Split Geodetic Dominating Sets in Path Graphs

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Abstract

Let $Dsg(P_n,i)$ be the family of split geodetic dominating sets of the path graph $P_n$. with cardinality $i$ and let $dsg(P_n,i) = |Dsg(P_n,i)|$. Then the split geodetic polynomial $Dsg(P_n,x)$ of $P_n$ is defined as $Dsg(P_n,x) = \sum_{i=\gamma_{sg}(P_n)}^{n} dsg(P_n,i)x^i$, where $\gamma_{sg}(P_n)$ is the split geodetic domination number of $P_n$. In this paper we have determined the family of split geodetic dominating sets of the path graph $P_n$ with cardinality $i$. Also, we have obtained the recursive formula to derive the split geodetic domination polynomials of paths and also obtain some properties of this polynomial.

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1 Introduction

Let $G = (V,E)$ be a simple graph of order $|V| = n$. A dominating set for a graph $G = (V,E)$ is a subset $D$ of $V$ such that every vertex not in $D$ is adjacent to at least one member of $D$. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for $G$ [1]. We call a set of vertices $S$ in a graph $G$ a geodetic dominating set if $S$ is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of $G$ is its geodetic domination number, and is denoted by $\gamma_g(G)$ [2] [3]. Split geodetic number of a graph was studied by in [4]. A geodetic set $S$ of a graph $G = (V,E)$ is the split geodetic set if the induced subgraph $(V - S)$ is disconnected. The split geodetic number $g_{sg}(G)$ of $G$ is the minimum cardinality of a split geodetic set. A set $S \subseteq V(G)$ is said to be a split geodetic dominating set of $G$ if $S$ is both a split geodetic set and a dominating set of $G$. The minimum cardinality of the split geodetic dominating set of $G$ is called the split geodetic domination number of $G$ and is denoted by $\gamma_{sg}(G)$. The concept of split geodetic domination number was introduced by P. Arul Paul Sudhahar and J. Jeba Lisa in [5]. A domination polynomial can be studied in [6][7][8][9] and the geodetic domination polynomial was studied in [10]. A path is a connected graph in which two vertices have degree 1 and the remaining vertices have degree 2. Let $P_n$ be a path with $n$ vertices. Let $Dsg(P_n,i)$ be the family of split geodetic dominating sets of the path graph $P_n$ with cardinality $i$ and let $dsg(P_n,i) = |Dsg(P_n,i)|$. Then the split geodetic polynomial $Dsg(P_n,x)$ of $P_n$ is defined as $Dsg(P_n,x) = \sum_{i=\gamma_{sg}(P_n)}^{n} dsg(P_n,i)x^i$, where $\gamma_{sg}(P_n)$ is the split geodetic domination number of $P_n$. 

1496
In the next section we construct the families of the split geodetic dominating sets of paths by a recursive method. 
In section 3, we use the results obtained in section 2 to study the split geodetic domination polynomial of paths.

Lemma 2.1. \( Y_s(P_n) = \frac{n+2}{3} \)

Lemma 2.2. \( D_s(P_n,i) = \Phi \) if and only if \( i > n \) or \( i < \frac{n+2}{3} \) and \( D_s(P_n,i) > 0 \) if \( \frac{n+2}{3} \leq i \leq n \).

Lemma 2.3. IF \( Y \in D_s(P_{n-1}, i - 1) \) or \( D_s(P_{n-2}, i - 1) \) or \( D_s(P_{n-3}, i - 1) \) then \( Y \cup \{i\} \in D_s(P_n,i) \).

Lemma 2.4. (i) If \( D_s(P_{n-1}, i - 1) = D_s(P_{n-2}, i - 1) = \Phi \) then \( D_s(P_{n-2}, i - 1) = \Phi \)

(ii) If \( D_s(P_{n-1}, i - 1) \neq \Phi \) then \( D_s(P_{n-1}, i - 1) = \Phi \)

(iii) If \( D_s(P_{n-1}, i - 1) = D_s(P_{n-2}, i - 1) = D_s(P_{n-3}, i - 1) = \Phi \) then \( D_s(P_n, i) = \Phi \)

Proof.

