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Local Stability and Chaotic Dynamics of Discrete Time Population Model With and Without Allee Effect

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Abstract

In addressing the shortcomings identified in the work of Gumus, this paper focuses on rectifying the mathematical concepts employed in this paper. Furthermore, an extension of the paper's core idea is undertaken for the nth generation of discrete time population models, incorporating considerations with and without Allee effect.

The correction of mathematical concepts is executed with precision, and the extension to nth generation models is rigorously explored. Analytical techniques are deployed to assess the local stability around fixed points in each generation, with a specific emphasis on the impact of Allee effect on stability regions. In the 2nd generation, it is empirically observed that the presence of Allee effect significantly diminishes the region of local stability surrounding the equilibrium point. Notably, this reduction is more pronounced compared to the generation without Allee effect. For the 3rd generation, a parallel trend is identified, revealing a shrinking stability region and the subsequent emergence of chaos beyond this region. Results for the nth generation echo those of the 2nd generation, both in terms of stability and instability regions. The identified trends in stability and chaos provide valuable information for policymakers and researchers involved in population modeling, aiding in the development of more accurate and predictive models for ecological and demographic studies. The analytical results are further validated through numerical simulations, enhancing the credibility of the findings and facilitating a comprehensive comparison between the 1st and 2nd generations.

Keywords: Population model, Allee effect, Equilibrium point, Period of n -cycle, discrete time population model.

1. Introduction

In ecological study the conservation and restoration of ecosystem is of great concern with the increasing biodiversity loss and extinction of species in population. Conservation of species of plants or animals are affected by excessive use of plant or animal-based products [1,2,3]. Population has been defined as a group of individuals that belong to same species and can interbreed within an area. In population study the most important parameter is the number of individuals and the population size. A clear view of population size is the population density, which is defined as number of individuals per unit area or volume. The population density is dynamic in nature which changes depending upon natality, mortality, emigration, and immigration [4]. Age structure, fecundity and clutch size are important factors that impact the density [4,5]. Thus, in any population these factors have direct impact on population density while other factors like flood, weather, forest fire, etc have indirect impact on population density. The population size can be large or small depending on various factors that control size.

Whether populations are small or large they are governed by factors like predation, competition, mortality, natality, immigration, emigration, food resources, fecundity, etc. On the basis of size, the populations show peculiar behaviour. There is an increased risk of extinction in case of small population size [6].
The individuals in small populations have difficulty in finding suitable partners and there also rises the issue of inbreeding [7]. The study of the factors that affect the structure and size of population over time is termed as population dynamics [8]. Thus, in ecology the population dynamics is one of the fundamental areas of study. The basics of population dynamics is the better understanding of importance of competition for resources and predation. Begon has explained population dynamics as that part in ecology which deals with the variation in population density and size among species [9]. To study the population dynamics, population models are worked out. Models are made to have more clear insight of any system. Maltby has described population models as conceptual models that associate responses (individual level) to effects in population structure and density [10].

An intriguing phenomenon within population dynamics is the Allee effect, named after the seminal work of Wander Clyde Allee [11]. It is observed that, beyond competition, reduced crowding can also curtail or limit populations in numerous species, constituting a classic phenomenon in population ecology studies [12-18]. A profound understanding of the Allee effect assumes significance in the realms of population dynamics, ecology, and environmental management. Mathematical models are harnessed to explicate the Allee effect in population studies.

To unravel the intricacies of population dynamics, stability analysis becomes imperative. A model, formulated through difference equations, enables the identification of points where the system remains unaltered, and stability in the proximity of these points is subsequently explored [12,13, 19-29]. Local asymptotic analysis is pursued for solutions converging to points where the system exhibits stability [30]. This paper endeavours to scrutinize the stability and chaos of stationary points in the context of extended work by Gumus [1,2] to nth generations, demonstrating that the dynamics of third-generation populations may exhibit chaos beyond the established stability region.

The structure of this paper is compartmentalized into five sections. The introductory section lays the foundation, followed by the material and methods section elucidating the population model and fundamental terminologies. Section three substantiates the investigation through the proof of theorems concerning stability. The fourth section employs graphical representations, employing suitable functions to validate the analytical results. The conclusive section synthesizes the findings, drawing overarching conclusions from this scientific inquiry.

2. Material and methods

2.1 Population Model

A population model [25] which is linear and depending upon birth and death rates are introduced as

$$D_{t+1} = rD_t$$

Where, D_t represents the population density at time t and $r\,(r = 1 + b - d)$ be the growth rate. This model neglects many important aspects of biological reality. Since most of the parameters connected to the interactions among individuals are overlooked. Thus, a non-linear population model offers a more realistic approach than linear population models.

In this paper, we consider the following non-linear discrete time population model-

$$D_{t+1} = rD_t + f(D_t) = g(r, D_t) \qquad (1)$$

Where $f(D_t)$ interactions among mature individuals, and r is growth rate such that $r \in (0,1)$.

The following assumptions are imposed on the function f :

(i) $f\,'(D_t) < 0$ for $D_t \in [0, \infty)$: i.e. the function f decreases continuously when the population density increases.

(ii) when $D_t = 0$, $f(D_t)$ will give finite positive value.

If the population model (1) is subject to the Allee effect, the following Non-linear population model is used-

$$D_{t+1} = r^* h\,(D_t)\,D_t + f(D_t) = g_h(r^*, h, D_t) \qquad (2)$$

Where f satisfies the conditions (i) and (ii). Note that, since $r^* > 0$ is related to the normalized per capita growth rate given by $r^* = \dfrac{r}{h(D^*)}$, also (1) and (2) have the same equilibrium point. Here $h = h(D_t)$ is the Allee function at time t.

The following assumptions on the Allee function are-

(iii) Reproduction will not happen in the absence of parameters. It means Allee function will be a constant function with the constant value is zero when the population density is zero i.e. $h(D_t) = 0$ if $D_t = 0$

(iv) Population density is directly proportional to the Allee function. It means when one will increase another will also increases i.e. $h'(D_t) > 0 \;\; \forall D_t > 0$

(v) After certain value of population density, the Allee function will approaches to one. It means at high densities function will disappear [1,2].

