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Some Fixed-Point Theorem for Multiplicative Convex Contraction Mappings with an Application

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Abstract:

The study introduces multiplicative convex contraction mappings of order 2, which extends the traditional notions of convex contractions to multiplicative metric spaces. Furthermore, a generalization of convex contraction mapping of order 2 introduces and establishes theorem regarding the existence and uniqueness of fixed points. To exemplify the practical utility of these theoretical developments, concrete examples discusses and apply to find the existence of solution for non linear Fredholm integral equation.

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Introduction

Istratescu [6] introduce the concept of convex contraction and established the existence and uniqueness of fixed points for such mappings on complete metric spaces. Expanding on this line of inquiry, In non-normal cone metric space, Mohammad A. Alghamdia [1] extended the Banach theorem to encompass convex contractions. Additionally, in 2013, Mehdi Amir Miandaragh [7] initiated the concept of generalized convex contractions, further contributing to this area of study. Lately, a group of researchers delved into the generalization of this category of mappings within diverse spatial contexts (For example Nallaselli et al. [6], Mitrovic et al. [7], Bisht, R. K., Rakocevic, V. [4]) In 2008, Bashirov et al. introduced a novel metric called multiplicative distance, utilizing the notion of multiplicative absolute value. Ozavsar [11] further extended this by introducing the concept of multiplicative contraction mappings on multiplicative metric spaces.

Definition 1.1 [11] A function $\rho: \varrho \times \varrho \to \mathbb{R}_+$ is considered a multiplicative metric on the nonempty set ϱ if it satisfies the following conditions for any $x^*, y^*, z^* \in \varrho$,

$(m1)$ $\rho(x^*, y^*) \geq 1$ if and only if $x^* = y^*$;

$(m2)$ $\rho(x^*, y^*) = \rho(y^*, x^*)$;

$(m3)$ $\rho(x^*, y^*) \leq \rho(x^*, z^*)\rho(z^*, y^*)$

The pair (ϱ, ρ) is called a multiplicative metric space.

Definition 1.2 [11] Let (ϱ, ρ) be a multiplicative metric space, $x^* \in \varrho$ and $\epsilon > 1$,

$$B_\epsilon(x^*) = \{y^* \in \varrho | \rho(x^*, y^*) < \epsilon\},$$

This is referred to as multiplicative open with a radius of ϵ centered at x^*.

Similarly, multiplicative closed ball as

$$\bar{B}_\epsilon(x^*) = \{y^* \in \varrho | \rho(x^*, y^*) \leq \epsilon\}$$

Definition 1.3 [11] Let (ϱ, ρ) be a multiplicative metric space and $A \subset \varrho$. When every point within set A is considered a multiplicative interior point, denoted as $A = int(A)$, then A is commonly termed as a multiplicative open set.

Definition 1.4 [11] In a multiplicative metric space (ϱ, ρ), a sequence (x_n^*) is deemed a multiplicative Cauchy sequence if, for every $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $\rho(x_i^*, x_j^*) < \epsilon$ for $i, j \in \mathbb{N}$.

Definition 1.5 [11] A multiplicative metric space is deemed complete if any multiplicative Cauchy sequence contained within it converges multiplicatively towards a limiting point $x^* \in \varrho$.

Definition 1.6 [7] Let ϱ be a nonempty set. If $\Gamma: \varrho \to \varrho$ and $\alpha^*: \varrho \times \varrho \to [0, \infty)$ be a mappings, we assert that Γ is an α^* −admissible if $x^*, y^* \in \Gamma, \alpha^*(x^*, y^*) \geq 1$ implies that $\alpha^*(\Gamma x^*, \Gamma y^*) \geq 1$.

Definition 1.7 [7] Assuming Γ is an α^* −admissible on ϱ, we assert that ϱ possesses property (H) provided that for any x^*, y^* belonging to the fixed points of Γ, there exists $z^* \in \varrho$ such that $\alpha^*(x^*, z^*) \geq 1$ and $\alpha^*(y^*, z^*) \geq 1$.

