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Results in Fixed Point Theorems for Relational-Theoretic Contraction Mappings in Metric Spaces

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Abstract:-The purpose of this article is to extend the classical Banach contraction principle to a complete metric space endowed with a binary relation where the contraction condition is relatively weaker than usual contraction as it is required to hold only on those elements which are related under the underlying relation rather than the whole space. Finally, we apply our results to prove the existence and uniqueness of solution of a certain class of nonlinear contraction in a complete metric space. The presented theorems extend and subsumes various known comparable results from the current literature. Some illustrative examples and applications are provided to demonstrate the main results and to show the genuineness of our results.

Keywords*: Fixed point, binary relations, complete metric spaces, contraction mappings.*

1. Introduction

The metric space, first developed by French mathematician M. Frechet in 1906, has been an important concept in functional analysis and mathematical analysis. Since its introduction, many researchers have generalized and extended the metric space concept in various ways, leading to the development of complex valued metric spaces, rectangular metric spaces, semi metric spaces, quasimetric spaces, and others. These generalizations have been motivated by the wide-ranging impact of the original metric space on mathematics, with applications in areas such as probability theory, game theory, and computer science.A significant amount of research has been devoted to exploring the generalizations and improvements of fixed point results, many of which focus on establishing the existence and uniqueness of fixed points. One of the most notable of these results is that of Ran and Reurings [12], who studied the existence of fixed points for certain mappings in partially ordered metric spaces, with applications to matrix equations. This work was later extended by Nieto and Lopez [11], who considered non-decreasing mappings and derived solutions to partial differential equations with periodic boundary conditions. In this direction several authors obtained further results [2,5,7,9,10,20,23].

Turinici [1] introduced the novel idea of an order-theoretic fixed point result, and this concept was later expanded upon by Samet and Turinici [17], who developed fixed point results using the symmetric closure of an amorphous binary relation. Alam and Imdad [13] further extended these results, deriving a relation-theoretic analogue of the Banach contraction principle that unifies many existing order-theoretic results. These results represent significant contributions to the field of fixed point theory, building upon the work of Turinici [1] and other researchers.

In addition to their previous work [13], Alam and Imdad [14] presented a new variant of the Banach contraction principle on complete metric spaces with binary relations. This variant utilized relation-theoretic analogues of various metrical notions such as contraction, completeness, and continuity, all of which were reduced to their corresponding usual notions under the universal relation. This variant represents an important step forward in developing fixed point theory in the context of relations.

In view of the above considerations, we extend the classical Banach contraction principle to a complete metric space endowed with a binary relation where the contraction condition is relatively weaker than usual contraction as it is required to hold only on those elements which are related under the underlying relation rather than the whole space. The presented theorems extend and subsumes various known comparable results from the current literature. Some illustrative examplesand applications are provided to demonstrate the main results.

Throughout this paper, R stands for a nonempty binary relation, but forthe sake of simplicity, we write onlybinary relation instead of nonemptybinary relation, $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ respectively denote the sets of natural numbers, rationalnumbers and real numbers wherein $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also, we use the following notations:
(i)$F(T)$ = the set of all fixed points of T,
(ii)$X(T, R) := \{x \in X: (x, Tx) \in R\}$,
(iii) $Y(x, y, R)$:= the class of all paths in R from x to y.

2Preliminaries

We start this section by presenting some basic relevant definitions andpropositions.

Definition 2.1 [14] Let X be a nonempty set. A subset R of X^2 is called a binary relation on X.

Notice that for each pair$x, y \in X$, one of the following holds:

(i) $(x, y) \in R$; means that "x is R-related to y" or "x relates to y under R". Sometimes, we write xRy instead of $(x, y) \in R$.

(ii) $(x, y) \notin R$; means that "x is not R-related to y" or "x doesn't relate to y under R".

Trivially, X^2 and $\emptyset$ being subsets of X^2 are binary relations on X, which are respectively called the universal relation (or full relation) and empty relation.

Definition 2.2 [19] Let R be a binary relation on a nonempty set X and $x, y \in X$. Then$x$ and y are R-comparative if either $(x, y) \in R$ or $(y, x) \in R$. Denoted by$[x, y] \in R$.

