrix}\alpha, & \beta, & \gamma; \\ 1+\alpha-\beta, & 1+\alpha-\gamma, & ;\end{matrix} a\right] =$$

$$(1-a)^{-\alpha}\,{}_3F_2\left[\begin{matrix}\dfrac{\alpha}{2}, & \alpha+\dfrac{1}{2}, & 1+\alpha-\beta-\gamma; \\ 1+\alpha-\beta\text{'}1+\alpha-\gamma & & ;\end{matrix}\right]$$

thus

$$\sigma = (1-a)^{1-\alpha-\beta-r-s} F^{2:\ 1;2}_{2:\ 0:1}\left[\begin{matrix}\alpha+\beta+r \\ \hline \alpha+s\end{matrix}\right]$$

$$\frac{a^f}{f!}.\frac{(-f)_g(r)_g.(\alpha+\beta-\gamma+r)_g}{g!(\alpha+r)_g(\beta+r)_g}$$ (Checked first sir)

$$\Rightarrow F\begin{matrix}\ell:1;2\\ i:0;1\end{matrix}\left[\begin{matrix}\alpha_1,...,\alpha_\ell:u-r-s;r,s;\\ \beta_1,...,\beta_i: \quad ;\ u\ ;\end{matrix} a,a\right]=$$

$$_{\ell+2}F_{i+1}\left[\begin{matrix}\alpha_1,\cdots,\alpha_\ell,u-r,u-s;\\ \beta_1,\cdots,\beta_i,\quad u \quad ;\end{matrix} a\right]$$

gives the right hand side of equation (18).

For cases reducibility:

(1) Put $r=\gamma-\beta$ and $s=1-\beta$ in (18)

Then

$$_2F_1\left[\begin{matrix}\alpha,\beta;\\ \gamma\ ;\end{matrix} a\right] {}_2F_1\left[\begin{matrix}\gamma-\beta,1-\beta;\\ 1+\alpha-\beta\ ;\end{matrix} a\right]=(1-a)^{\beta-\alpha-\gamma}$$

$$_4F_3\left[\begin{matrix}\alpha\ ,\gamma-\beta\ ,1+\alpha-\beta+\gamma,\frac{\alpha-\beta+\gamma}{2};\\ 1+\alpha-\beta\ ,\quad \gamma\quad ,\alpha+\gamma-\beta \quad ;\end{matrix} \frac{4a}{(1-a)^2}\right]$$

which is the required result of Bailey see [see [32]; p. 383 (74)] (24)

(2) Put $\gamma=\beta+r$ in equation (18),

$$F\begin{matrix}1:2;2\\ 1:1;1\end{matrix}\left[\begin{matrix}\alpha+\beta+r+s-1:\alpha,\beta\ ;\ r,s\ ;\\ \alpha+r \quad :\beta+r\ ;\alpha+s;\end{matrix} a,a\right]=$$

$$(1-a)^{1-\alpha-\beta-r-s}\ {}_4F_3\left[\begin{matrix}\frac{\alpha+\beta+r+s-1}{2},\alpha+\beta+r+s,\ r,a\ ;\\ \alpha+s \quad ,\quad \alpha+r \quad ,\beta+r;\end{matrix} \frac{-4a}{(1-a)^2}\right] \tag{25}$$

which is the required new result.

(3) Put $\gamma=\alpha+\beta$ and $r=0$ in equation (18), we get our previous equation (22).

Theorem: $F^{(3)}\left[\begin{matrix}\alpha::-;(\beta);-;s-\beta-1,\ 2+\beta-s;(r);(u);\\ -::-;(\gamma);-;1+\beta-s,\quad s \quad ;(s);(v);\end{matrix} a,b,c\right]=$

$$(1-a)^{-\alpha}\ F^{(3)}\left[\begin{matrix}\alpha::-;(\beta);-:\beta,1+\frac{\beta}{2};(r);(u);\\ -::-;(\gamma);-:\gamma,\ \frac{\beta}{2}\ ;(s);(v);\end{matrix} \frac{-a}{1-a},\frac{b}{1-a},\frac{c}{1-c}\right] \tag{26}$$