Definition 1.8 [7] A self mapping Γ on a multiplicative metric space (ϱ, ρ) is said to be asymptotically regular at $x^* \in \varrho$ if

$$\rho(\Gamma^j x^*, \Gamma^{j+1} x^*) \to 1 \text{ as } n \to \infty$$

Where $\Gamma^j x^*$ denotes the j^{th} iterate of Γ at x^*.

Lemma 1.1 [7] Let (ϱ, ρ) be a multiplicative metric space and Γ be an asymptotically regular selfmap on ϱ that is $\rho(\Gamma^j x^*, \Gamma^{j+1} x^*) \to 1$ for all $x^* \in \varrho$.

2. Main Result

Definition 2.1 A mapping Γ from the set ϱ to itself, defined on a multiplicative metric space, is classified as a multiplicative convex contraction mapping of order 2 when there exists two constants, denoted as $'\delta'$ and $'\kappa'$, both belonging to the open interval $(0,1)$, such that the mapping Γ satisfies the condition

$$\rho(\Gamma^2(x^*), \Gamma^2(y^*) \leq \rho(\Gamma(x^*), \Gamma(y^*))^\delta \rho(x^*, y^*)^\kappa$$

For all $x^*, y^* \in \varrho$, and additionally, the sum of $'a'$ and $'b'$ is strictly less than 1.

Definition 2.2 In a multiplicative metric space (ϱ, ρ), with Γ as a self map on ϱ and $\alpha^*: \varrho \times \varrho \to [0, \infty)$ a mapping, we define the property (H) as follows: for any $x^*, y^* \in \varrho$, there exists $z^* \in \varrho$ such that

$$\alpha^*(x^*, z^*) \geq 1, \ \alpha^*(y^*, z^*) \geq 1.$$

A self map Γ acting on the space ϱ is termed a generalized multiplicative convex contraction if there exists a mapping $\alpha^*: \varrho \times \varrho \to [0, \infty)$, along with constants $\delta, \kappa \in [0,1)$ such that $\delta + \kappa < 1$, satisfying the condition

$$\alpha^*(x^*, y^*)\rho(\Gamma^2 x^*, \Gamma^2 y^*) \leq \rho(\Gamma x^*, \Gamma y^*)^\delta \rho(x^*, y^*)^\kappa$$

For all $x^*, y^* \in \varrho$.

Definition 2.3. In a multiplicative metric space (ϱ, ρ), with Γ as a self map on ϱ and $\alpha^*: \varrho \times \varrho \to [0, \infty]$ a mapping, we define the property (H) as follows: For any $x^*, y^* \in \varrho$, there exists $z^* \in \varrho$ such that $\alpha^*(x^*, z^*) \geq 1$

and $\alpha(y^*, z^*) \geq 1$. A self map Γ acting on the space ϱ is termed a generalized multiplicative convex contraction of order 2 if there exists a mapping $\alpha^*: \varrho \times \varrho \rightarrow [0, \infty)$, along with constant $\delta_1, \delta_2, \kappa_1, \kappa_2 \in [0,1)$ such that $\delta_1 + \delta_2 + \kappa_1 + \kappa_2 < 1$, satisfying the condition

$$\alpha^*(x^*, y^*)\rho(\Gamma^2 x^*, \Gamma^2 y^*) \leq d(x^*, \Gamma x^*)^{\delta_1} d(\Gamma x^*, \Gamma^2 x^*)^{\delta_2} d(y^*, \Gamma y^*)^{\kappa_1} d(\Gamma y^*, \Gamma^2 y^*)^{\kappa_2}$$

for all $x^*, y^* \in \varrho$.

Theorem 2.1 Consider a complete multiplicative metric space denoted by (ϱ, ρ), and Let $\Gamma: \varrho \rightarrow \varrho$ be a multiplicative convex contraction mapping of order 2 with respect to the multiplicative metric. Then Γ possesses a unique fixed point within ϱ, and for any $x^* \in \varrho$, the iterative sequence $\Gamma^n x^*$ converges towards this fixed point.