Proposition 2.3[13] If (X, d) is a metric space, R is a binary relation on X, $T$is a self-mapping on X and $\lambda \in [0,1)$, then the following contractive conditions are equivalent:

(a) $d(Tx, Ty) \leq \lambda d(x, y), \forall x, y \in X$ with $(x, y) \in R$,

(b) $d(Tx, Ty) \leq \lambda d(x, y), \forall x, y \in X$ with $[x, y] \in R$.

Proof The implication (b)⇒(a) is trivial. On the other hand, suppose that (a) holds. Take $x, y \in X$ with $[x, y] \in R$. If $(x, y) \in R$, then (b) is directly follow from (a). Otherwise, in case $(y, x) \in R$, using symmetry of d and (a), we obtain

$$d(Tx, Ty) = d(Ty, Tx) \leq \lambda d(y, x) = \lambda d(x, y) \qquad (2.1)$$

Implies that (a)⇒(b).

Proposition 2.4 If (X,d) is a metric space, R is a binary relation on X, $T$is a self-mapping on X and $\lambda \in [0,1)$, then the following contractivity conditions are equivalent:

(a) $d(Tx,Ty) \leq \lambda\big(M(x,y)\big), \forall x,y \in X$ with $(x,y) \in R$,

where

$$M(x,y) = max\left\{\begin{matrix} d(x,y), \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)}, \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \\ \dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \end{matrix}\right\} \quad (2.2)$$

(b) $d(Tx,Ty) \leq \lambda\big(M(x,y)\big), \forall x,y \in X$ with $[x,y] \in R$,

where

$$M(x,y) = max\left\{\begin{matrix} d(x,y), \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)}, \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \\ \dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \end{matrix}\right\} \quad (2.3)$$

ProofThe implications follow from the proof of proposition 2.3.

Definition 2.5 [13] A binary relation R defined on a nonempty set $X$is called

- reflexive if$(x,x) \in R\ \forall x \in X$,
- irreflexive if$(x,x) \notin R\ \forall x \in X$,
- symmetric if $(x,y) \in R$ implies$(y,x) \in R$,
- antisymmetric if$(x,y) \in R$ and$(y,x) \in R$ implies $x = y$,
- transitive if$(x,y) \in R$ and$(y,z) \in R$ implies$(x,z) \in R$,
- complete, connected or dichotomous if$[x,y] \in R\ \forall x,y \in X$,
- weakly complete, weakly connected or trichotomous if $[x,y] \in R$ or $x = y\ \ \forall x,y \in X$.

Definition 2.6 [13] A binary relation R defined on a non-empty set X is called

- strict order or sharp order if R is irreflexive and transitive,
- near-order if R is antisymmetric and transitive,
- pseudo-order if R is reflexive and antisymmetric,
- quasi-order or preorder if R is reflexive and transitive,
- partial order if R is reflexive, antisymmetric and transitive,
- simple order if R is weakly complete strict order,
- weak order if R is complete preorder,
- total order, linear order or chain if R is complete partial order,
- tolerance if R is reflexive and symmetric,
- equivalence if R is reflexive, symmetric and transitive.

Remark 2.7 Clearly, universal relation X^2 defined on a nonempty set Xremains a complete equivalence relation.

Definition 2.8 [14] Let X be a nonempty set and R a binary relation on X.

(1) The inverse or transpose or dual relation of$R$, denoted byR^{-1} andis defined by

$R^{-1} \coloneqq \{(x,y) \in X^2 : (x,y) \in R\}$. (2.4)

(2) The symmetric closure of R, denoted by R^s, is defined to be the set$R \cup R^{-1}$by

$R^s \coloneqq R \cup R^{-1}$. (2.5)

Indeed, R^s is the smallest symmetric relation on X containing R.

Proposition 2.9 [14] For a binary relation R on a nonempty set X,

$(x, y) \in R^s \Leftrightarrow [x, y] \in R.$ (2.6)

Definition 2.10 [4] Let X be a nonempty set and R a binary relation on X. A sequence$\{x_n\} \subset X$ is called R-preserving if

$(x_n, x_{n+1}) \in R \; \forall \, n \in \mathbb{N}_0.$ (2.7)

Definition 2.11 [4]Let (X, d) be a metric space and R a binary relation on X.A binary relation R on X iscalled d-self-closed if whenever $\{x_n\}$ is an R-preserving sequence and $x_n \xrightarrow{d} x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with

$[x_{n_k}, x] \in R \;\; \forall \, k \in \mathbb{N}_0.$ (2.8)

Definition 2.12 [15] Let X be a nonempty set and T a self-mapping on X. A binary relation R on X is called T-closed if for all $x, y \in X$,

$(x, y) \in R \Rightarrow (Tx, Ty) \in R.$ (2.9)

Proposition 2.13[6] Let X be a nonempty set, R a binary relation on X and $T$is a self-mapping on X. If R is$T$-closed, thenR^s is also T-closed.