$$F^{(3)}\left[\begin{matrix} -::-;(\beta);\alpha:s-\beta+1,\ 2+\beta-s;\ (r);(u); \\ -::-;(\gamma);-:1+\beta-s, \qquad s \qquad ;\ (s);(v); \end{matrix} a,b,c\right]=$$

$$(1-a)^{-\beta}F^{(3)}\left[\begin{matrix} -::-;(\beta),-;\beta,\ 1+\frac{\beta}{2};\ (r);(u); \\ -::-;(\gamma)-;s, \quad \frac{\beta}{2} \quad ;\ (s);(v); \end{matrix} \frac{-a}{1-a},b,\frac{c}{1-a}\right] \tag{27}$$

and

$$G\begin{matrix}1;\ 2;\ D\\0;\ 2;\ H\end{matrix}\left[\begin{matrix} \alpha:s-\beta-1,\ 2+\beta-s;\ (r); \\ -:s \qquad ,\ 1+\beta-s;\ (v); \end{matrix} a,\ c\right]$$

$$(1-a)^{-\alpha}G\begin{matrix}1;\ 2;\ D\\0;\ 2;\ H\end{matrix}\left[\begin{matrix} \alpha:\beta,\ 1+\frac{\beta}{2};\ (r); \\ -:0, \quad \frac{\beta}{2} \quad ;\ (v); \end{matrix} \frac{-a}{1-a},c(1-a)\right] \tag{28}$$

Proof: First we express right hand side of equation (26) in power series with the help of the equations (6), (7), (8) and (9), we have

$$\sigma=\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_{f+g+\ell}(\beta)_{g+\ell}(\beta)_f\left(1+\frac{\beta}{2}\right)_f(r)_g(u)_\ell}{f!g!\ell!(\gamma)_{g+\ell}(s)_f\left(\frac{\beta}{2}\right)_f(s)_g(v)_\ell}.(1-a)^f(b)^g(c)^\ell(1-a)^{-(f+g+\ell+\alpha)} \tag{29}$$

$$\Rightarrow\ \sigma=\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_{f+g+\ell+i}(\beta)_{g+\ell}(\beta)_f\left(1+\frac{\beta}{2}\right)_f(r)_g(u)_\ell}{i!f!g!\ell!(s)_{g+\ell}(s)_f\left(\frac{\beta}{2}\right)_f(s)_g(v)_\ell}.(-1)^f.b^g c^\ell a^{f+1} \tag{30}$$

with the help of the result of author ([33], p. 58 (1))

$$\sum_{g=0}^{\infty}\sum_{t=0}^{\infty}B(t,g)=\sum_{g=0}^{\infty}\sum_{t=0}^{f}B(t,g-t) \tag{31}$$

in equation (30), we may easily obtain

$$\sigma=\sum_{g,\ell,i=0}^{\infty}\frac{(\alpha)_{g+\ell+i}(\beta)_{g+\ell}(r)_g(u)_\ell}{(\gamma)_{g+\ell}(s)_g(v)_\ell}.\frac{a^i}{i!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!}.$$

$$3F_2\left[\begin{matrix}-i,\ \beta,\ 1+\frac{\beta}{2};\\ s,\ \frac{\beta}{2}\qquad ;\end{matrix}\ i\right] \tag{32}$$

$$3F_2\left[\begin{matrix}\alpha,\ 1+\frac{\alpha}{2},\ -f;\\ \frac{\alpha}{2},\ \eta_3\qquad ;\end{matrix}\ i\right]=\frac{\left(\eta_3-\alpha-i-f\right)\left(\eta_3-\alpha\right)_{f-1}}{\left(\eta_3\right)_f} \tag{33}$$

which gives the left hand side of equation (26).

Similarly, we can show that the equation (27).

Now, with the help of the equation (10) we get the equation (28).

