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A Study On Fuzzy Topological Algebraic System

M. Srividya
Department of Mathematics
Vels Institute of Science, Technology and advanced studies (VISTAS), Chennai, Tamil Nadu, India.
srividyamohan38@gmail.com

G. Jayalalitha
Department of Mathematics
Vels Institute of Science, Technology and advanced studies (VISTAS), Chennai, Tamil Nadu, India.
g.jayalalithamaths.sbs@velsuniv.ac.in

Abstract-This paper is aimed at introducing Fuzzy topological algebraic system by considering TM algebra on a fuzzy topological system. Some properties of fuzzy topological algebraic system are investigated.

Keywords-Fuzzy topological algebraic system, t-transition, transition group, TM algebra, balanced.

I. INTRODUCTION

The fuzzy counterpart of basic topology was commenced by Chang [3]. The fuzzy TM algebra was proposed by Tamilarasi and Megalai [2]. They studied the relation between TM algebra and other algebras. A classical dynamical system and the Fuzzy topological group was introduced by Wieslaw szlenk in 1984 [1]. An algebraic system is a structure (ζ, G, X), where G is a fuzzy topological group, X is a fuzzy TM algebra and topological space and ζ is a continuous function from $G \times X \to X$.

II. PRELIMINERIES

DEFINITION 2.1 [2]

A TM algebra $(X, *, 0)$ is a non-empty set X with constant 0 and a binary operation $*$ satisfying the following axioms

i. $x * 0 = x = 0 * x$
ii. $(x * y) * (x * z) = z * y$

DEFINITION 2.2 [1]

Let X be a fuzzy topological space; G be a fuzzy topological group. If $\pi: G \times X \to X$ satisfies the following:

i) $\pi(0, x) = x$
ii) $\pi(s, (t, x)) = \pi(s + t, x)$
iii) π is fuzzy continuous

Then (π, G, X) is called a fuzzy topological dynamical system.

III. FUZZY TOPOLOGICAL ALGEBRAIC SYSTEM

DEFINITION 3.1

Let X be a TM algebra and fuzzy topological space and let G be a fuzzy topological group. If $\zeta: G \times X \to X$ satisfies the following properties:

i) $\zeta(o, x) = x$

ii) $\zeta(s,(t,x)) = \zeta(s+t,x)$

iii) ζ is fuzzy continuous.

Then (ζ, G, X) is called a fuzzy topological algebraic system.

Throughout this system X will denote fuzzy topological algebraic system.

DEFINITION 3.2

The t-transition of (ζ, G, X) denoted by ζ^t for any $t \epsilon G$, is the mapping $\zeta^t: X \to X$ such that $\zeta^t(x) = \zeta(t, x)$

RESULT 3.3

i) ζ^0 is the identity mapping of X

ii) $\zeta^s\zeta^t = \zeta^{s+t}$ for $s, t\epsilon G$

iii) ζ^t is 1-1 mapping of X onto X and $-(\zeta^t) = \zeta^{-t}$

iv) ζ^t is a fuzzy homeomorphism of X onto X for $t\epsilon G$.

DEFINITION 3.4

The transition group of (ζ, G, X) is the set $G^t = (\zeta^t : t\epsilon G)$. The transition projection of (ζ, G, X) is the mapping $\psi: G \to G^t$ defined as $\psi(t) = \zeta^t$.

DEFINITION 3.5

(ζ, G, X) is said to be effective if $t\epsilon G$ with $t \neq 0$ then $\zeta'(x) \neq x$ for some x.

RESULT 3.6

i) G^t is a group homeomorphism of X onto X.

ii) ψ is a group homeomorphism of G onto G^t.

iii) ψ is 1-1 if and only if (ζ, G, X) is effective.

DEFINITION 3.7

The X-motion of (ζ, G, X) is the mapping $\zeta_x: G \to X$ such that $\zeta_x(t) = \zeta(t, x)$ for any $x\epsilon X$.

RESULT 3.8:

ζ_x is a fuzzy continuous mapping of G into X.

NOTATION

We will denote $\zeta(\alpha \times \beta)$ by $\alpha\beta$.

RESULT 3.9

i) For $t\epsilon G$ and a fuzzy subset β of X, $cl\ \zeta(t \times \beta) = \zeta(t \times cl\ \beta)$

ii) Let G and X be product related, the for a fuzzy subset α of G and a fuzzy subset β of X, $\zeta(cl\ \alpha \times cl\ \beta) \subseteq cl\ \zeta(\alpha \times \beta)$ and $cl\ \zeta(cl\ \alpha \times \beta) = cl\ \zeta(\alpha \times cl\ \beta) = cl\ \zeta(\alpha \times \beta)$
.

iii) $\zeta^t\beta = \beta\alpha^{-1}$ for any $t\epsilon G$.

iv) $\zeta^t\beta^C = 1 - \alpha^t\beta$.

v) If $\beta\epsilon I^x$ is a fuzzy open (closed), then $t\beta$ is fuzzy open (closed).