Proposition 2.14[4] Let X be a nonempty set, R a binary relation on X and $T$is a self-mapping on X. If R is$T$-closed, then, for all $n \in \mathbb{N}_0$. R is also T^n-closed, where T^n denotes nth iterate of T.

Definition 2.15[3] Let (X, d, R) be a metric space and R a binary relation on X. Then(X, d) is R-complete if every R-preserving Cauchy sequence in X converges to a point in X.

Definition 2.16[8] Let (X, d) be a metric space and R a binary relation on X. A subset E of X is called R-closed if every R-preserving convergent sequence in E converges to a point of E.

Remark 2.17 Every closed subset of a metric space is R-closed, for any binary relation R. Particularly, under the universal relation the notion of R-closedness coincides with usual closedness.

Definition 2.18 [17]Let (X, d) be a metric space and R a binary relation on X. Then a subset E of X is called R-connected if for each pair $x, y \in E$, there exists a path in R from x to y.

Definition 2.19 [16] Let X be a nonempty set and R a binary relation on X. A subset E of X is called R-directed if for each pair $x, y \in E$, there exists $z \in X$ such that$(x, z) \in R$ and$(y, z) \in R$.

Definition 2.20 [6] Let X be a nonempty set and R a binary relation on X. For $x, y \in X$, a path of length k (where k is a natural number) in R from x to y is a finite sequence$\{z_0, z_1, z_2, \ldots, z_k\} \subset X$ satisfying the following conditions:

(i) $z_0 = x$ and $z_k = y$,

(ii) $(z_i, z_{i+1}) \in R$ for each $i(0 \le i \le k-1)$.

Notice that a path of length k involves $k+1$ elements of X, although they are not necessarily distinct.

3Results

Theorem 3.1 Let $X$and Ybe nonempty set equipped with a binary relation R and a metric d such that (X, d) is an R-complete subspace of X. Let $T: X \to X$ be a self-mapping. Suppose that the following conditions hold:

(a) $X(T, R)$ is nonempty,

(b) R is T-closed,

(c) either T is continuous or R is d-self-closed,

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(d) there exists $\lambda \in [0,1)$ such that

$d(Tx,Ty) \leq \lambda\big(M(x,y)\big), \forall x,y \in X$ with $(x,y) \in R$,

where

$$M(x,y) = max\left\{\begin{matrix} d(x,y), \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)}, \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \\ \dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \end{matrix}\right\} \quad (3.1)$$

Then T has a fixed point.

ProofIn view of the assumption (d), we choose x_0 an arbitrary element of $X(T,R)$. Construct a sequence $\{x_n\}$of Picard iterates, i.e.

$x_n = T^n(x_0)$ for all $n \in \mathbb{N}_0$. (3.2)

Since$(x_0, Tx_0) \in R$, using assumption (b), we get

$$(Tx_0, T^2x_0), (T^2x_0, T^3x_0), \dots, (T^nx_0, T^{n+1}x_0), \dots \in R$$

Observe that

$(x_n, x_{n+1}) \in R$for all $n \in \mathbb{N}_0$ (3.3)

Thus, the sequence $\{x_n\}$is R-preserving. Applying the contractive condition (d), we have

$d(x_{n+1}, x_{n+2}) \leq \lambda\big(M(x_n, x_{n+1})\big)$ for all $n \in \mathbb{N}_0$ (3.4)

where

$$M(x_n, x_{n+1}) = max\left\{\begin{matrix} d(x_n,x_{n+1}), \dfrac{d(x_n,Tx_n)d(x_{n+1},Tx_{n+1})}{d(x_n,x_{n+1})}, \dfrac{d(x_n,Tx_n)d(x_{n+1},Tx_{n+1})}{d(x_n,x_{n+1})+d(x_n,Tx_{n+1})+d(x_{n+1},Tx_n)} \\ \dfrac{d(x_n,Tx_n)d(x_n,Tx_{n+1})+d(x_{n+1},Tx_n)d(x_{n+1},Tx_{n+1})}{d(x_{n+1},Tx_n)+d(x_n,Tx_{n+1})} \end{matrix}\right\} \quad (3.5)$$

$= max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$ (3.6)

If for some $n \geq 1$, we have $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$.