2.2 Basic Terminology:

Definition-2.2.1: A point D^* is said to be equilibrium point of non-linear discrete time population model (1) if it satisfies the condition- $g(D^*) = D^*$

Definition2.2.2: A point D^* is said to be periodic point of period n if

$g^n(D^*) = D^*$ and $g^i(D^*) \neq D^*$ for $i = 1, 2, 3.......n-1$

Theorem-2.2.1 Let D^* is an equilibrium point of (1). Also let g' be a continuous function on open interval J containing D^* as an equilibrium point. Then D^* is locally asymptotically stable if

$\left|g'(D^*)\right| < 1$ and unstable if, $\left|g'(D^*)\right| > 1$

Theorem-2.2.2(a) Let D^* be the equilibrium point with respect to r. Then D^* is locally asymptotically stable if the inequality

$f'(D^*) > -1 - r$ holds. (see Theorem 4 Gumus A.K. [1])

Theorem-2.2.2(b) Let D^* be an equilibrium point of (1) with respect to r. Then D^* is locally asymptotically unstable if the inequality-

$-\infty < g'(D^*) < -1 - r$ holds.

Theorem-2.2.2(c) Let D^* be the equilibrium point of (1) with respect to r. Then D^* may be locally stable or unstable if the equality hold-

$$\left|f'(D^*)\right| = 1$$

$$\Rightarrow r + f'(D^*) = \pm 1$$

$$\Rightarrow f'(D^*) = \pm 1 - r$$

But

$$f'(D^*) < 0 \therefore f'(D^*) \neq 1 - r \ (\because 1 - r > 0)$$
$$\therefore f'(D^*) = -1 - r$$

This case will be doubtful. In this situation we can discuss about its stability by cobweb-diagram (see [26]).

Theorem-2.2.3(a) Suppose g' is continuous on an open interval I and the n-cycle $\left\{x_1, f(x_1), f^2(x_1), \ldots\ldots\ldots f^{n-1}(x_1)\right\}$ of the difference equation (1) is contained in I then the n-cycle is locally asymptotically stable if

$$\left|\frac{d}{dx} g^n(x_k)\right| < 1 \quad \text{for some k}$$

And unstable if,

$$\left|\frac{d}{dx} g^n(x_k)\right| > 1 \quad \text{for some k}$$

The above result can also be written as-

Theorem-2.2.3(b) Suppose $\left\{x_1, x_2, x_3 \ldots\ldots x_n\right\}$ is an n-cycle of the difference equation (1), then n-cycle is locally asymptotically stable if

$$\left|g'(x_1).g'(x_2).\cdots g'(x_n)\right| < 1$$

And unstable if

$\left|g'(x_1).g'(x_2).g'(x_n)\right| > 1$ (See [26,27,30]).

3. Results

3.1 Local stability of an equilibrium points without Allee effect for 2-cycle, 3-cycle to n-cycle

Theorem-3.1(a) Let $\{D_1, D_2\}$ be 2- cycle of the difference equation (1) with respect to r then 2-cycle is locally asymptotically stable if the inequality -

$0 < f'(D_1).\, f'(D_2) < 1 - r^2 - r\,k_1$ hold.

where $k_1 = f'(D_1) + f'(D_2)$

Proof: From assumption (i)-(ii) of our model it is clear that g' is continuous function.

The population model (1) can be linearized which can be written as-

$V_{t+1} = g'(D_t^*)V_t$ where $g'(D_t^*) = r + f'(D_t^*)$ and $V_t = D_t - D^*$

By result 3(b), as the 2-cycle will be locally asymptotically stable if-

$$\left| g'(D_1).\, g'(D_2) \right| < 1$$

$$\Rightarrow \left| (r + f'(D_1)).(r + f'(D_2) \right| < 1$$

$$\Rightarrow \left| r^2 + r(f'(D_1) + f'(D_2)) + f'(D_1).\, f'(D_2) \right| < 1$$

Let, $k_1 = f'(D_1) + f'(D_2)$

$$-1 - r^2 - rk_1 < f'(D_1).\, f'(D_2) < 1 - r^2 - rk_1$$

$$\Rightarrow f'(D_1).\, f'(D_2) \in (-1 - r^2 - rk_1, 1 - r^2 - rk_1) \quad (3)$$

$$\because f'(D_1) < 0 \ \& \ f'(D_2) < 0$$

$$\Rightarrow f'(D_1) + f'(D_2) < 0$$

$$\Rightarrow f'(D_1).\, f'(D_2) \in (0, \infty) \quad (4)$$

From equation (3) and (4), we get

$$f'(D_1).\, f'(D_2) \in (-1 - r^2 - rk_1, 1 - r^2 - rk_1)$$
$$\& \ f'(D_1).\, f'(D_2) \in (0, \infty)$$

$$\Rightarrow f'(D_1).\, f'(D_2) \in (0, 1 - r^2 - rk_1)$$
$$i.e. \ 0 < f'(D_1).\, f'(D_2) < (1 - r^2 - rk_1)$$

which is the required inequality.

Theorem-3.1(b): Let $\{D_1, D_2\}$ be 2-cycle of the difference equation (1) with respect to r. Then 2-cycle is unstable if the inequality-

$$(1-r^2-rk_1) < f'(D_1).f'(D_2) < \infty \text{ holds.}$$

Theorem- 3.2(a): Let $\{D_1, D_2, D_3\}$ be the three-cycle of the difference equation (1) with respect to r then the 3-cycle is locally asymptotically stable if the inequality holds-

$$(-1-r^3-r^2k_1-rk_2) < f'(D_1).f'(D_2).f'(D_3) < 0$$

Theorem-3.2(b): Let $\{D_1, D_2, D_3\}$ be the three cycle of the difference equation (1) with respect to r then the 3-cycle is chaotic if the inequality holds-

$$-\infty < f'(D_1).f'(D_2).f'(D_3) < (-1-r^3-r^2k_1-rk_2)$$

Proof: By the above results, the three cycle $\{D_1, D_2, D_3\}$ will be unstable if $|g'(D_1).g'(D_2).g'(D_3)| > 1$

Also, we know that period three implies chaos [28,29]. On combining these two results, we can say that the three cycle $\{D_1, D_2, D_3\}$ will be chaotic if $|g'(D_1).g'(D_2).g'(D_3)| > 1$

Solving this inequality as above we get

$$-\infty < f'(D_1).f'(D_2).f'(D_3) < (-1-r^3-r^2k_1-rk_2)$$

which is required inequality.