Cases of reducibility:

(i) put $c=0$ in equation (26), then

$$F^{1:2;D}_{0:2;E}\left[\begin{matrix}\alpha:s-\beta-1,\ 2+\beta-s;\ (r);\\ -:1+\beta-s,\qquad s\qquad ;\ (s);\end{matrix}\ a,\ b\right]=(1-a)^{-\alpha}.$$

$$F^{1:2;D}_{0:2;E}\left[\begin{matrix}\alpha:\beta,\ 1+\frac{\beta}{2};\ (r);\\ -:s,\quad \frac{\beta}{2}\ ;\ (s);\end{matrix}\ \frac{-a}{1-a},\ \frac{b}{1-a}\right] \tag{34}$$

put $b=0$ in equation (34), we get

$$3F2\left[\begin{matrix}\alpha\quad ,s-\beta-1,\ 2+\beta-s;\\ 1+\beta-s,\qquad ,\qquad s\qquad ;\end{matrix}\ a\right]=$$

$$(1-a)^{-\alpha}\,3F_2\left[\begin{matrix}\alpha,\beta,\ 1+\frac{\beta}{2};\\ s,\frac{\beta}{2}\qquad ;\end{matrix}\ \frac{-a}{1-a}\right] \tag{35}$$

It is only valid when $|a|<1$ and also when $a=1$ provided $\operatorname{Re}(1+\beta-\alpha)>0$, and when $a=-1$ provided that $\operatorname{Re}(s-\alpha-\beta)>0$

which gives the result of author [34].

(ii) put $D=E=2$ in equation (34), we get

$$F^{1:2;2}_{0:2;2}\left[\begin{matrix}\alpha:s-\beta-1,\ 2+\beta-s;\ \in,\in;\\ -:1+\beta-s,\qquad s\qquad ;\ \mu,e;\end{matrix}\ a,b\right]=$$

$$(1-a)^{-\alpha}\ F\begin{matrix}1:2\ ;\ 2\\0:2\ ;\ 2\end{matrix}\left[\begin{matrix}\alpha:\beta,\ 1+\frac{\beta}{2};\ \in,\ \in_1;\ \frac{-a}{1-a},\frac{b}{1-a}\\-:s,\ \frac{\beta}{2}\qquad;\ \mu,e;\end{matrix}\right] \tag{36}$$

which gives the result of author 33 [p. 56 (2.2)].

(iii) putting $B=E=H=1,\ C=D=G=0$

and then replacing β by s in equation (26), we get

$$F^{(3)}\left[\begin{matrix}\alpha::-:\Delta:-:-1,2;-;-;\\-::-:-:-:\ 1\ ,s,\mu;e;\end{matrix}a,b,c\right]=$$

$$(1-a)^{-\alpha}\ F_E\left[\alpha,\alpha,\alpha,1+\frac{s}{2},\Delta,\Delta;\frac{s}{2},\mu,e,\frac{-e}{1-a},\frac{b}{1-a},\frac{c}{1-a}\right] \tag{37}$$

where F_E is represented by the equation (ii).

(iv) put $B=E=H=0,\ C=D=G=1$ and then replacing s by β in equation (26), we obtain

$$F^{(3)}\left[\begin{matrix}\alpha::-;-;-;-1,\ 2;\mu,e;\\-::-;\Delta;-;\ 1\ ,\beta;-,-;\end{matrix}a,b,c\right]=$$

$$(1-a)^{-\alpha}\ F_G\left[\alpha,\alpha,\alpha,1+\frac{\beta}{2},\mu,e;\frac{\beta}{2},\Delta,\Delta;\frac{-a}{1-a},\frac{b}{1-a},\frac{c}{1-a}\right] \tag{38}$$

where F_G is represented by the equation (12).

(v) put $B=C=D=1,G=E=H=0$ and replace s by β in equation (27), we get

$$F^{(3)}\left[\begin{matrix}\alpha::-;-;-;,\ 2;\mu,-;\\-;-;-;1\ \ ,\beta;-,\ -;\end{matrix}a,b,c\right]=$$

$$(1-a)^{-\alpha}\ F_M\left[1+\frac{\beta}{2},\ \in,\in,\in,\mu,\alpha;\frac{\beta}{2},\theta,\theta;\frac{-a}{1-a},b,\frac{c}{1-a}\right] \tag{39}$$

where F_M is represented by the equation (14).

(vi) put $B=C=0$ and $D=G=E=H=1$ and put $s=\beta$ in equation (26), we get

$$F^{(3)}\left[\begin{matrix}\alpha::-;-;-;-1,\ \ 2;\in;\mu;\\-::-;-;-;1,\ \ \beta;\ Q;\ e;\end{matrix}a,\ b,\ c\right]=$$

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$$(1-a)^{-\alpha} F_A\left[\alpha;1+\frac{\beta}{2}, \in, \mu; \frac{\beta}{2}, Q_1, e, \frac{-a}{1-a}, \frac{b}{1-a}, \frac{c}{1-a}\right] \quad (40)$$

where F_A is represented by the equation (16).