From (3.6), we get

$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$ (3.7)

a contradiction. Thus, for all $n \geq 1$, we have

$$max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_n, x_{n+1}) \quad (3.8)$$

Using (3.4) and (3.8), we get

$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1})$ for all $n \geq 1$ (3.9)

Using an inductive process, we obtain

$d(x_{n+1}, x_{n+2}) \leq \lambda^{n+1} d(x_0, Tx_0)$ for all $n \in \mathbb{N}_0$. (3.10)

Now, consider (3.10) and triangular inequality for all $n \in \mathbb{N}_0, k \in \mathbb{N}$ with $k \geq 2$, we have

$$d(x_{n+1}, x_{n+k}) \leq d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{n+k-1}, x_{n+k})$$

$$\leq (\lambda^{n+1} + \lambda^{n+2} + \cdots + \lambda^{n+k-1}) d(x_0, Tx_0)$$
$$= \lambda^n d(x_0, Tx_0) \sum_{j=1}^{k-1} \lambda^j$$

(3.11)

as $n \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists$x \in X$ such that

$$x_n \xrightarrow{d} x.$$

Now, we consider the following two cases:

Case1 On using assumption(c), T is continuous, we have

$$x_{n+1} = T(x_n) \xrightarrow{d} T(x). \qquad (3.12)$$

Owing to the uniqueness of limit, we get $T(x) = x$. Thus,x is a fixed point of T.

Case 2 Now, assume that R is d-self-closed. Since$\{x_n\}$ is an Rpreserving sequence and $x_n \xrightarrow{d} x$, there exists a subsequence$\{x_{n_r}\}$of $\{x_n\}$ with

$$[x_{n_r}, x] \in R \ \ \forall r \in \mathbb{N}_0. \qquad (3.13)$$

Using (d), Proposition 2.4, $[x_{n_r}, x] \in R$ and$x_{n_r} \xrightarrow{d} x$, we have

$$d(x_{n_r+1}, Tx) = d(Tx_{n_r}, Tx) \leq \lambda\left(M(x_{n_r}, x)\right),$$

where

$$M(x_{n_r}, x) = max \left\{ \begin{matrix} d(x_{n_r}, x), \dfrac{d(x_{n_r}, Tx_{n_r}) d(x, Tx)}{d(x_{n_r}, x)}, \dfrac{d(x_{n_r}, Tx_{n_r}) d(x, Tx)}{d(x_{n_r}, x) + d(x_{n_r}, Tx) + d(x, Tx_{n_r})} \\ \dfrac{d(x_{n_r}, Tx_{n_r}) d(x_{n_r}, Tx) + d(x, Tx_{n_r}) d(x, Tx)}{d(x, Tx_{n_r}) + d(x_{n_r}, Tx)} \end{matrix} \right\}$$

(3.14)

$$d(x_{n_r+1}, Tx) = d(Tx_{n_r}, Tx) \leq \lambda\left(M(x_{n_r}, x)\right) \to 0 \qquad (3.15)$$

as $r \to \infty$ so that $x_{n_r+1} \xrightarrow{d} T(x)$. Again, owing to the uniqueness of limit, we get

$T(x) = x$.

Thus,x is a fixed point of$T$.

Now, we prove a corresponding uniqueness result.

Theorem 3.2 In addition to the hypotheses of Theorem 3.1, suppose that the following condition holds:

(e) $Y(x, y, R^s)$ is nonempty, for each $x, y \in X$,

Then T has a unique fixed point.

Proof In view of Theorem 3.1, $F(T) \neq \emptyset$. Take $x, y \in F(T)$, then for all $n \in \mathbb{N}_0$, wehave $T^n(x) = x$ and $T^n(y) = y$. (3.16)

Clearly$x, y \in T(X)$. By assumption (e), there exists a path (say $\{z_0, z_1, z_2, \ldots, z_k\}$)of some finite length k in R^s from x to y so that

$z_0 = x, z_k = y$ and $[z_i, z_{i+1}] \in R$ for each $i(0 \leq i \leq k-1).(3.17)$
since R is T-closed, using Propositions 2.13, we have