Theorem-3.3(a): Let $\{D_1, D_2, D_3 .... D_n\}$ be n-cycle of the difference eq (1) with respect to r then n-cycle is locally asymptotically stable if the inequality

Case 1. When n is odd.

$$-1-r^n-r^{n-1}k_1-r^{n-2}k_2-\ldots\ldots-rk_{n-1} < f'(D_1).f'(D_2)\ldots\ldots f'(D_n) < 0$$

Case 2. When n is even.

$$-\infty < f'(D_1).f'(D_2).f'(D_3)\ldots\ldots f'(D_n) < -1-r^{n-1}-r^{n-2}k_1\ldots\ldots-rk_{n-1}$$

Proof: Let us suppose that

$$k_1 = f'(D_1)+f'(D_2)+f'(D_3)+\ldots\ldots+f'(D_n)$$

$$k_2 = f'(D_1).f'(D_2)+f'(D_2).f'(D_3)+f'(D_{n-1}).f'(D_n)$$
$$k_3 = f'(D_1).f'(D_2).f'(D_3)+\ldots\ldots+f'(D_{n-2}).f'(D_{n-1}).f'(D_n)$$
$$\ldots\ldots\ldots$$
$$k_{n-1} = f'(D_1).f'(D_2)\ldots\ldots f'(D_{n-1})+f'(D_1).f'(D_2)\ldots\ldots f'(D_{n-2}).f(D_n)+\ldots\ldots$$
$$\ldots\ldots+f'(D_2)\,f'(D_3)\ldots\ldots f'(D_n)$$

By theorem-3(b) n-cycle will be locally asymptotically stable if-

$$\left|g'(D_1).g'(D_2).g'(D_3)........................g'(D_n)\right|<1$$

$$\Rightarrow\left|(r+f'D_1)).(r+f'(D_2)).(r+f'(D_3)).(r+f'(D_4))\ldots\ldots(r+f'(D_n))\right|<1$$

$$\Rightarrow\left|\begin{array}{l} r^n+r^{n-1}\left(f'(D_1)+f'(D_2)+\ldots\ldots+f'(D_n)\right) \\ +r^{n-2}\left(f'(D_1).f'(D_2)+f'(D_2).f'(D_3)+\ldots\ldots+f'(D_{n-1}).f'(D_n))\ \right)+ \\ r^{n-3}(f'(D_1).f'(D_2).f'(D_3)+\ldots.....+f'(D_{n-2}).f'(D_{n-1}).f'(D_n))+..... \\ \ldots\ldots\ \ldots+r\left(f'(D_1).f'(D_2)\ldots f'(D_{n-1})+..........+f'(D_1)f'(D_2)\ldots f'(D_{n-2})f'(D_n)\right) \\ ....+(f'(D_1).f'(D_2)\ldots.....f'(D_n)) \end{array}\right|<1$$

So,

$$\left|r^n+r^{n-1}k_1++r^{n-2}k_2+\ldots\ldots+r^{n-(n-1)}k_{n-1}+f'(D_1).f'(D_2)\ldots....f'(D_n)\right|<1$$

$$\Rightarrow-1<r^n+r^{n-1}k_1++r^{n-2}k_2+\ldots\ldots+rk_{n-1}+f'(D_1).f'(D_2)\ldots.....f'(D_n)<1 \qquad (5)$$

Here, two cases are arising -

Case 1. When n is odd.

In this case, $k_1<0,k_2>0,k_3<0,\ldots k_{n-1}>0$

And $f'(D_1)f'(D_2)\ldots f'(D_n)<0$ (6)

So, from (5) and (6), we get

$$\Rightarrow-1-r^n-r^{n-1}k_1-\ r^{n-2}k_2-\ldots\ldots-rk_{n-1}<f'(D_1).f'(D_2)....\ldots f'(D_n)<0$$

This is the required inequality for asymptotically stable for n-cycle when n is odd.

Case 2. When n is even.

In this case, $k_1<0,k_2>0,k_3<0,k_4>0,\ldots k_{n-1}>0$

And $f'(D_1)f'(D_2)\ldots f'(D_n)>0$ (7)

So, from (5) and (7), we get

$$\Rightarrow\ 0<f'(D_1).f'(D_2)....\ldots f'(D_n)<1-r^n-r^{n-1}k_1-\ r^{n-2}k_2-\ldots\ldots-rk_{n-1}$$

This is the required inequality for asymptotically stable for n-cycle when n is even.

Theorem 3.3(b): Let $\{D_1,D_2,D_3........D_n\}$ be n-cycle of the difference eq (1) with respect to r .Then the n-cycle is unstable if the inequality.

Case 1. When n is even

$$1-r^n-r^{n-1}k_1-\ r^{n-2}k_2-\ldots\ldots-rk_{n-1}<f'(D_1).f'(D_2)....\ldots f'(D_n)<\infty$$

Case 2. When n is odd.

$$-\infty < f'(D_1).f'(D_2).f'(D_3) \ \ldots\ldots f'(D_n) < \ -1 - r^{n-1} - r^{n-2}k_1 \ldots\ldots - rk_{n-1})$$

(Also at n=3 it will be chaotic.)

3.4: Effect of Allee Function

Theorem 7 is taken from A. K. Gumus [1] theorem no. 5. Here the concept of this theorem is corrected. In this theorem the double derivative of $f(x)$ is taken always positive which is not true in general.

Theorem-3.4(a): Let $D*$ be the equilibrium point of difference eq (2) by the Allee effect at time with respect to $r*$. Then $D*$ is locally asymptotically stable if the inequality

$$-r - rD^* \frac{h'(D*)}{h(D*)} < f'(D*) < 1 - r - rD^* \frac{h'(D*)}{h(D*)}$$ and unstable otherwise. .

Proof: From assumption (i) -(v) of our model, it is clear that $\mathrm{g}_h{}'$ is continuous function the population model (2) can be linearized in neighborhood of $D*$ is given by

$V_{t+1} = g_h{}'(D_t*)V_t$ where $V_t = D_t - D*$

By theorem (1) we get $D*$ is locally asymptotically stable if

$$|g'(D*)| < 1$$

$$\Rightarrow \left| r*\ h'(D*)D* + r*h(D*) + f'(D*) \right| < 1$$

$$\Rightarrow\ -1 < \frac{r}{h(D*)} h'(D*)D* + \ \frac{r}{h(D*)} h(D*) + f'(D*) < 1$$

$$\Rightarrow\ -1 < r\frac{h'(D*)}{h(D*)} D* + \ r + f'(D*) < 1$$

$$\Rightarrow\ -1 - r < r\frac{h'(D*)}{h(D*)} D* + f'(D*) < 1 - r \qquad (8)$$

Here $r = r*h'(D*)$ & $h(D*) < 1$

We know that,

$$f''(x) = \lim_{h \to o} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

By our assumption, we have f decreases

So, $(x+h) < (x+2h) \Rightarrow f(x+h) > f(x+2h) \Rightarrow f(x+2h) - f(x+h) < 0$

Also, $(x+h) > x \Rightarrow f(x) > f(x+h) \Rightarrow f(x) - f(x+h) > 0$

Let, us consider

$$f\left(x+2h\right)-f\left(x+h\right)=k_1\ \left(k_1<0\right)$$
$$f\left(x\right)-f\left(x+h\right)=k_2\ (k_2>0)$$

Now,

$$f''\left(x\right)=\lim_{h\to o}\frac{[f\left(x+2h\right)-f\left(x+h\right)]+[f\left(x\right)-f\left(x+h\right)]}{h^2}$$

In this paper, $f''\left(x\right)$ considered as always positive which is not true in general. Which may be positive, negative or zero.