Proof

Consider $x_0^* \in \varrho$, $x_1^* = \Gamma x_0^* \ldots x_{j+1}^* = \Gamma^{j+1} x_0^*$ for all $j \geq 1$. Set $v = \rho(\Gamma^2 x_0^*, \Gamma x_0^*)\rho(\Gamma x_0^*, x_0^*)$.

Then $\rho(\Gamma^i x_0^*, \Gamma^{i+1} x_0^*) \leq \rho(\Gamma^2 x_0^*, \Gamma x_0^*)\rho(\Gamma x_0^*, x_0^*)^{\psi^l}$ when $i = 2l, j = 2p$ with $p \geq 2$ and $l \geq 1, i < j$.

$$\rho(\Gamma^i x_0^*, \Gamma^j x_0^*) \leq \rho(\Gamma^i x_0^*, \Gamma^{i+1} x_0^*)\rho(\Gamma^{i+1} x_0^*, \Gamma^{i+2} x_0^*)\rho(\Gamma^{i+2} x_0^*, \Gamma^{i+3} x_0^*) \ldots \rho(\Gamma^{i-1} x_0^*, \Gamma^i x_0^*)$$

$$= \rho(\Gamma^{2l} x_0^*, \Gamma^{2l+1} x_0^*)\rho(\Gamma^{2l+1} x_0^*, \Gamma^{2l+2} x_0^*)\rho(\Gamma^{2l+2} x_0^*, \Gamma^{2l+3} x_0^*) \ldots$$

$$\leq v^{\frac{2\psi^l}{1-\psi}}$$

Similarly, for $i = 2l, j = 2p + 1$ with $p \geq 1,\ l \geq 1, i < j$.

$$\rho(\Gamma^i x_0^*, \Gamma^j x_0^*) < v^{\frac{2\psi^l}{1-\psi}}$$

Also for $i = 2l + 1$ and $j = 2p$ with $p \geq 2$ and $l \geq 1, i < j$

$$\rho(\Gamma^i x_0^*, \Gamma^j x_0^*) \leq \rho(\Gamma^i x_0^*, \Gamma^{i+1} x_0^*)\rho(\Gamma^{i+1} x_0^*, \Gamma^{i+2} x_0^*)\rho(\Gamma^{i+2} x_0^*, \Gamma^{i+3} x_0^*) \ldots \rho(\Gamma^{i-1} x_0^*, \Gamma^i x_0^*) \leq v^{\frac{2\psi^l}{1-\psi}}$$

For $i = 2l + 1, j = 2p + 1$ with $p \geq 2, l \geq 1, i < j, \rho(\Gamma^i x_0^*, \Gamma^j x_0^*) \leq v^{\frac{2\psi^l}{1-\psi}}$

Notice that $v^{\frac{2\psi^l}{1-\psi}} \rightarrow 1$ as $l \rightarrow \infty$

Now if $i < j$, we have that $\rho(\Gamma^i x_0^*, \Gamma^j x_0^*) < \epsilon\ (l \rightarrow \infty \Leftrightarrow m \rightarrow \infty)$. Hence x_n^* forms a Cauchy sequence in (ϱ, ρ). As ϱ is complete, there exists a point $z^* \in \varrho$ such that $x_n^* \rightarrow z^*$. Given that $x_n^* \rightarrow z^*$ and Γ is continuous, it follows that $\Gamma x_n^* \rightarrow \Gamma z^*$, which implies $x_{n+1}^* \rightarrow \Gamma z^*$. Consequently, we can conclude that $\Gamma z = z^*$.

Let $y^* \neq z^*$ be another fixed point.

Then

$$\rho(z^*, y^*) = \rho(\Gamma^2 y^*, \Gamma^2 z^*)$$

$$\leq \rho(\Gamma z^*, \Gamma y^*)^{\delta}\rho(z^*, y^*)^{\kappa}$$

$$= \rho(z^*, y^*)^{\delta+\kappa}$$

$$< \rho(y^*, z^*)$$

Which is contraction. Γ has a unique fixed point.