$[T^n z_i, T^n z_{i+1}] \in R$ for each $i(0 \leq i \leq k-1)$ and for each $n \in \mathbb{N}_0$. (3.18)
Consider (3.16),(3.17), (3.18), assumption (d) and Proposition 2.4, we have

$d(x,y) \leq \lambda\big(M(T^n z_i, T^n z_{i+1})\big),$

where

$$M(T^n z_i, T^n z_{i+1}) = max\left\{\begin{array}{c} d(T^n z_i, T^n z_{i+1}), \frac{d(T^n z_i, TT^n z_i)d(T^n z_{i+1}, TT^n z_{i+1})}{d(T^n z_i, T^n z_{i+1})}, \frac{d(T^n z_i, TT^n z_i)d(T^n z_{i+1}, TT^n z_{i+1})}{d(T^n z_i, T^n z_{i+1}) + d(T^n z_i, TT^n z_{i+1}) + d(T^n z_{i+1}, TT^n z_i)} \\ \frac{d(T^n z_i, TT^n z_i)d(T^n z_i, TT^n z_{i+1}) + d(T^n z_{i+1}, TT^n z_i)d(T^n z_{i+1}, TT^n z_{i+1})}{d(T^n z_{i+1}, TT^n z_i) + d(T^n z_i, TT^n z_{i+1})} \end{array}\right\}$$

$$= max\left\{\begin{array}{c} d(T^n z_i, T^n z_{i+1}), \frac{d(T^n z_i, T^{n+1} z_i)d(T^n z_{i+1}, T^{n+1} z_{i+1})}{d(T^n z_i, y)}, \frac{d(T^n z_i, T^{n+1} z_i)d(T^n z_{i+1}, T^{n+1} z_{i+1})}{d(T^n z_i, T^n z_{i+1}) + d(T^n z_i, T^{n+1} z_{i+1}) + d(T^n z_{i+1}, T^{n+1} z_i)} \\ \frac{d(T^n z_i, T^{n+1} z_i)d(T^n z_i, T^{n+1} z_{i+1}) + d(T^n z_{i+1}, T^{n+1} z_i)d(T^n z_{i+1}, T^{n+1} z_{i+1})}{d(T^n z_{i+1}, T^{n+1} z_i) + d(T^n z_i, T^{n+1} z_{i+1})} \end{array}\right\}$$

$$d(x,y) \leq \lambda\big(M(T^n z_i, T^n z_{i+1})\big) \to 0 \quad (3.19)$$

as $n \to \infty$.

Thus, $x = y$.

Hence T has a unique fixed point.

If R is complete or X is R^s-directed, then, we have the following Corollary.

Corollary 3.3 Theorem 3.1 remains true if we replace condition (e) by one of the following conditions and retaining the rest of the hypotheses:

(f) R is complete,

(g) X is R^s-directed

ProofSuppose R is complete, then for each $x, y \in X$, $[x, y] \in R$, then we have that $\{x, y\}$ is a path of length 1 in R^s from x to y so that $Y(x, y, R^s)$ is nonempty. By Theorem 3.1 we can gives the conclusion. Otherwise, X is R^s-directed, then for each $x, y \in X$, there exists $z \in X$ such that $[x, z] \in R$ and $[y, z] \in R$ so that $\{x, z, y\}$ is a path of length 2 in R^s from x to y. Hence, $Y(x, y, R^s)$ is nonempty, for each $x, y \in X$ and in light of Theorem 3.1 the conclusion is immediate.

Now, we consider the following examples in support of Theorem 3.1 and Theorem 3.2.

Example 3.4 Consider $X = R$ with the usual metric $d = |x - y|$, then (X, d) is a complete metric space. Define a binary relation

$$R = \{(x, y) \in R^2 : x - y \geq 0, x \in \mathbb{Q}\}$$

on X. Let $T: X \to X$ be a mapping defined by

$$T(x) = 4 + \frac{1}{3}x.$$

Observe that R is T-closed and T is continuous. Now, for $x, y \in X$ with (x, y) ∈ R, we have

$$d(Tx,Ty) = \left|\left(4+\frac{1}{3}x\right)-\left(4+\frac{1}{3}y\right)\right| = \frac{1}{3}|x-y| = \frac{1}{3}d(x,y) < \frac{2}{5}d(x,y),$$

i.e., T satisfies assumption (d) of Theorem 3.1 for$\lambda = \frac{2}{5}$. Thus, all the conditions (a)-(d) of Theorem 3.1 are satisfied and T has a fixed point in X. Moreover, here assumption (e) of Theorem 3.2 also holds and therefore, T has a unique fixed point ($x = 6$).