For example: If we consider

$$f(x)=r-x\ ,\ r\in(0,1) \qquad f'(x)=-1 \qquad f''(x)=0$$

$$f(x)=r-x^2\ ,\ r\in(0,1) \quad ,\ f'(x)=-2x \qquad f''(x)=-2<0$$

and

$$f(x)=\frac{1}{x+1}\ ,\ r\in(0,1)\ \&\ x\in(0,\infty)$$
$$f'(x)=-\frac{1}{(x+1)^2}$$
$$f''(x)=\frac{2}{(x+1)^3}>0\ \forall\ x\in(0,\infty)$$

which confirms that the double derivative of $f\left(x\right)$ may be positive, negative or zero.

Case1- When $k_1+k_2>0$

then in this case $f''\left(x\right)>0$

Now let us consider the first and the 2nd derivative of the function g_h as follow-

$$\Rightarrow g_h=r^*h(D^*)D^*+f\left(D^*\right)$$
$$\therefore\ g'_h=r^*\ h'(D^*)D^*+r^*h(D^*)+f'\left(D^*\right)$$
$$\Rightarrow\ g''_h=r^*\ h''(D^*)D^*+2r^*h'(D^*)+f''\left(D^*\right)$$

such that $\left|h''\left(D^*\right)\right|<h'\left(D^*\right)$ in the case of $h''\left(D^*\right)<0$.

Consider, if $f''\left(x\right)>0$ so $f''\left(D^*\right)>0,\ h'\left(D^*\right)>0\ ,\ r^*>0$

Therefore, $g''_h>0$, so g_h is convex function.

So, g_h' is increasing.

Also, $g_h\to\infty\ as\, D_t\to\infty$

$$\Rightarrow g_h=r^*\ h(D^*)D^*+f\left(D^*\right)$$
$$\therefore\ g'_h>0\Rightarrow r^*\ h'(D^*)D^*+r^*h(D^*)+f'\left(D^*\right)>0$$
$$\Rightarrow r\,D^*\frac{h'(D^*)}{h(D^*)}+f'\left(D^*\right)+r>0$$

$$\Rightarrow r\,D^*\frac{h'(D^*)}{h(D^*)}+f'(D^*)>-r \tag{9}$$

From (8) and (9), we get

$$-r-rD^*\frac{h'(D^*)}{h(D^*)}<f'(D^*)<1-r-rD^*\frac{h'(D^*)}{h(D^*)}\ .$$ which is required inequality

Case2- When $k_1+k_2=0$

then in this case, $f''=0$ so f' is constant. This implies f' may be positive or negative but in our case f' is negative. So $f'>0$ is not possible.

If we take $r^*\ h''(D^*)D^*+2r^*h'(D^*)>0$ then $g_h''(x)>0$
Now we can solve as case1. We get the required result.

Case3- When $k_1+k_2<0$
then in this case $f''(x)<0$.
If we take $r^*\ h''(D^*)D^*+2r^*h'(D^*)+f''(D^*)>0$ then $g_h''(x)>0$
Now we can solve as case1. We get the required result.

Theorem 3.4(b). Let $\{D_1,D_2\}$ is 2-cycle of the difference eq (2) with respect to r^* then 2-cycle is locally stable if the inequality

$$-r^2-r^2\left(P_1P_2+P_1+P_2\right)-r\left(P_1f'(D_1)+P_2f'(D_2)+f'(D_1)+f'(D_2)\right)<$$
$$f'(D_1).f'(D_2)<1-r^2-r^2\left(P_1P_2+P_1+P_2\right)-r\left(P_1f'(D_1)+P_2f'(D_2)+f'(D_1)+f'(D_2)\right)$$

and unstable otherwise

where $P_1=\frac{h'(D_1)}{h(D_1)}D_1\ \&\ P_2=\frac{h'(D_2)}{h(D_2)}D_2$

Proof: From (i) to (v) we can say that g_h' is continuous function. The linearized form of (2) in the neighborhood of D^* is given by $V_{t+1}=g_h'(D^*)V_t$, where $V_t=D_t-D^*$

We know that, the 2-cycle will be locally asymptotically stable if

$$|g_h'(D_1).g_h'(D_2)|<1$$

$$\Rightarrow\left|(r^*h'(D_1)D_1+r^*h(D_1)+f'(D_1))(r^*h'(D_2)D_2+r^*h(D_2)+f'(D_2))\right|<1$$

$$\Rightarrow\left|\left(\frac{r}{h(D_1)}h'(D_1)D_1+\frac{r}{h(D_1)}h(D_1)+f'(D_1)\right)\left(\frac{r}{h(D_2)}h'(D_2)D_2+\frac{r}{h(D_2)}h(D_2)+f'(D_2)\right)\right|<1$$

$$\Rightarrow\left|(r\frac{h'(D_1)}{h(D_1)}D_1+r+f'\left(D_1\right))(r\frac{h'(D_2)}{h(D_2)}D_2+r+f'\left(D_2\right))\right|<1$$

$$\Rightarrow\left|\begin{array}{l} r^2\left(\dfrac{h'(D_1)}{h(D_1)}r\dfrac{h'(D_2)}{h(D_2)}D_1.D_2+r^2\dfrac{h'(D_1)}{h(D_1)}N_1+hD_2\right)+r^2 \\ +r\left(\dfrac{h'(D_1)}{h(D_1)}D_1.f'\left(D_1\right)+\dfrac{h'(D_2)}{h(D_2)}D_2.f'\left(D_2\right)+f'\left(D_1\right)f'\left(D_2\right)\right)+f'\left(D_1\right).f'\left(D_2\right)\end{array}\right|<1 \qquad (10)$$

As previous theorem, we have already seen that

$$g_h'\left(D^*\right)>0\Rightarrow g_h'\left(D_1\right)>0\,\&\,g_h'\left(D_2\right)>0\Rightarrow g_h'\left(D_1\right).g_h'\left(D_2\right)>0$$