Example 2.1 Consider $\varrho = \{(v,0) \in \mathbb{R}^2 | 0 \leq v \leq 1\}$. Define $\rho: \varrho \times \varrho \to \varrho$ by $\rho\big((v,0),(\varphi,0)\big) = e^{\gamma|v-\varphi|}$ where $\gamma \in (0,1)$. Then (ϱ, ρ) is a complete multiplicative metric space. Let $\Gamma: \varrho \times \varrho$ defined be $\Gamma(v,0) = \left(\frac{v^2}{2}, 0\right)$. Then Γ is continuous and

$$\rho(\Gamma(v,0),\Gamma(\varphi,0) =\leq \rho\left(\left(\frac{v^2}{2},0\right),\left(\frac{\varphi^2}{2},0\right)\right)$$

$$= e^{\gamma\left|\frac{v^2}{2}-\frac{\varphi^2}{2}\right|}$$

$$= e^{\gamma\left(\frac{v+\varphi}{2}\right)|v-\varphi|}$$

$$= \rho((v,0),(\varphi,0))^{\frac{v+\varphi}{2}}$$

Then $$\rho(\Gamma(v,0),\Gamma(\varphi,0) \leq \rho((v,0),(\varphi,0))^{\lambda}$$

does not hold for all $(v,0),(\varphi,0) \in \varrho$ and any fixed $\lambda \in [0,1)$. If we assume $\frac{v+\varphi}{2} \leq \lambda$, it leads to a contradiction when v and φ approach 1 for a fixed $\lambda \in [0,1)$. Consequently, we conclude that Γ cannot be a contraction on ϱ. Notice that

$$\rho(\Gamma^2(v,0),\Gamma^2(\varphi,0) = \rho(\Gamma(v,0),\Gamma(\varphi,0))$$

$$= \rho\left(\Gamma\left(\frac{v^2}{2},0\right),\Gamma\left(\frac{\varphi^2}{2},0\right)\right)$$

$$= \rho\left(\left(\frac{v^4}{8},0\right),\Gamma\left(\frac{\varphi^2}{8},0\right)\right)$$

$$= e^{\gamma\left|\frac{v^4}{8}-\frac{\varphi^4}{8}\right|}$$

$$= e^{\gamma\left(\frac{v^2+\varphi^2}{4}\right)\left|\frac{v^2-\varphi^2}{2}\right|}$$

$$= \rho((v,0),(\varphi,0))^{\frac{1}{2}}$$

Thus Γ is a convex contraction of order 2. It is easy to see that Γ has a unique fixed point $(0,0) \in \varrho$.

Theorem 2.2. Consider a multiplicative metric space (ϱ, ρ) and a generalized multiplicative convex contraction Γ on ϱ with the based mapping α^*. We examine the following criteria:

(i) Γ is α^* −admissible

(ii) There exists an element $x_0^* \in \Gamma$ such that $\alpha^*(x_0^*, \Gamma x_0^*) \geq 1$.

Under these conditions, the following results hold:

Firstly, Γ possesses an approximate fixed point.

Furthermore, if Γ is continuous, (ϱ, ρ) is a complete multiplicative metric space, and ϱ satisfies the property (H), then Γ has a fixed point, and ϱ has a unique fixed point.

Proof.

Consider $x_0^* \in \varrho \ni \alpha^*(x_0^*, \Gamma x_0^*) \geq 1$ by $x_{j+1} = \Gamma^{j+1} x_0^*$ for all $j \geq 0$.

If $x_j^* = x_{j+1}^*$ for some j, we need not prove anything.

Assume $x_j^* \neq x_{j+1}^*$ for all $j \geq 0$. As Γ is α^* −admissible, it can be readily verified that $\alpha^*(x_j^*, x_{j+1}^*) \geq 1$ for all j.