Example 3.5 Let$X = [0,2]$equipped with usual metric $d = |x-y|$ so that (X,d) is a complete metric space. Define a binary relation

$$R = \{(0,0),(0,1),(1,0),(1,1),(0,2)\}$$

on X and $T: X \to X$be a mapping defined by

$$T(x) = \begin{cases} 0, if\ 0 \le x \le 1, \\ 1,\ \ if\ 1 < x \le 2. \end{cases}$$

Clearly, R is T-closed but T is not continuous. Take an R-preserving sequence $\{x_n\}$ such that$x_n \xrightarrow{d} x$ so that $(x_n, x_{n+1}) \in R$for all $n \in \mathbb{N}_0$. Observe that$(x_n, x_{n+1}) \notin \{(0,2)\}$ so that$(x_n, x_{n+1}) \in \{(0,0),(0,1),(1,0),(1,1)\}\ \forall n \in \mathbb{N}_0$, which gives rise to$\{x_n\} \subset \{(0,1)\}$. Since $\{(0,1)\}$ is closed, we have$[x_n, x] \in R$. Therefore, R is d-self-closed. By a routine calculation, one can verify assumption (d) of Theorem 3.1 with$\lambda = \frac{1}{2}$. Thus, all the conditions (a)-(d) of Theorem 3.1 are satisfied and T has a fixed point in $X(x = 0)$.

4 Applications

In this section, we consider some special cases, wherein our result deduces several well-known fixed point theorems of the existing literature.

4.1 Fixedpoint theorems in ordered metric spaces via comparable mappings.

We first considerthe type of results involving comparable mappings which are contained inTurinici [1], Nieto and RodriguezLopez [11]and Alam and Imdad [19].

Definition 4.1 Let the pair$(X, \preccurlyeq)$, stands for a nonempty set X equippedwith a partial order $\preccurlyeq$often called an ordered set wherein we generally write $x \succcurlyeq y$instead of$y \preccurlyeq x$. Two elements x and y in an ordered set $(X, \preccurlyeq)$ are said tobe comparable if either$x \preccurlyeq y$ or$y \preccurlyeq x$ and denote it as$x \prec\succ y$. A subset $E$of an ordered set is called totally ordered if $x \prec\succ y$ for all$x, y \in E$.

Nieto andRodrıguez-Lopez [11] replaced this condition by preservation of comparable elementsand proved the following Theorem as follows:

Theorem 4.2 (Theorem 7, Nieto and Rodrıguez-Lopez [11]). Let $(X, d, \preccurlyeq)$bean ordered metric space and T a self-mapping on X. Suppose that the followingconditions hold:

(a) (X, d) is complete,

(b) for $x, y \in X$∈X with $x \preccurlyeq y \Rightarrow T(x) \preccurlyeq T(y)\ or\ T(x) \succcurlyeq T(y)$,

(c) either T is continuous or $(X, d, \preccurlyeq)$ satisfies the following property:

if $\{x_n\}$is a sequence in X such that$x_n \xrightarrow{d} x$ whose consecutive terms arecomparable, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that everyterm is comparable to the limit x,

(d) there exists$x_0 \in X$ such that$x_0 \prec\succ T(x_0)$,

(e) there exists $\lambda \in [0,1)$ such that

$$d(Tx,Ty) \le \lambda d(x,y), \forall x,y \in X \text{ with } x \prec\succ y, \qquad (4.1)$$

(f) for every pair$x, y \in X$ there exists$z \in X$ which is comparable to x and y.

Then T has a unique fixed point x. Moreover, for every$x \in X, \lim_{n\to\infty} T^n(x) = x$.