So, equation (15) becomes

$$\begin{array}{l}\Rightarrow -r^2\left(\dfrac{h'(D_1)}{h(D_1)}\dfrac{h'(D_2)}{h(D_2)}D_1D_2+r^2\dfrac{h'(D_1)}{h(D_1)}D_1+\dfrac{h'(D_2)}{h(D_2)}D_2\right)-r^2 \\ -r\left(\dfrac{h'(D_1)}{h(D_1)}D_1.f'\left(D_1\right)+\dfrac{h'(D_2)}{h(D_2)}D_2.f'\left(D_2\right)+f'\left(D_1\right)+f'\left(D_2\right)\right) \\ <f'\left(D_1\right)f'\left(D_2\right)<1-r^2\left(\dfrac{h'(D_1)}{h(D_1)}\dfrac{h'(D_2)}{h(D_2)}D_1D_2+r^2\dfrac{h'(D_1)}{h(D_1)}D_1+\dfrac{h'(D_2)}{h(D_2)}D_2\right) \\ -r^2-r\left(\dfrac{h'(D_1)}{h(D_1)}D_1.f'\left(D_1\right)+\dfrac{h'(D_2)}{h(D_2)}D_2.f'\left(D_2\right)+f'\left(D_1\right)+f'\left(D_2\right)\right)\end{array} \qquad (11)$$

Let $P_1=\dfrac{h'\left(D_1\right)}{h\left(D_1\right)}D_1\,\&\,P_2=\dfrac{h'\left(D_2\right)}{h\left(D_2\right)}D_2$

Then equation (11) becomes -

$$\begin{array}{l}-r^2-r^2\left(P_1P_2+P_1+P_2\right)-r\left(P_1f'\left(D_1\right)+P_2f'\left(D_2\right)+f'\left(D_1\right)+f'\left(D_2\right)\right)<f'\left(D_1\right).f'\left(D_2\right)< \\ 1-r^2-r^2\left(P_1P_2+P_1+P_2\right)-r\left(P_1f'\left(D_1\right)+P_2f'\left(D_2\right)+f'\left(D_1\right)+f'\left(D_2\right)\right)\end{array}$$

Let us assume that, $-r^2-r^2\left(P_1P_2+P_1+P_2\right)-r\left(P_1f'\left(D_1\right)+P_2f'\left(D_2\right)+f'\left(D_1\right)+f'\left(D_2\right)\right)\in(0,1)$

Hence, the two cycle of the difference equation (2) will be locally asymptotically stable if the inequality

$$\begin{array}{l}-r^2-r^2\left(P_1P_2+P_1+P_2\right)-r\left(P_1f'\left(D_1\right)+P_2f'\left(D_2\right)+f'\left(D_1\right)+f'\left(D_2\right)\right)< \\ f'\left(D_1\right).f'\left(D_2\right)<1-r^2-r^2\left(P_1P_2+P_1+P_2\right)-r\left(P_1f'\left(D_1\right)+P_2f'\left(D_2\right)+f'\left(D_1\right)+f'\left(D_2\right)\right)\end{array}$$

holds.

Theorem 3.4(c). Let $\left\{D_1,D_2,D_3\right\}$ is 3-cycle of the difference equation (2) with respect to r^* then 3-cycle is locally asymptotic stable if the inequality-

$$-r^3-r^3\left(P_1P_2P_3+P_1P_2+P_2P_3+P_3P_1+P_1+P_2+P_3\right)-$$

$$r^2\left(\begin{array}{l} P_1P_2f'(D_3)+P_2P_3f'(D_1)+P_1P_3f'(D_2)+(P_1(f'(D_2)+f'(D_3)) \\ +(P_2(f'(D_1)+f'(D_3))+(P_3(f'(D_1)+f'(D_2))+ \\ (f'(D_1)+f'(D_2)+f'(D_3)) \end{array}\right)-$$

$$r\left(\begin{array}{l} P_1(f'(D_2)f'(D_3)+P_2(f'(D_1)f'(D_3)+P_3(f'(D_1)f'(D_2)+f'(D_3)+ \\ f'(D_1)f'(D_2)+f'(D_2)f'(D_3)+f'(D_1)f'(D_3) \end{array}\right)<$$

$$f'(D_1).f'(D_2).f'(D_3)<$$

$$1-r^3-r^3\left(P_1P_2P_3+P_1P_2+P_2P_3+P_3P_1+P_1+P_2+P_3\right)-$$

$$r^2\left(\begin{array}{l} P_1P_2f'(D_3)+P_2P_3f'(D_1)+P_1P_3f'(D_2)+(P_1(f'(D_2)+f'(D_3)) \\ +(P_2(f'(D_1)+f'(D_3))+(P_3(f'(D_1)+f'(D_2))+ \\ (f'(D_1)+f'(D_2)+f'(D_3)) \end{array}\right)-$$

$$r\left(\begin{array}{l} P_1(f'(D_2)f'(D_3)+P_2(f'(D_1)f'(D_3)+P_3(f'(D_1)f'(D_2)+f'(D_3)+ \\ f'(D_1)f'(D_2)+f'(D_2)f'(D_3)+f'(D_1)f'(D_3) \end{array}\right)$$

holds otherwise chaotic.

Theorem 3.4(d). Let $\{D_1, D_2, D_3.......D_n\}$ is m-cycle of the difference eq (2) with respect to r^* then n-cycle is locally stable if the inequality-

$$-r^n-r^n\left[P_1P_2P_3........P_n+\sum P_1P_2P_3........P_{n-1}+\sum P_1P_2P_3........P_{n-2}+........+\sum P_1\right]-$$

$$r^{n-1}\left[\sum P_1P_2P_3........P_{n-1}\left(\underset{j\neq 1,2,....n-1}{f'(Dj)}\right)+\sum P_1P_2P_3........P_{n-2}\left(\sum_{i\neq j}^{n}\{f'(D_i)+f'(D_j)\}\right)+\sum P_1P_2........P_{n-1}\right]..............$$

....

$$r\left[\sum P_1\{f'(D_2)f'(D_3).........f'(D_n)\}+\sum\{f'(D_1)f'(D_2)........f'(D_{n-1})\}\right]<$$

$$f'(D_1)f'(D_2)........f'(D_n)$$

$$<1-r^n-r^n\left[P_1P_2P_3........P_n+\sum P_1P_2P_3........P_{n-1}+\sum P_1P_2P_3........P_{n-2}+........+\sum P_1\right]-$$

$$r^{n-1}\left[\sum P_1P_2P_3........P_{n-1}\left(\underset{j\neq 1,2,....n-1}{f'(Dj)}\right)+\sum P_1P_2P_3........P_{n-2}\left(\sum_{i\neq j}^{n}\{f'(D_i)+f'(D_j)\}\right)+\sum P_1P_2........P_{n-1}\right]..............$$

....

$$r\left[\sum P_1\{f'(D_2)f'(D_3).........f'(D_n)\}+\sum\{f'(D_1)f'(D_2)........f'(D_{n-1})\}\right]$$

4. Discussion

In this part, we show the above results for period one and two of discrete time population model (1) and (2) numerically using MATLAB.