Consider $v = \rho(\Gamma x_0^*, \Gamma^2 x_0^*)\rho(x_0^*, \Gamma x_0^*)$ and $v = \delta + \kappa$. Then $\rho(\Gamma x_0^*, \Gamma^2 x_0^*) \leq v$. Now $x^* = \Gamma x_0^*, y^* = x_0^*$

Then

$$\rho(\Gamma^3 x_0^*, \Gamma^2 x_0^*) \leq \alpha^*(\Gamma x_0^*, x_0^*)\rho(\Gamma^3 x_0^*, \Gamma^2 x_0^*)$$

$$\leq \rho(\Gamma^2 x_0^*, \Gamma x_0^*)^{\delta}\rho(x_0^*, \Gamma x_0^*)^{\kappa}$$

$$\leq v^{\psi}$$

By the use of similar technique $\rho(\Gamma^{i+1} x_0^*, \Gamma^i x_0^*) \leq v^{2\psi^{l-1}}$

Continuing with this procedure

When $i = 2l$ or $i = 2l - 1$ for all $l \geq 2$.

$$\rho(\Gamma^i x_0^*, \Gamma^j x_0^*) \leq \rho(\Gamma^i x_0^*, \Gamma^{i+1} x_0^*)\rho(\Gamma^{i+1} x_0^*, \Gamma^{i+2} x_0^*)\rho(\Gamma^{i+2} x_0^*, \Gamma^{i+3} x_0^*) \ldots \rho(\Gamma^{j-1} x_0^*, \Gamma^j x_0^*)$$

$$= \rho(\Gamma^{2l} x_0^*, \Gamma^{2l+1} x_0^*)\rho(\Gamma^{2l+1} x_0^*, \Gamma^{2l+2} x_0^*)\rho(\Gamma^{2l+2} x_0^*, \Gamma^{2l+3} x_0^*) \ldots$$

$$\leq v^{\frac{4\psi^l}{1-\psi}}, \text{ for all } j > i$$

From this, it follows that the sequence x_n^* is Cauchy. Given the continuity of Γ and the completeness of the multiplicative metric (ϱ, ρ), there exists an element $x^{**} \in \varrho$ such that $x_n^* \to x^{**}$. Consequently, we have $\Gamma x_n^* \to \Gamma x^*$, which implies $\Gamma x^{**} = x^{**}$.

Assuming that ϱ also satisfies property (H), we aim to establish the existence of a UFP for Γ. Let x^{**} and y^{**} denote Fixed points of Γ. We select $z^* \in \varrho$ such that $\alpha^*(x^{**}, z^*) \geq 1, \alpha^*(y^{**}, z^*) \geq 1$. Given that $\alpha^* - admissibility$ of Γ, it follows that $\alpha^*(x^{**}, \Gamma^i z^*) \geq 1$ and $\alpha^*(y^{**}, \Gamma^i z^*) \geq 1$ for all $i \geq 1$. Put $\psi = \delta + \kappa$ and $v = \rho(x^{**}, \Gamma^2 z^*)\rho(x^{**}, \Gamma z^*)$

Then we have,

$$\rho(x^{**}, \Gamma^3 z^*) = \rho(\Gamma^2 x^{**}, \Gamma^2(\Gamma z^*))$$

$$\leq \alpha(x^{**}, \Gamma z^*)\rho(\Gamma^2 x^{**}, \Gamma^2(\Gamma z^*))$$

$$\leq \rho(x^{**}, \Gamma^2 z^*)^{\delta}\rho(x^{**}, \Gamma z^*)^{\kappa}$$

$$\leq v^{\psi}$$

$$\rho(x^{**}, \Gamma^4 z^*) = \rho(\Gamma^2 x^{**}, \Gamma^2(\Gamma^2 z^*))$$

$$\leq \alpha(x^{**}, \Gamma^2 z^*)\rho(\Gamma^2 x^{**}, \Gamma^2(\Gamma^2 z^*))$$

$$\leq \rho(x^{**}, \Gamma^3 z^*)^{\delta}\rho(x^{**}, \Gamma^2 z^*)^{\kappa}$$

$$= \rho(x^{**}, \Gamma^2 z^*)^{\delta^2}\rho(x^{**}, \Gamma z^*)^{\delta\kappa}\rho(x^{**}, \Gamma^2 z^*)^{k}$$