Theorem 4.3 Let $(X, d, \preccurlyeq)$bean ordered metric space and T a self-mapping on X. Suppose that the followingconditions hold:

(a) (X, d) is complete,

(b) for $x, y \in X$∈X with $x \preccurlyeq y \Rightarrow T(x) \preccurlyeq T(y)$ or $T(x) \succcurlyeq T(y)$,

(c) either T is continuous or $(X, d, \preccurlyeq)$ satisfies the following property:

if $\{x_n\}$is a sequence in X such that$x_n \xrightarrow{d} x$ whose consecutive terms arecomparable, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that everyterm is comparable to the limit x,

(d) there exists$x_0 \in X$ such that$x_0 \prec\succ T(x_0)$,

(e) there exists $\lambda \in [0,1)$ such that

$d(Tx, Ty) \leq \lambda\big(M(x, y)\big), \forall x, y \in X$ with $(x, y) \in R$,

where

$$M(x,y) = max\left\{\begin{matrix} d(x,y), \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)}, \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \\ \dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \end{matrix}\right\} \quad (4.2)$$

(f) for every pair$x, y \in X$ there exists$z \in X$ which is comparable to x and y.

Then T has a unique fixed point x. Moreover, for every$x \in X, \lim_{n\to\infty} T^n(x) = x$.

Turinici [22, 24] proved similar results besides observing that these results are particular cases of Banach Contraction Principle [25]and it an important generalization due to Maia [23]. Following Turinici [22,24],given $x, y \in X$, any subset$\{z_1, z_2, \ldots, z_k\}$ (for $k \geq 2$) in X with $z_1 = x$, $z_k = y$ and$z_i \prec\succ z_{i+1}$ for each $i(1 \leq i \leq k - 1)$is called a $\prec\succ$-chain between x and y. Theclass of such chains is denoted by$C(x, y, \prec\succ)$.

Theorem 4.4 (Theorem 2.1, Turinici [22]). Let $(X, d, \preccurlyeq)$ be an ordered metric spaceand T a self-mapping on X. Suppose that the following conditions hold:

(a) (X, d) is complete,

(b) for $x, y \in X$∈X with $x \preccurlyeq y \Rightarrow T(x) \preccurlyeq T(x)$,

(c) T is continuous,

(d) there exists$x_0 \in X$ such that$x_0 \prec\succ T(x_0)$,

(e) there exists $\lambda \in [0,1)$ such that

$d(Tx, Ty) \leq \lambda\big(M(x, y)\big), \forall x, y \in X$ with $(x, y) \in R$,

where

$$M(x,y) = max\left\{\begin{matrix} d(x,y), \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)}, \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \\ \dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \end{matrix}\right\} \quad (4.3)$$

(f) $C(x, y, \prec\succ)$ is nonempty for each $x, y \in X$.

Then T has a unique fixed point z. Moreover, for each $x \in X$, the sequence$\{T^n x\}$isconvergent and $\lim_{n\to\infty} T^n(x) = z$.

Theorem 4.5 Let $(X, d, \preccurlyeq)$ be an ordered metric spaceand T a self-mapping on X. Suppose that the following conditions hold:

(a) (X, d) is complete,

(b) for $x, y \in X$∈X with $x \preccurlyeq y \Rightarrow T(x) \preccurlyeq T(x)$,

(c) T is continuous,

(d) there exists$x_0 \in X$ such that$x_0 \prec\succ T(x_0)$,

(e) there exists $\lambda \in [0,1)$ such that

$d(Tx, Ty) \leq \lambda d(x, y), \forall x, y \in X$ with $x \prec\succ y$, (4.4)

(f) $C(x, y, \prec\succ)$ is nonempty for each $x, y \in X$.

Then T has a unique fixed point z. Moreover, for each $x \in X$, the sequence$\{T^n x\}$is

convergent and $\lim_{n\to\infty} T^n(x) = z$.

Theorem 4.6 (Theorem 2.1, Turinici [24]) Let $(X, d, \preccurlyeq)$ be an ordered metric spaceand T a self-mapping on X. Suppose that the following conditions hold:

(a) (X, d) is complete,

(b) for $x, y \in X$∈X with $x \preccurlyeq y \Rightarrow T(x) \preccurlyeq T(x)$,

(c) $(X, d, \preccurlyeq)$ satisfies the following property:

if $\{x_n\}$is a sequence in X such that$x_n \xrightarrow{d} x$ whose consecutive terms arecomparable, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that everyterm is comparable to the limit x,

(d) there exists$x_0 \in X$ such that$x_0 \prec\succ T(x_0)$,

(e) there exists $\lambda \in [0,1)$ such that

$d(Tx, Ty) \leq \lambda d(x, y), \forall x, y \in X$ with $x \prec\succ y$, (4.5)

(f) $C(x, y, \prec\succ)$ is nonempty for each $x, y \in X$.