RESULT 3.10

Let λ be a constant fuzzy subset of G and $\beta\epsilon I^X$ be fuzzy open. Then $\zeta(\lambda \times \beta)$ is fuzzy open.

PROOF:

We have for any $a\epsilon X$,

$$\zeta(\lambda \times \beta)(a) = \sup\{(\lambda \times \beta)(t,x) : \zeta(t,x) = a\}$$
$$= \sup\{(\lambda(t) \wedge \beta(x) : \zeta(t,x) = a\}$$
$$= \sup\{(\lambda \wedge \beta(x) : \zeta(t,x) = a\}$$
$$= \lambda \wedge \sup\{\beta(x) : \zeta^t(x) = a\}$$
$$= \lambda \wedge \sup\{\beta(\zeta^{-t}(a)) : \zeta^{-t}(a) = x\}$$
$$= \lambda \wedge \sup\{\zeta^t\beta(a) : \zeta^{-t}(a) = x\}, \quad \because \beta\zeta^{-t} = \zeta^t\beta$$
$$= \{\lambda \wedge \{\vee(\zeta^t\beta)\}\}(a), \text{ where } \zeta^{-t}(a) = x$$

Thus $\zeta(\lambda \times \beta) = \lambda \wedge \{\vee(\zeta^t\beta)\}$. Now each ζ^t is open and β is open so $\zeta^t\beta$ is open. Also, by definition of fuzzy topology, λ is open. Consequently $\lambda \wedge \{\vee(\zeta^t\beta)\}$ is open. Hence $\zeta(\lambda \times \beta)$ is open.

COROLLARY 3.11

Let β be a fuzzy open subset of X, then for any fuzzy point t_λ of G, $\zeta(t_\lambda \times \beta)$ is fuzzy open.

COROLLARY 3.12

Let α be any fuzzy subset of G and $\beta\epsilon I^x$ be fuzzy open, then $\zeta(\lambda \times \beta)$ is fuzzy open.

CORROLLARY 3.13

Let $t\epsilon G$ and $\beta\epsilon I^x$ be a fuzzy neighbourhood of X, for some $x\epsilon X$. Then $\zeta(t_\lambda \times \beta)$ is a fuzzy neighbourhood of $\zeta(t_\lambda \times \beta)$.

RESULT 3.14

Let β be a fuzzy closed subset of X then for any fuzzy point t_λ of G, $\zeta(t_\lambda \times \beta)$ is fuzzy closed.

PROOF:

$\zeta(t_\lambda \times \beta)(a) = \sup\{t_\lambda \times \beta)(s,x) : \zeta(s,x) = a\}$ $for\ any\ a\epsilon X.$

$= \sup\{t_\lambda(s) \wedge \beta(x)) : \zeta(s,x) = a\}$

$= \lambda \wedge \beta(x) : \zeta(t,x) = a$ $\{\because t_\lambda(s) \neq 0\ only\ when\ s = t\}$

$= \lambda \wedge \beta(x) : \zeta^t(x) = a$

$= \lambda \wedge \beta(\zeta^{-t}(a))$

$= \lambda \wedge \zeta^t\beta(a)$ $\because \beta\zeta^{-t} = \zeta^t\beta$

$= (\lambda \wedge \zeta^t\beta)(a)$, considering λ as a constant fuzzy subset of X.

Thus $\zeta(t_\lambda \times \beta) = \lambda \wedge \zeta^t\beta$. Now ζ^t is closed and β is closed so $\zeta^t\beta$ is closed. Also λ is closed. Consequently $\lambda \wedge \zeta^t\beta$ is fuzzy closed. Hence $\zeta(t_\lambda \times \beta)$ is closed.

COROLLARY 3.15

Let α be any fuzzy subset of G and $\beta\epsilon I^X$ be fuzzy closed. If $\sup\alpha$ is finite, then $\zeta(\alpha \times \beta)$ is fuzzy closed.

PROOF:

We have $\alpha = \vee t_\lambda$, where $\lambda = \alpha(x)$. So, $\zeta(\alpha \times \beta) = \zeta(\vee t_\lambda \times \beta) = \vee\ \zeta(t_\lambda \times \beta)$.