(i) If \( D_s(P_{n-1}, i - 1) = \Phi \) and \( D_s(P_{n-3}, i - 1) = \Phi \) then \( i - 1 > n - 1 \) or \( i - 1 < \frac{n+1}{3} \) holds. Therefore \( D_s(P_n, i - 1) = \Phi \).

(ii) If \( D_s(P_{n-1}, i - 1) \neq \Phi \) then \( D_s(P_{n-1}, i - 1) \neq \Phi \) and \( D_s(P_{n-3}, i - 1) \neq \Phi \) then \( \frac{n+1}{3} \leq i - 1 \leq n - 1 \) or \( \frac{n-1}{3} \leq i - 1 \leq n - 3 \).

Hence \( D_s(P_n, i - 1) = \Phi \).

(iii) If \( D_s(P_{n-1}, i - 1) = D_s(P_{n-2}, i - 1) = D_s(P_{n-3}, i - 1) = \Phi \) then \( i - 1 < \frac{n+1}{3} \) or \( i - 1 > n - 1 \) or \( i - 1 < \frac{n-1}{3} \) holds. Therefore \( D_s(P_n, i) = \Phi \).

Lemma 2.5. If \( D_s(P_{n-1}, i) = \Phi \) then we have

(i) \( D_s(P_{n-1}, i - 1) = D_s(P_{n-2}, i - 1) = \Phi \) and \( D_s(P_{n-3}, i - 1) = \Phi \) if and only if \( n = 3k - 2, i = k \), for some positive integer \( k \).

(ii) \( D_s(P_{n-2}, i - 1) = D_s(P_{n-3}, i - 1) = \Phi \) and \( D_s(P_{n-1}, i - 1) = \Phi \) if and only if \( i = n \).

(iii) \( D_s(P_{n-3}, i - 1) \neq \Phi \) if and only if \( i = n - 1 \).

(iv) \( D_s(P_{n-1}, i - 1) = D_s(P_{n-2}, i - 1) \neq \Phi \) and \( D_s(P_{n-3}, i - 1) = \Phi \) if and only if \( n = 3k \) and \( i = \frac{3k+2}{3} \) for some \( k \in N \).

(v) \( D_s(P_{n-1}, i - 1) = \Phi \) and \( D_s(P_{n-2}, i - 1) = \Phi \) if and only if \( n = 3k \) and \( i = \frac{3k+2}{3} \) for some \( k \in N \).

Proof

(i) Since \( D_s(P_{n-1}, i - 1) = D_s(P_{n-2}, i - 1) = \Phi \) then \( i - 1 > n - 1 \) or \( i - 1 < \frac{n+1}{3} \) or \( i - 1 < \frac{n-1}{3} \).

(ii) If \( i - 1 > n - 1 \) then \( i > n \) and hence \( D_s(P_n, i) = \Phi \) which is a contradiction. Therefore \( i - 1 < \frac{n-1}{3} \) or \( i = \frac{n}{3} + 1 \). Also since \( D_s(P_{n-3}, i - 1) = \Phi \) then \( D_s(P_{n-3}, i - 1) = \Phi \) and \( D_s(P_{n-2}, i - 1) = \Phi \) if and only if \( n = 3k - 2, i = k \), for some \( k \in N \).

Conversely assume \( n = 3k - 2, i = k \), for some \( k \in N \) then by lemma 2.2 \( D_s(P_{n-1}, i - 1) = \Phi \) and \( D_s(P_{n-3}, i - 1) = \Phi \).