(vii) put $B=E=H=0,\ C=G=1,\ D=2$ and $\beta=s$ in equation (27), we get

$$F^{(3)}\left[\begin{matrix}-::-;-;\alpha:-1,\ 2;\ \alpha_2,\beta_2;\theta_1; \\ -::-;\in\ ;-;1,\ s;\ \ -\ -;\ -;\end{matrix}\ a,\ b,\ c\right]=$$

$$(1-a)^{-\alpha} F_N\left[1+\frac{\alpha}{2},\frac{\alpha}{2},\theta_1,\alpha,\beta_2,\alpha;\ \frac{s}{2},\ \alpha,\alpha;\frac{-a}{1-a},b,\ \frac{c}{1-a}\right] \quad (41)$$

where F_N is represented by the equation (15).

(viii) put $b=0$ in equation (40),

$$F\begin{matrix}1:2,1\\0:2,1\end{matrix}\left[\begin{matrix}\alpha:-1,2;\mu; \\ -:1,\ \beta;\ e;\end{matrix}\ a,\ c\right]=$$

$$(1-a)^{-\alpha} F_2\left[\alpha;1+\frac{\beta}{2},\mu;\frac{\beta}{2},e;\frac{-a}{1-a},\frac{c}{1-a}\right] \quad (42)$$

where F_2 is represented by the equation

$$F_2[\alpha;\beta,\gamma;r,s;\ a,b]=\sum_{f,g=0}^{\infty}\frac{(\alpha)_{f+g}(\beta)_f(\gamma)_g}{(r)_f(s)_g}.\frac{a^f}{f!}.\frac{b^g}{g!}$$

(ix) replacing b by $\frac{b}{\alpha_2}$ letting $\alpha_2\to\infty$, the function F_N in equation (41) reduces to Jain's confluent triple hypergeometric function $3\psi_N^{(1)}$ then

$$F^{(B)}\left[\begin{matrix}-::-;-;\alpha:-1,\ 2;\beta_2;\theta_1; \\ -::-1;\in;-;1,\ \ \ s;-;\ -;\end{matrix}\ a,b,c\right]=$$

$$(1-a)^{-\alpha}\, 3\psi_N^{(1)}\left[1+\frac{\alpha}{2},\ \beta_2,\ \theta_1,\ \alpha,\ \alpha,;\ \frac{s}{2},\in,\ \in;\frac{-a}{1-a},b,\frac{c}{1-a}\right] \quad (43)$$

(x) put $C=G=0,\ B=E=D=H=1,\ s=\beta$ in equation (27), we get

$$F^{(3)}\left[\begin{matrix}-::-;\theta_1;\alpha:-1,\ \ 2;\alpha_1;-; \\ -::-;-;-:1,\ \ \ b:\alpha_2;\ \ \Delta;\end{matrix}\ a,\ b,\ c\right]=$$

$$(1-a)^{-\alpha} F_K\left[1+\frac{\beta}{2},\theta_1,\theta_1,\alpha_1,\alpha;\frac{\beta}{2}, \in_2,\ \Delta,\ \frac{-a}{1-a},\ b,\ \frac{c}{1-a}\right]$$

where F_K is represented by the equation (15).

(xi) put $D=2,\ H=0$ and $s=\beta$ in equation (28), we obtain

$$G\begin{matrix}1:2;\ 2\\0:2;\ 0\end{matrix}\left[\begin{matrix}\alpha::-1.2;\in,\theta_1;\\-::\beta,1;\quad -;\end{matrix}\ a,\ b\right]=$$

$$(1-a)^{-\alpha} H_2\left[\alpha,\ 1+\frac{\beta}{2},\ \in,\ \theta_1;\ \frac{\beta}{2};\frac{-a}{1-a},\ c(1-a)\right] \tag{44}$$

Conclusion:

In this research paper, we have proved two theorems and obtained new relations which are very useful in in function spaces and integral transformation.

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