Since each $\zeta(t_\lambda \times \beta)$ is closed and $\sup\alpha$ is finite, the union is over finite number of closed fuzzy subsets. Hence $\zeta(\alpha \times \beta)$ is closed.

RESULT 3.16

Let β be a neighbourhood of $b = \zeta(t,x)$ in X. Then for each real number χ with $0 < \chi < \beta(b)$ there exists open neighbourhoods β_1, β_2 of the points t, x respectively such that $\zeta(\beta_1 \times \beta_2) \subseteq \beta$ and $\min\{\beta_1(t), \beta_2(t)\} > \chi$.

PROOF:

We assume that β is open without loss of generality. Since the map $\zeta : G \times X \to X$ is continuous, the fuzzy set $\zeta^{-1}(\beta)$ is open in $G \times X$.

Since, $\zeta^{-1}(\beta)(t,x) = \beta(b) > \chi$ there exist open fuzzy sets β_1, β_2 in G and X respectively with $\beta_1 \times \beta_2 \leq \zeta^{-1}(\beta)$ and $(\beta_1 \times \beta_2)(t,x) > \chi$.

Clearly β_1, β_2 are open neighbourhood of t, x respectively and $\zeta(\beta_1 \times \beta_2) \subseteq \beta$.

REMARK 3.17

Let (ζ, G, X) be a fuzzy topological algebraic system. Then for any $\beta\epsilon I^X$, $\zeta(s, \zeta(t,\beta)) = \zeta(s+t, \beta), for\ s,t\epsilon G,$

$\zeta(s,(t,\beta))(x) = \sup\{(s,\zeta(t,\beta))(r,a); \zeta(r,a) = x\}$

$= \sup\{\zeta(t,\beta)(a); \zeta(s,a) = x\}$

$= \sup[\sup\{(t,\beta)(p,y) : \zeta(p,y) = a\} : \zeta(s,a) = x]$

$$= \sup\left[\sup\{\beta(y) : \zeta(t,y) = a\} : \zeta(s,a) = x\right]$$

$$= \sup\left[\sup\{\beta(y) : \zeta(s,\zeta(t,y)) = x\}\right]$$

$$= \sup\left[\sup\{\beta(y) : \zeta(s+t,y) = x\}\right]$$

$$= \zeta(s+t,\beta)(x).$$

RESULT 3.18

Let X be a fuzzy topological algebraic system, $h \epsilon X$ and β a neighborhood of h. Then for each $0 < \psi < \beta(h)$ there exist an open neighborhood of zero in $\mathcal{S}$ such that $\lambda(0) > \psi$ and $\lambda(t) \le \beta\zeta(t,h)$ for all $t \epsilon \mathcal{S}$.

PROOF:

We have $\zeta : \mathcal{S} \times X \to X$ is continuous. Since $\zeta(0,h) = h$, $\zeta^{-1}(\beta)$ is a neighborhood of $(0,h)$ in $\mathcal{S} \times X$. Hence there exist an open neighborhood λ_1 of zero in $\mathcal{S}$ and an open neighborhood β_1 of h in X such that $\lambda_1 \times \beta_1 \le \zeta^{-1}(\beta)$ and $\min\{\lambda_1(0), \beta_1(h) > \psi$. Choose a real number ψ_1 such that $\psi < \psi_1 < \min\{\lambda_1(0), \beta_1(h)\}$.

The set $H = \{t \epsilon\, \mathcal{S} : \lambda_1(t)) > \psi_1$ is an open subset of $\mathcal{S}$ for the usual topology of $\mathcal{S}$. Since $0 \epsilon H$, there exist $\varepsilon > 0$ such that $\{t : |t| < \varepsilon\} \subset H$.

Let $\lambda : \mathcal{S} \to I$ be a continuous function, $0 \le \lambda \le \psi_1$, $\lambda(0) = \psi_1$, $\lambda(t) = 0$ if $|t| \ge \varepsilon$. We will show that $\lambda(t) \le \beta(th)$ for all $t \epsilon\, \mathcal{S}$. In fact, if $|t| \ge \varepsilon$, then $\lambda(t) = 0$. For $|t| < \varepsilon$ we have $\lambda_1(t) > \psi_1$ and hence,

$\beta(th) = \beta(\zeta(t,h)) = \zeta^{-1}(\beta)(t,h) \ge \min\{\lambda_1(t), \beta_1(h)\}$

$$> \psi_1$$

$$\ge \lambda(t).$$

COROLLARY 3.19

Given a neighbourhood β of h in a fuzzy topological algebraic system X and $0 < \psi < \beta(h)$, there exists $\varepsilon > 0$ such that $\beta(th) > \psi$ if $|t| \le \varepsilon$.