Then T has a unique fixed point z. Moreover, for each $x \in X$, the sequence$\{T^n x\}$isconvergent and $\lim_{n\to\infty} T^n(x) = z$.

Theorem 4.7 Let $(X, d, \preccurlyeq)$ be an ordered metric spaceand T a self-mapping on X. Suppose that the following conditions hold:

(a) (X, d) is complete,

(b) for $x, y \in X$∈X with $x \preccurlyeq y \Rightarrow T(x) \preccurlyeq T(x)$,

(c) $(X, d, \preccurlyeq)$ satisfies the following property:

if $\{x_n\}$is a sequence in X such that$x_n \xrightarrow{d} x$ whose consecutive terms arecomparable, then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that everyterm is comparable to the limit x,

(d) there exists$x_0 \in X$ such that$x_0 \prec\succ T(x_0)$,

(e) there exists $\lambda \in [0,1)$ such that

$d(Tx,Ty) \leq \lambda(M(x,y)), \forall x,y \in X$ with $(x,y) \in R$,

where

$$M(x,y) = max\left\{\begin{matrix} d(x,y), \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)}, \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \\ \dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \end{matrix}\right\} \quad (4.6)$$

(f) $C(x,y,\prec\succ)$ is nonempty for each $x,y \in X$.

Then T has a unique fixed point z. Moreover, for each $x \in X$, the sequence$\{T^n x\}$isconvergent and $\lim_{n\to\infty} T^n(x) = z$.

4.2 Fixed point theorems under symmetric closure of a binary relation.

Now we consider the type of results involving symmetric closure of a binary relationwhich are contained inSamet and Turinici [17] which is also pursued in Berzig [18]. In this context, R stands for an arbitrary binary relation on a nonempty set X and$S := R^s$.

Definition 4.8 [17]. Let Tbe a self-mapping on X. We say that$T$ is comparative if for any $x,y \in X$,

$(x,y) \in S \Rightarrow (Tx,Ty) \in S.$ $\qquad (4.7)$

Remark 4.9 It is clear that T is comparative iff S is$T$-closed.

Definition 4.10 [21]. (X,d,S) is regular if the following condition holds: if the sequence $\{x_n\}$ in X and the point$x \in X$ are such that

$(x_n, x_{n+1}) \in S$ for all n and$\lim_{n\to\infty}(x_{n_k}, x) = 0,$ $\qquad (4.8)$

then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that$(x_{n_k}, x) \in S$ for all k.

Remark 4.11 Clearly, $(X,d,S)$is regular iff$S$ is d-self-closed.

Corollary 4.12 [18] Let(X,d) be a metric space, R a binary relation on X and Ta self-mapping on X. Suppose that the following conditions hold:

(a)$T(X) \subseteq X$,

(b) T is comparative,

(c) there exists$x_0 \in X$ such that$(x_0, x_0) \in S$,

(d) there exists $\lambda \in [0,1)$ such that

$d(Tx,Ty) \leq \lambda d(x,y), \forall x,y \in X$ with $(x,y) \in S,$ $\qquad (4.9)$

(e) (X,d) is complete and$X$ is closed,

(f) (X,d,S) is regular.

Then T has a fixed point.

Corollary 4.13 Let(X,d) be a metric space, R a binary relation on X and T a self-mapping on X. Suppose that the following conditions hold:

(a)$T(X) \subseteq X$,

(b) T is comparative,

(c) there exists$x_0 \in X$ such that$(x_0, x_0) \in S$,

(d) there exists $\lambda \in [0,1)$ such that

$d(Tx,Ty) \leq \lambda(M(x,y)), \forall x,y \in X$ with $(x,y) \in R$,

where

$$M(x,y) = max\left\{\begin{matrix} d(x,y), \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)}, \dfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \\ \dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \end{matrix}\right\} (4.10)$$

(e) (X,d) is complete and$X$ is closed,

(f) (X,d,S) is regular.

Then T has a fixed point.

5Conclusion

In this article, we present a generalization of the classical Banach contraction principle for complete metric spaces with a binary relation, where the contraction condition is weaker than usual. The contraction condition is required to hold only on those elements that are related by the underlying relation, rather than the entire space. We apply our results to prove the existence and uniqueness of solutions to a certain class of nonlinear contractions in a complete metric space. Our results extend and build upon existing work in the literature, and are supported by illustrative examples and applications that demonstrate their validity of the results.

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