$$\leq v^{\psi}$$

Furthermore, we possess

$$\rho(x^{**}, \Gamma^5 z^*) = \rho(\Gamma^2 x^{**}, \Gamma^2(\Gamma^3 z^*))$$

$$\leq \alpha(x^{**},\Gamma^3 z^*)\rho(\Gamma^2 x^{**},\Gamma^2(\Gamma^3 z^*))$$

$$\leq \rho(x^{**},\Gamma^4 z^*)^{\delta}\rho(x^{**},\Gamma^3 z^*)^{\kappa}$$

$$= \rho(x^{**},\Gamma^2 z^*)^{\delta^3}\rho(x^{**},\Gamma z^*)^{\delta^2\kappa}\rho(x^{**},\Gamma^2 z^*)^{2\delta k}\rho(x^{**},\Gamma z^*)^{\kappa}$$

$$\leq v^{\psi^2}$$

$$\rho(x^{**},\Gamma^6 z^*) = \rho(\Gamma^2 x^{**},\Gamma^2(\Gamma^4 z^*))$$

$$\leq \alpha(x^{**},\Gamma^4 z^*)\rho(\Gamma^2 x^{**},\Gamma^2(\Gamma^4 z^*))$$

$$\leq \rho(x^{**},\Gamma^5 z^*)^{\delta}\rho(x^{**},\Gamma^4 z^*)^{\kappa}$$

$$= \rho(x^{**},\Gamma^2 z^*)^{\delta^4}\rho(x^{**},\Gamma z^*)^{\delta^3\kappa}$$

$$\leq v^{2\psi^2}$$

Through the continuation of this process, we arrive at the result that $\rho(x^{**},\Gamma^i z^*) \leq 2\psi^{l-1}v$ where $i = 2l, i = 2l-1$, for all $l \geq 2$. Hence $\Gamma^i z^* \to x^{**}$.

Likewise, it can be demonstrated that as $\Gamma^i z^*$ converges to y^{**}, we obtain $x^{**} = y^{**}$. This establishes that Γ possesses a unique fixed point.

In order to demonstrate the theorem's validity, let's consider the following example,

Example 2.2. Let $\varrho = \{2,4,6\}$, $\rho(x^*,y^*) = e^{|x^*-y^*|}$, Γ be a selfmap on ϱ defined by $\Gamma 2 = 4, \Gamma 4 = 2, \Gamma 6 = 6$. Then by putting $\delta = \frac{1}{4}, \kappa = \frac{1}{4}, x^* = 2, y^* = 4$. 7=$\rho(\Gamma^2 2,\Gamma^2 4) \leq \rho(\Gamma 2,\Gamma 4)^{\frac{1}{4}}\rho(2,4)^{\frac{1}{4}}$ =2.7. Thus, Γ is not a contraction. While by putting $\alpha^*(x^*,y^*) = \frac{1}{4}$ whenever $x^* \leq y^*$ and $\alpha^*(x^*,y^*) = 0$. Otherwise $\delta = \frac{1}{4}$ and $\kappa = \frac{1}{4}$. It is apparent Γ exhibits the generalized multiplicative convex contraction.

Theorem 2.3 Consider (ϱ,ρ) be a multiplicative metric space, let Γ be a generalized multiplicative convex contraction of order 2, associated with the base mapping α^*. Assume that Γ satisfies α^* −admissibility and there exists an element $x_0^* \in \varrho$ such that $\alpha^*(x_0^*,\Gamma x_0^*) \geq 1$. Under these conditions, it can be established that Γ possesses an approximate fixed point. Furthermore, if Γ is continuous, and (ϱ,ρ) is a complete multiplicative metric space, then Γ admits a fixed point. Additionally, if Γ satisfies property (H), it ensures the uniqueness of the fixed point for Γ.

Proof.

Set $x_0^* \in \varrho$ be such that $\alpha^*(x_0^*,\Gamma x_0^*) \geq 1$. Define the sequence $\{x_j^*\}$ by $x_{j+1} = \Gamma^{j+1}x_0^*$ for all $j \geq 0$. If $x_j^* = x_{j+1}^*$ for some j, there is no need for further demonstration. Given that x_j is distinct from x_{j+1} for all $j \geq 0$, and considering that Γ is $\alpha^* - admissible$, it can be readily confirmed that $\alpha^*(x_j^*,x_{j+1}^*) \geq 1$ for all j.