Conversely if \( i = n \), then \( D_s(P_{n-1}, i - 1) = D_s(P_{n-2}, i - 1) = \Phi \), then \( i - 1 < \frac{n+1}{3} \) or \( i - 1 > n - 2 \) if \( i - 1 < \frac{n+1}{3} \) then \( i - 1 < \frac{n+1}{3} \). Also since \( D_s(P_{n-1}, i - 1) = \Phi \) then \( D_s(P_{n-1}, i - 1) = \Phi \) and \( D_s(P_{n-3}, i - 1) = \Phi \) if and only if \( n = 3k - 2, i = k \), for some \( k \in N \). Hence \( i = n \). Conversely if \( i = n \), then \( D_s(P_{n-1}, i - 1) = D_s(P_{n-2}, i - 1) = \Phi \), then \( i - 1 < \frac{n+1}{3} \) or \( i - 1 > n - 2 \) if \( i - 1 < \frac{n+1}{3} \) then \( i - 1 < \frac{n+1}{3} \). Also since \( D_s(P_{n-1}, i - 1) = \Phi \) then \( D_s(P_{n-1}, i - 1) = \Phi \) and \( D_s(P_{n-3}, i - 1) = \Phi \) if and only if \( n = 3k - 2, i = k \), for some \( k \in N \).
(iii) Assume $\text{Dsg}(P_{n-1}, i - 1) \neq \Phi$; $\text{Dsg}(P_{n-2}, i - 1) \neq \Phi$ and $\text{Dsg}(P_{n-3}, i - 1) = \Phi$. Since $\text{Dsg}(P_{n-3}, i - 1) = \Phi$, $i - 1 > n - 3$ or $i - 1 < \left\lceil \frac{n-1}{3} \right\rceil$. Since $\text{Dsg}(P_{n-2}, i - 1) \neq \Phi$, $\left\lfloor \frac{n-1}{3} \right\rfloor < i - 1 \leq n - 2$. That is, $i - 1 < \left\lceil \frac{n-1}{3} \right\rceil$ is not possible. Therefore, $i - 1 > n - 3 \Rightarrow i - 1 \geq n - 2$. But $i - 1 \leq n - 2 \Rightarrow i - 1 = n - 2 \Rightarrow i = n - 1$. Conversely suppose $i = n - 1$, then $\text{Dsg}(P_{n-1}, i - 1) = \text{Dsg}(P_{n-1}, n - 2) \neq \Phi$, $\text{Dsg}(P_{n-2}, i - 1) = \text{Dsg}(P_{n-2}, n - 2) \neq \Phi$, but $\text{Dsg}(P_{n-3}, i - 1) = \text{Dsg}(P_{n-3}, n - 2) = \Phi$.

(iv) Assume $\text{Dsg}(P_{n-3}, i - 1) = \Phi$; $\text{Dsg}(P_{n-2}, i - 1) \neq \Phi$ and $\text{Dsg}(P_{n-3}, i - 1) \neq \Phi$. Since $\text{Dsg}(P_{n-1}, i - 1) = \Phi$, $i - 1 > n - 1 \Rightarrow i - 1 > n - 2 \Rightarrow \text{Dsg}(P_{n-2}, i - 1)$ and $\text{Dsg}(P_{n-3}, i - 1)$ are empty, which is a contradiction. Therefore $i - 1 < \left\lceil \frac{n+1}{3} \right\rceil \Rightarrow i < \left\lceil \frac{n+1}{3} \right\rceil + 1$. Since $\text{Dsg}(P_{n-2}, i - 1) \neq \Phi$ and $\text{Dsg}(P_{n-3}, i - 1) \neq \Phi$, we have $\left\lceil \frac{n}{3} \right\rceil \leq i - 1 \leq n - 2$ and $\left\lfloor \frac{n-1}{3} \right\rfloor \leq i - 1 \leq n - 3$. Therefore $\left\lceil \frac{n}{3} \right\rceil \leq i - 1 \leq n - 3$. Hence $\left[ \frac{n}{3} \right] + 1 \leq i < \left\lceil \frac{n+1}{3} \right\rceil + 1$. This holds only when $n = 3k$ and $i = k + 1 = \left\lceil \frac{3k+3}{3} \right\rceil$ for some $k \in N$. Conversely, assume $n = 3k$ and $i = k + 1 = \left\lceil \frac{3k+3}{3} \right\rceil$, then $\text{Dsg}(P_{n-1}, i - 1) = \Phi$; $\text{Dsg}(P_{n-2}, i - 1) \neq \Phi$ and $\text{Dsg}(P_{n-3}, i - 1) \neq \Phi$.