For example, we consider the function-

$$f(D_t)=\frac{1}{D_t+1}\quad,\ h(D_t)=\frac{D_t}{D_t+1}$$

In figure 1 we see that Allee Effect decreases the local stability of discrete time population model (1) and (2) with and without Allee effect of period one for $r=0.5$ & $r=0.62$. In figure 2 we see that Allee Effect decreases the local stability of discrete time population model (1) and (2) with and without Allee effect with period two for different values of r. In figure 3 & 4 graph of the function $g(D_t), g_h(D_t), g^2(D_t)$ and $g_h{}^2(D_t)$ with the same co-ordinate plane of D_t.

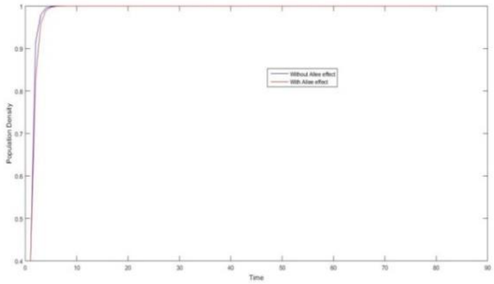

Fig. 1(a)

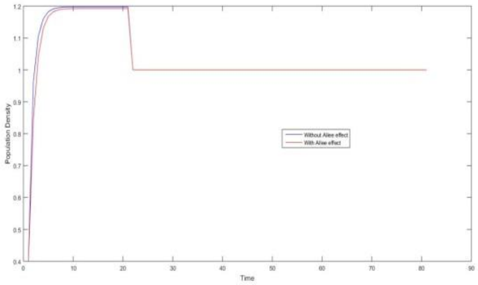

Fig. 1(b)

Fig. 1(a) Local stability for the population model with and without Allee effect for the function $g(r,D_t)$ and $g_h(r^*,D_t)$. Density time graph for the population models - $D_{t+1}=rD_t+\frac{1}{D_t+1}$ and $D_{t+1}=r^*\frac{D_t^2}{D_t+1}+\frac{1}{D_t+1}$, where $h(D_t)=\frac{D_t}{D_t+1}$ and $r=r^*h(D_t)$ with initial conditions $N_0=0.4, r=0.5, r^*=1, N^*=1$.

Fig. 1(b) Local stability for the population model with and without Allee effect for the function $g(r,D_t)$ and $g_h(r^*,D_t)$. Density time graph for the population models - $D_{t+1}=rD_t+\frac{1}{D_t+1}$ and $D_{t+1}=r^*\frac{D_t^2}{D_t+1}+\frac{1}{D_t+1}$, where $h(D_t)=\frac{D_t}{D_t+1}$ and $r=r^*h(D_t)$ with initial conditions $N_0=0.4, r=0.62, r^*=1.13, N^*=1.2$

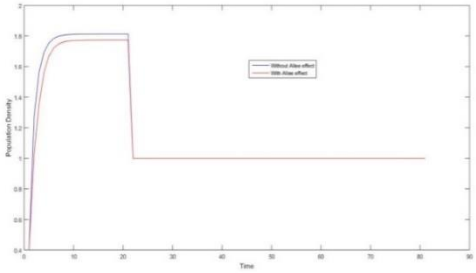

Fig. 2(a)

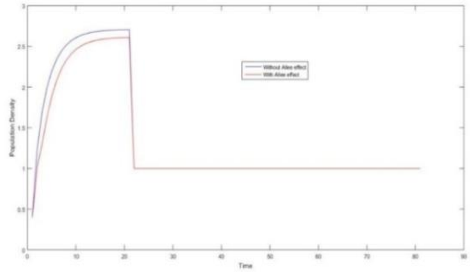

Fig. 2(b)

Fig. 2(a) Local stability for the population model with and without Allee effect for the function $g^2(r,D_t)\,\&\,g_h{}^2(r,D_t)$ .Density-time graph for the population models-

$$D_{t+1}=\frac{r^2D_t^{\,2}+r^2D_t+1}{D_t+1}+\frac{D_t+1}{rD_t^{\,2}+(1+r)D_t+2}\ \text{ and }\ D_{t+1}=\frac{(r^*)^3D_t^{\,4}+[(2r^*)^2+1]D_t^{\,2}+2D^t+(r^*+1)}{+r^*D_t^{\,3}+(1+r^*)D_t^{\,2}+3D_t+2}$$

with initial condition $N_0=0.4, r=0.82, r^*=1.23, N^*=2$

Fig. 2(b) Local stability for the population model with and without Allee effect for the function $g^2(r,D_t)\,\&\,g_h{}^2(r,D_t)$ .Density-time graph for the population models-

$$g^2(D_t)=\frac{r^2D_t^{\,2}+r^2D_t+1}{D_t+1}+\frac{D_t+1}{rD_t^{\,2}+(1+r)D_t+2}\ \text{ and }\ g_h{}^2(D_t)=\frac{(r^*)^3D_t^{\,4}+[(2r^*)^2+1]D_t^{\,2}+2D^t+(r^*+1)}{r^*D_t^{\,3}+(1+r^*)D_t^{\,2}+3D_t+2}$$

with initial condition $N_0=0.4, r=0.91, r^*=1.21, N^*=3$

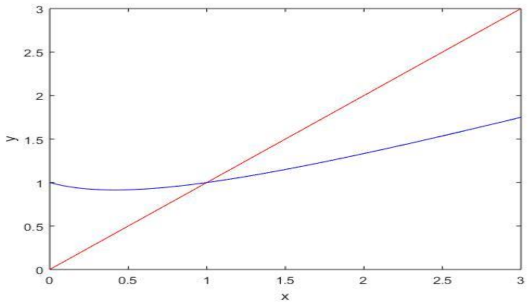

Fig. 3(a)

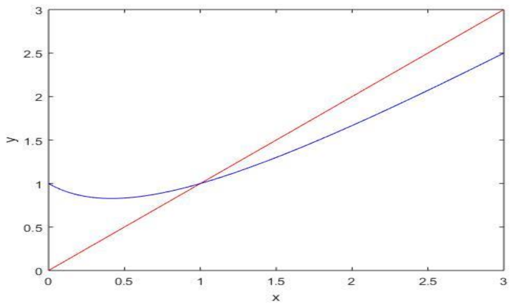

Fig. 3(b)