Let $v = \rho(\Gamma x_0^*,\Gamma^2 x_0^*)\rho(x_0^*,\Gamma x_0^*), \gamma = 1-\kappa_2, \psi = \delta_1+\delta_2+\kappa_1$. Then we have

$$\rho(\Gamma^3 x_0^*,\Gamma^2 x_0^*) \leq \alpha^*(\Gamma x_0^*,x_0^*)\rho(\Gamma^3 x_0^*,\Gamma^2 x_0^*)$$

$$\leq \rho(x_0^*,\Gamma x_0^*)^{\delta_1}\rho(\Gamma x_0^*,\Gamma^2 x_0^*)^{\delta_2}(\Gamma x_0^*,\Gamma^2 x_0^*)^{\kappa_1}\rho(\Gamma^3 x_0^*,\Gamma^2 x_0^*)^{\kappa_2}$$

$$\leq v^{\delta_1}v^{\delta_2+\kappa_1}\rho(\Gamma^3 x_0^*,\Gamma^2 x_0^*)^{\kappa_2}$$

$$\rho(\Gamma^3 x_0^*,\Gamma^2 x_0^*) \leq v^{\frac{\psi}{\gamma}}$$

If $x^* = \Gamma x_0^*$ and $y^* = \Gamma^2 x_0^*$. Then

$$\rho(\Gamma^3 x_0^*, \Gamma^4 x_0^*) \leq \alpha^*(\Gamma^2 x_0^*, \Gamma x_0^*)\rho(\Gamma^3 x_0^*, \Gamma^4 x_0^*)$$

$$\leq v^{\delta_1} v^{\delta_2+\kappa_1 \frac{\delta_1+\delta_2+\kappa_1}{1-\kappa_2}} \rho(\Gamma^3 x_0^*, \Gamma^2 x_0^*)^{\kappa_2}$$

$$\rho(\Gamma^3 x_0^*, \Gamma^4 x_0^*) \leq v^{\frac{\psi}{\gamma}}$$

Similarly, we obtain $\rho(\Gamma^5 x_0^*, \Gamma^4 x_0^*) \leq v^{\frac{\psi^2}{\gamma}}$ and $\rho(\Gamma^5 x_0^*, \Gamma^6 x_0^*) \leq v^{\frac{\psi^2}{\gamma}}$. By continuing this process and we get that $\rho(\Gamma^{i+1} x_0^*, \Gamma^i x_0^*) \leq v^{\frac{\psi^l}{\gamma}}$, where $i = 2l$ or $i = 2l+1$ for $l \geq 1$. Thus $\rho(\Gamma^{i+1} x_0^*, \Gamma^i x_0^*) \to 1$. Through the utilization of lemma 1, Γ has an approximate fixed point.

Now, let us assume that Γ is continuous function, and consider (ϱ, ρ) as a complete multiplicative metric space. Given that x_j^* is a Cauchy sequence, we can select $x^{**} \in \varrho$ such that $x_j^* \to x^{**}$. Assuming that ϱ satisfies property (H), employing a technique akin to the one used in theorem 2.2, we can establish the UFP of Γ.

For instance, consider the following scenario to illustrate the demonstrate started theorem.

Example 2.3

Let $\varrho = \{1, 3, 5\} \leq = \{(1,1),(3,3),(5,5),(1,3)\}$, $\rho(x^*, y^*) = e^{|x^*-y^*|}$and Γ be a self map on ϱ defined by $\Gamma1=3$, $\Gamma3=1$, $\Gamma5=5$. Then by putting $\delta_1 = \delta_2 = \kappa_1 = \kappa_2 = \frac{1}{8}$, $x^* = 1$, $y^* = 3$ we have

$$2 = \rho(\Gamma^2 1, \Gamma^2 3) > \rho(1,1)^{\delta_1}\rho(\Gamma 1, \Gamma^2 1)^{\delta_2}\rho(3, \Gamma 3)^{\kappa_1}\rho(\Gamma 3, \Gamma^2 3)^{\kappa_2} = 0.8$$

It can be observed that Γ does not satisfy the condition of being a generalized multiplicative convex contraction of order 2. However by defining $\alpha^*(x^*, y^*) = \frac{1}{4}$ for $x^* \leq y^*$, $\alpha^*(x^*, y^*) = 0$ and setting $\delta_1 = \delta_2 = \kappa_1 = \kappa_2 = \frac{1}{8}$, it can be easily verified that self-map Γ indeed qualifies as generalized multiplicative convex contraction of order 2.