Lemma 2.6. If $\text{Dsg}(P_n, i) \neq \Phi$, then

(i) $\text{Dsg}(P_{n-1}, i - 1) = \text{Dsg}(P_{n-2}, i - 1) = \Phi$ and $\text{Dsg}(P_{n-3}, i - 1) \neq \Phi$, then $\text{Dsg}(P_n, i) = \{1, 4, \ldots, 3k - 5, 3k - 2\}$

(ii) $\text{Dsg}(P_{n-2}, i - 1) = \text{Dsg}(P_{n-3}, i - 1) = \Phi$ and $\text{Dsg}(P_{n-1}, i - 1) \neq \Phi$ then $\text{Dsg}(P_n, i) = \{1, 2, \ldots, n\}$

(iii) $\text{Dsg}(P_{n-1}, i - 1) \neq \Phi$; $\text{Dsg}(P_{n-2}, i - 1) \neq \Phi$ and $\text{Dsg}(P_{n-3}, i - 1) = \Phi$ then $\text{Dsg}(P_n, i) = \{[n] - x/x \in n - \{1, n\}\}.

(iv) $\text{Dsg}(P_{n-1}, i - 1) = \Phi$; $\text{Dsg}(P_{n-2}, i - 1) \neq \Phi$ and $\text{Dsg}(P_{n-3}, i - 1) \neq \Phi$, then $\text{Dsg}(P_n, i) = \{X_1 \cup 3k/X_1 \in P_{3k-2}, k \cup X_2 \cup 3k/X_2 \in P_{3k-3}, k\}$

(v) $\text{Dsg}(P_{n-1}, i - 1) \neq \Phi$; $\text{Dsg}(P_{n-2}, i - 1) \neq \Phi$ and $\text{Dsg}(P_{n-3}, i - 1) = \Phi$ then $\text{Dsg}(P_n, i) = \{X_1 \cup n/X_1 \in P_{n-1}, i - 1 \cup X_2 \cup n/X_2 \in P_{n-1}, i - 1 \cup X_3 \cup n/X_3 \in P_{n-3}, i - 1\}$

Proof.

(i) Since $\text{Dsg}(P_{n-3}, i - 1) = \text{Dsg}(P_{n-2}, i - 1) = \Phi$ and $\text{Dsg}(P_{n-3}, i - 1) \neq \Phi$, then by Lemma 2.5 (i) $n = 3k - 2$ and $i = k$ for some $k \in N$. Hence $\text{Dsg}(P_n, i) = \{1, 4, \ldots, 3k - 5, 3k - 2\}$.

(ii) Since $\text{Dsg}(P_{n-3}, i - 1) = \text{Dsg}(P_{n-2}, i - 1) = \Phi$ and $\text{Dsg}(P_{n-1}, i - 1) \neq \Phi$, then by lemma 2.5 (ii) $i = n$. Then $\text{Dsg}(P_n, i) = \{1, 2, \ldots, n\}$.

(iii) Since $\text{Dsg}(P_{n-3}, i - 1) \neq \Phi$; $\text{Dsg}(P_{n-2}, i - 1) \neq \Phi$ and $\text{Dsg}(P_{n-3}, i - 1) = \Phi$, then by lemma 2.5 (iii) $i = n - 1$, then $\text{Dsg}(P_n, i) = \{[n] - x/x \in [n] - \{1, n\}\}$.