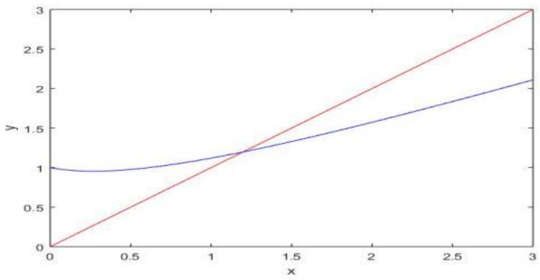

Fig. 3(c)

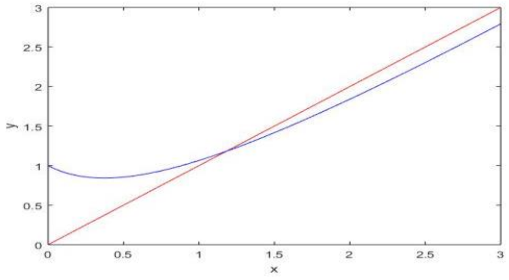

Fig. 3(d)

Graph of the function $g(D_t)$ and $g_h(D_t)$ with the same coordinate plane of D_t for the discrete time population model in

Fig.3(a) $g(D_t)=rD_t+\dfrac{1}{D_t+1}$ for $r=0.5$;

For **Fig.3(b)** $g_h(D_t)=r^*\dfrac{D_t^{\,2}}{D_t+1}+\dfrac{1}{D_t+1}, h(D_t)=\dfrac{D_t}{D_t+1}$ for r = 0.5;

For **Fig.3(c)** $g(D_t) = rD_t + \dfrac{1}{D_t + 1}$ for $r = 0.62$

and for **Fig.3(d)** $g_h(D_t) = r^* \dfrac{D_t^2}{D_t + 1} + \dfrac{1}{D_t + 1}$ for $r = 0.62$.

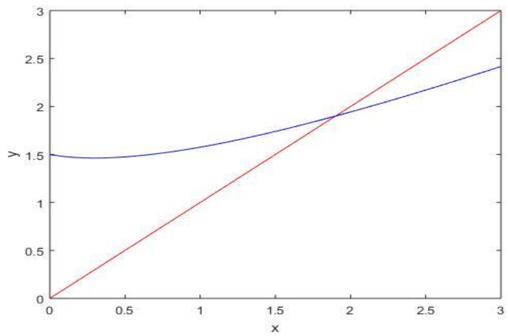

Fig.4(a)

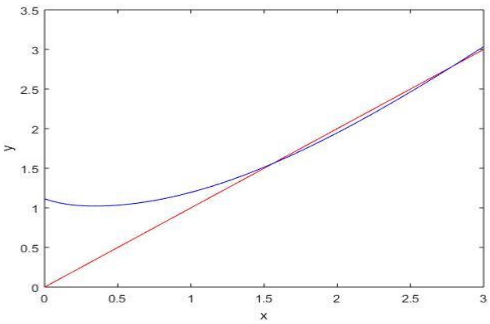

Fig.4(b)

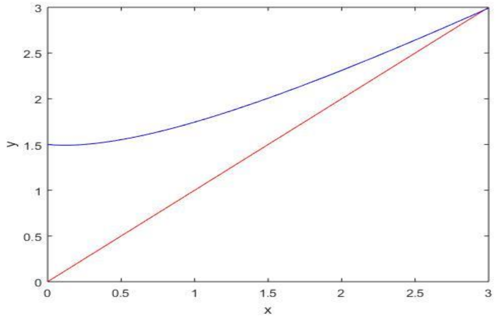

Fig.4(c)

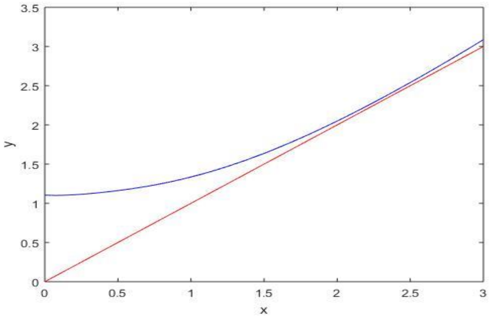

Fig.4(d)

Graph of the function $g^2(D_t)$ and $g^2{}_h(D_t)$ with the same coordinate plane of D_t for the discrete time population model in

Fig.4(a) $g^2(D_t) = \dfrac{r^2 D_t^2 + r^2 D_t + 1}{D_t + 1} + \dfrac{D_t + 1}{rD_t^2 + (1+r)D_t + 2}$ for $r = 0.82$;

Fig.4(b) $g_h{}^2(D_t) = \dfrac{(r^*)^3 D_t^4 + [(2r^*)^2 + 1]D_t^2 + 2D^t + (r^* + 1)}{r^* D_t^3 + (1 + r^*)D_t^2 + 3D_t + 2}$ for $r = 0.82, r^* = 1.23$;

Fig.4(c) $g^2(D_t) = \dfrac{r^2 D_t^2 + r^2 D_t + 1}{D_t + 1} + \dfrac{D_t + 1}{rD_t^2 + (1+r)D_t + 2}$ for $r = 0.91$;

Fig.4(d) $g_h{}^2(D_t) = \dfrac{(r^*)^3 D_t^4 + [(2r^*)^2 + 1]D_t^2 + 2D^t + (r^* + 1)}{r^* D_t^3 + (1 + r^*)D_t^2 + 3D_t + 2}$ for $r = 0.91, r^* = 1.21$

5. Conclusion

The investigation into the local stability of equilibrium points in discrete-time population models, represented by equations (1) and (2), has been conducted in this study, considering both the presence and absence of the Allee effect. The findings reveal that the region of local stability surrounding an equilibrium point exhibits a more rapid reduction as successive generations are considered in both scenarios. Notably, in the third generation, chaotic behaviour is observed beyond the established stability region.

Upon comparing figures 1(a) and 2(a) in the absence of the Allee effect, it is noted that the region of local stability around the fixed point diminishes more rapidly in the second generation. Likewise, with the Allee effect, figures 1(b) and 2(b) illustrate that in the subsequent generation, the region of local stability decreases around a fixed point. Additionally, a comparative analysis of figures 1(a), 1(b), 2(a), and 2(b) indicates that the region of local stability around an equilibrium point diminishes more rapidly in the presence of the Allee effect compared to its absence.