3. Application

Consider ϱ be a set of all real continuous function defined on the closed interval $[a, b]$, equipped with the multiplicative metric space $\rho(\phi, \chi) = sup\left|\frac{\phi(t)}{\chi(t)}\right|$ for all $\phi, \psi \in C[a, b]$. Then (ϱ, ρ) is a complete multiplicative metric space. The nonlinear Fredholm integral equation is

$$x^*(t) = \int_a^b k(t, s, x^*(t))^{ds} \tag{3.1}$$

Where $t, s \in [a, b]$. Suppose that $k: [a, b] \times [a, b] \times X \to \mathbb{R}$ and $v: [a, b] \to \mathbb{R}$ are continuous functions, with $v(t)$ being a specified function in the set ϱ.

Theorem 3.1

Assume that Nonlinear Fredholm integral equation as in (3.1) and there exists $a, b \in [0,1)$ such that for all $x^*, y^* \in \varrho$ with $x^* \neq y^*$and s, $t \in [a, b]$ satisfies the generalized multiplicative convex contraction of order 2. If Γ is orbitally continuous, then the integral operator defined by (3.1) possesses a unique solution $z^* \in \varrho$. For every $x^* \in \varrho$, it holds that $\Gamma x^* = x_n^*$ for all $n \in \mathbb{N} \cup 0$, and we can establish that $\lim_{n\to\infty} x_n^* = z^*$.

Proof. Let $\Gamma: \varrho \times \varrho \to \mathbb{R}_+$ by $\alpha^*(x^*, y^*) = 1$ for all $x^*, y^* \in \varrho$. Let $x_0^* \in \varrho$ and sequence $\{x_j^*\}$ in ϱ defined by $x_{j+1}^* = \Gamma^{j+1} x_0^*$ for all $j \geq 0$.

$\Gamma x^*(t) = \int_a^b k(t, s, x^*(s))^{ds}$ where $t, s \in [a, b]$

Define a sequence $x_{n+1}^* = \Gamma x_0(t) = \int_a^b k(t, s, x^*(s))^{ds}$

$$\left|\frac{\Gamma^2 x^*(t)}{\Gamma^2 y^*(t)}\right| \leq \left(\int_a^b \left|\frac{k(t,s,\Gamma x^*(s))}{k(t,s,\Gamma y^*(s))}\right|^{ds}\right)^2$$

$$\leq \int_a^b |\max\{|x^*(s)-y^*(s)|, |\Gamma x^*(s)-\Gamma^2 x^*(s)|, |y^*(s)-\Gamma y^*(s)|, |\Gamma y^*(s)-\Gamma^2 y^*(s)|\}|^{ds}$$

$= \max\{\rho(x^*,y^*), \rho(\Gamma x^*,\Gamma^2 x^*), \rho(y^*,\Gamma y^*), \rho(\Gamma y^*,\Gamma^2 y^*)\}^{b-a}$

Hence we have

$\alpha^*(x^*,y^*)\rho(\Gamma^2 x^*,\Gamma^2 y^*) \leq \max\{\rho(x^*,y^*), \rho(\Gamma x^*,\Gamma^2 x^*), \rho(y^*,\Gamma y^*), \rho(\Gamma y^*,\Gamma^2 y^*)\}^{b-a}$,

$\chi = b-a$. For each $(x^*,y^*) \in \varrho$ with $x^* \neq y^*$. Since $\varrho = C[a,b]$ is complete multiplicative metric space, the iteration scheme converges to some point $z^* \in \varrho$, i.e) $\lim_{n\to\infty} x_n^* \to z^*$. Thus z^* is a fixed point of Γ.

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