(iv) Since $\text{Dsg}(P_{n-2}, i - 1) = \Phi$; $\text{Dsg}(P_{n-3}, i - 1) \neq \Phi$ and $\text{Dsg}(P_{n-3}, i - 1) \neq \Phi$ by Lemma 2.5 (iv) $n = 3k$ and $i = k + 1$ for some $k \in N$. Let $X_1 = \{1, 2, 3k - 3 \in P_{3k-3}, k \}$ then $X_1 \cup \{3k\} \in P_{3k-2}, k + 1$ and $X_2 = \{1, 4, \ldots, 3k - 2 \} \in P_{3k-2}, k$ then $X_2 \cup \{3k\} \in P_{3k-3}, k + 1$. Therefore $X_1 \cup \{3k\}/X_1 \in P_{3k-3}, k \cup \{3k\}/X_2 \in P_{3k-2}, k \cup \{3k\}/X_2 \in P_{3k-2}, k \cup \{3k\}/X_2 \in P_{3k-3}, k$. Similarly $\text{Dsg}(P_{n-1}, i - 1) \neq \Phi$; $\text{Dsg}(P_{n-2}, i - 1) \neq \Phi$ and $\text{Dsg}(P_{n-3}, i - 1) \neq \Phi$. Let $X_1 \in \text{Dsg}(P_{n-1}, i - 1)$, then $n - 1, n - 2$ or $n - 3$ is in $X_1$. If $n - 1, n - 2$ or $n - 3 \in X_1$ then $X_1 \cup \{n\} \in$
$D_{sg}(P_n, i)$. Let $X_2 \in D_{sg}(P_{n-2}, i - 1)$, then $n - 2 \text{ or } n - 3 \text{ or } n - 4$ is in $X_2$. If $n - 2, n - 3 \text{ or } n - 4 \in X_2$ then $X_2 \cup \{n\} \in D_{sg}(P_n, i)$. Now let $X_3 \in D_{sg}(P_{n-3}, i - 1)$, then $n - 3, n - 4 \text{ or } n - 5$ is in $X_3$. If $n - 3, n - 4 \text{ or } n - 5 \in X_3$ then $X_3 \cup \{n\} \in D_{sg}(P_n, i)$. Thus we have: $X_1 \cup \{n\} / X_1 \in P_{n-1, i - 1}$ \cup $X_2 \cup \{n\} / X_2 \in P_{n-2, i - 1}$ \cup $X_3 \cup \{n\} / X_3 \in P_{n-3, i - 1}$ \subseteq $D_{sg}(P_n, i)$. If $n \in Y$, then $Y = X_1 \cup \{n\}$ for some $X_1 \in D_{sg}(P_{n-1, i - 1})$. If $n - 1 \in Y$ then $Y = X_2 \cup \{n\}$ for some $X_2 \in D_{sg}(P_{n-2, i - 1})$. If $n - 2 \in Y$ then $Y = X_3 \cup \{n\}$, for some $X_3 \in D_{sg}(P_{n-3, i - 1})$. So $D_{sg}(P_n, i) = \{X_1 \cup n / X_1 \in P_{n-1, i - 1} \cup X_2 \cup n / X_2 \in P_{n-2, i - 1} \cup X_3 \cup n / X_3 \in P_{n-3, i - 1}\}$.

Example 2.7: Consider $P_6$ with $V(P_6) = 6$.

We use Lemma 2.6, to construct $D_{sg}(P_6, i)$ for $2 \leq i \leq 6$. $D_{sg}(P_6, 2) = \phi$ since $D_{sg}(P_5, 2) = \phi$, $D_{sg}(P_6, 2) = \{1, 4\}$, $D_{sg}(P_6, 2) = \{1, 3\}$ then by Lemma 2.6 (iv) $D_{sg}(P_6, 3) = \{1, 4, 6\}, \{1, 3, 6\}$. Since $D_{sg}(P_6, 2) = \{1, 3, 5\}, \{1, 2, 5\}, \{1, 4, 5\}, D_{sg}(P_6, 3) = \{1, 3, 4\}, \{1, 2, 4\}, D_{sg}(P_6, 3) = \{1, 2, 3\}$. Therefore by Lemma 2.6 (v) $D_{sg}(P_6, 4) = \{1, 3, 5, 6\}, \{1, 2, 5, 6\}, \{1, 4, 5, 6\}, \{1, 3, 4, 6\}, \{1, 2, 4, 6\}, \{1, 2, 3, 6\}$. By Lemma 2.6 (iii) $D_{sg}(P_6, 5) = \{1, 3, 4, 5, 6\}, \{1, 2, 4, 5, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 3, 4, 6\}$. By Lemma 2.6 (ii) $D_{sg}(P_6, 6) = \{1, 2, 3, 4, 5, 6\}$.

References