This observed trend holds true for nth generations. Chaos occurs in unstable region when n equals 3. In the realm of population biology, it is noteworthy that population dynamics with density-dependent settlement may, in certain situations, exhibit characteristics akin to chaos, particularly when the growth rate is sufficiently small.

6. References:

1. Gumus.A.K.: Global and local stability analysis in a nonlinear discrete-time population model. Advances in Difference Equation. 2014;299.
   Available from:https://doi.org/10.1186/1687-1847-2014-299

2. Gumus.A.K: correction to: Global and local stability analysis in a nonlinear discrete-time population model. Advances in Difference Equation. Advances in Difference Equations, 2018; 357. Available from: https://doi.org/10.1186/s 13662-018-1812-7

3. Maxwell, S. L., Fuller, R. A., Brooks, T. M. and Watson, J. E. Biodiversity: The ravages of guns, nets and bulldozers. Nature News. 2016; *536*(7615):143-45. Available from: https://doi.org/10.1038/536143a

4. Lebreton,J.D, Kenneth. P. B, Jean. C, David.R.A. Modeling survival and testing biological hypotheses using marked animals: A unified approach with case studies. Ecological Monographs. 1992; 62:67-118. Available from: https://doi.org/10.2307/2937171

5. Girondot.M, Delmas.V, Rivalan.P, Courchamp.F, Prevot.A.C, Godfrey.M.H "Implications of temperature-dependent sex determination for population dynamics," Temperature-Dependent Sex Determination in Vertebrates. N. Valenzuela & V. Lance. Smithsonian Books. 2004: 148-55. Available from:https://doi.org/10.5479/SI.9781944466213

6. Martin, T.E. Avian life history evolution in relation to nest site, nest predation, and food. Ecological Monographs. 1995; 65**:**101-27. Available from: https://doi.org/10.2307/2937160

7. Caughley,G. Directions in conservation biology. Journal of Animal Ecology. 1994; 63:215-44.
   Available from: https://doi.org/10.2307/5542

8. Hamilton,W.D. Sex ratios. Science. 1967; 156**:**477-88. Available from:
   https://doi.org/10.1126/science.156.3774.477

9. Juliano, S.A. Population dynamics. Journal of the American Mosquito Control Association. 2007; *23*(2Suppl):265-75.
Available from:https://doi.org/10.2987%2F8756-971x(2007)23%5B265%3Apd%5D2.0.co%3B2

10. Jang, S.R. Allee effects in a discrete-time host-parasitoid model with stage structure in the host. Discrete Continuous Dynamical System.2007;8:145-59.
Available from:https://doi.org/10.3934/dcdsb.2007.8.145

11. Allee, W.C., Emerson, A.E., Park, O., Park, T. and Schmidt, K.P. Principles of Animal Ecology, Philadelphia. Saunders. 1949.

12. Begon M, Harper JL, Townsend CR. Ecology: Individuals, populations, and communities. 2. Blackwell Scientific Publications; Boston MA. 1986

13. Lopcz. R. R, Prunarct. P. D. Indirect Allee effect, bistability and chaotic oscillations in a predator-preydiscrete model of logistic type. Chaos Solitons fractals. 2005; 24: 85-101. Available from:https://doi.org/10.1016/j.chaos.2004.07.018

14. Merdan, H, Duman, O. On the stability analysis of a general discrete-time population model involving predation and Allee effects. Chaos Solitons Fractals. 2009; 40, 1169-75. Available from:https://doi.org/10.1016/j.chaos.2007.08.081

15. Gumus.A.K , On the stability of decay population dynamics related with Allee effects. Mathematical and Computational Applications. 2012;17(1),56-67. Available from:https://doi.org/10.3390/mca17010056

16. Fowler, M.S, Ruxton, G.D: Population dynamics consequences of Allee effects. J. Theor. Biol. 2002, 215, 39-46 Available from:https://doi.org/10.1006/jtbi.2001.2486

17. Stephens, P.A., Sutherland, W.J. and Freckleton, R.P. What is the Allee effect? Oikos. 1999; 87(1): 185-90. Available from: https://doi.org/10.2307/3547011

18. Kramer, A.M., Berec, L. and Drake, J.M. Allee effects in ecology and evolution. 2018; 87:7-10. Available from:https://doi.org/10.1111/1365-2656.12777

19. Luque, G.M., Vayssade, C., Facon, B., Guillemaud, T., Courchamp, F. and Fauvergue, X. The genetic Allee effect: a unified framework for the genetics and demography of small populations. Ecosphere. 2016 ;*7*(7). Available from:https://doi.org/10.1002/ecs2.1413

20. Wittmann, M. J., Stuis, H., & Metzler, D. Genetic Allee effects and their interaction with ecological Allee effects. Journal of Animal Ecology. 2016; 87: 11– 23. Available from: https://doi.org/10.1111/1365-2656.12598

21. Shaw, A. K., Kokko, H., & Neubert, M. G.: Sex differences and Allee effects shape the dynamics of sex-structured invasions. Journal of Animal Ecology.2017; 87, 36– 46. Available from:https://doi.org/10.1111/1365-2656.12658

22. Brassil, C. E. Mean time to extinction of a metapopulation with an Allee effect. Ecological Modelling. 2001; 143: 9–16. Available from:https://doi.org/10.1016/S0304-3800(01)00351-9

23. Boukal, D. S. &Berec.L Single-species models of the Allee effect: Extinction boundaries, sex ratios and mate encounters. Journal of Theoretical Biology. 2002; 218: 375–94. Available from:https://doi.org/10.1006/jtbi.2002.3084

24. Courchamp, F., Berec, L., Gascoigne, J. Allee Effects in Ecology and Conservation. Oxford University Press Inc., New York. 2008.

25. Braucer . F and Castillo-C.C. Mathematical Models in Population Biology and Epidemiology. Springer, New York, 2012.

26. Layak.G.C An introduction to Dynamical system and chaos. Springer, India, 2015.

27. Lindo J.S Allen. An Introduction to Mathematical Biology, Pearson, 2006.

28. Li.T.Y, and Yorke.J. A Period Three Implies Chaos: The American Mathematical Monthly. 1975;82, No. 10:985-92.https://doi.org/10.2307/2318254

29. May.R.M Biological populations obeying difference equations, Biology Department, Princeton University, Princeton. 1975;51: 511-24. Available from:https://doi.org/10.1016/0022-5193(75)90078-8

30. Andriano.V ,Bacciotti.A and Beccari.G , Global Stability and External Stability of Dynamical Systems, Nonlinear analysis, Theory, Methods and Applications, 1997, Vol. 28, No. 7, pp 1167-1185, available from: https://doi.org/10.1016/50362-545x(97)82867-2