PROOF:

By the previous result, there exists an open neighbourhood λ of zero in $\mathcal{S}$ such that $\lambda(0) > \psi$ and $\lambda(t) \le \beta(tx)$ for all $t \epsilon \mathcal{S}$. Since the set $H = \{t \epsilon \mathcal{S} : \lambda(t) > \psi\}$ is open and contains 0, there exists $\epsilon > 0$ such that $t \epsilon H$ whenever $|t| \le \varepsilon$.

Therefore, $\beta(th) > \psi$ if $|t| \leq \varepsilon$.

RESULT 3.20

Let X be a fuzzy topological algebraic system and β a neighborhood of h. Then, for each real number ψ with $0 < \psi < \beta(h)$ there exists an open neighborhood γ of $h\epsilon X$, with $\gamma \leq \beta$ and $\gamma(h) > \psi$ and a positive real number such that $\zeta(t,\gamma) \leq \beta$ foe each $t\epsilon\mathcal{S}$ with $|t| \leq \varepsilon$.

PROOF:

We may assume that β is zero. The function $\zeta: \mathcal{S} \times X \to X$ is continuous. Since $\zeta^{-1}(\beta)(0,h) = \beta(h) > \psi$, there exists an open neighbourhood of zero in $\mathcal{S}$ and an open neighbourhood β_1 of h in X such that $\min\{\lambda(0), \beta_1(h)\} > \psi$ and $\lambda \times \beta_1 \leq \zeta^{-1}(\beta)$. Let $\psi < \psi_1 < \lambda(0)$ and set $\gamma = \psi_1 \wedge \beta_1 \wedge \beta$. Then γ is open, $\gamma \subseteq \beta$ and $\gamma(h) > \psi$. Since λ is a lower semi continuous function on $\mathcal{S}$ when $\mathcal{S}$ has its usual topology, there exists a positive number ε such that $\{t\epsilon\mathcal{S} : |t| \leq \varepsilon\} \subset \{t : \lambda(t) > \psi_1\}$. Now let $|t| \leq \varepsilon$. For each $x\epsilon X$ we have,

$$\beta(tx) = \beta(\zeta(t,x)) = \zeta^{-1}(\beta)(t,x) \geq (\lambda \times \beta_1)(t,x)$$
$$\geq (\lambda \times \gamma)(t,x)$$
$$= \min\{\lambda(t), \gamma(x)\}$$
$$= \gamma(x)$$

Since $\gamma(x) \leq \psi_1 < \lambda(t)$ and since $\beta(\zeta(t,x)) \geq \gamma(x)$ for each $x\epsilon X$, it follows that $\zeta(t \times \gamma) \leq \beta$.

DEFINITION 3.21

Let $(\zeta, \mathcal{S}, X)$ be a fuzzy topological algebraic system. A fuzzy set β in X is called balanced if $\zeta(t,\beta) \leq \beta$ for each $t\epsilon\mathcal{S}$ with $|t| \leq 1$.

RESULT 3.22

β is balanced if and only if $\beta(tx) \geq \beta(x)$.

PROOF:

For any $r\epsilon X$,

We have

$$(t\beta)(r) = \zeta(t \times \beta)(r) = \sup\{(t \times \beta)(s,x) : \zeta(s,x) = r\}$$
$$= \sup\{\min(t(s), \beta(x)) : \zeta(t,x) = r\} \qquad = \sup(\beta(x) : \zeta(t,x) = r\}$$
……………..(1)

Suppose $\beta(tx) \geq \beta(x)$ for each $t\epsilon\ \mathcal{S}$ with $|t| \leq 1$.

That is $\beta(\zeta(t,x)) \geq \beta(x)$ for each $t\epsilon\ \mathcal{S}$ with $|t| \leq 1$.

Then for any $r \epsilon X$, $(t\beta)(r) = \sup\{\beta(x) : \zeta(t,x) = r\}$ (From (1)
$\leq \sup\{\beta(\zeta(t,x)) : \zeta(t,x) = r\}$ (given)

$= \beta(r)$.

Hence $t\beta \leq \beta$.

Consequently, let β be balanced. That is $\zeta(t,\beta) \leq \beta$ and $t\beta \leq \beta$ for each $t\epsilon\mathcal{S}$ with $|t| \leq 1.$ We have $t\beta \leq \beta \Rightarrow \zeta(t,\beta) \leq \beta$
$\Rightarrow \zeta(t,\beta)(r) \leq \beta(r)$ for all $r\epsilon X$ $\Rightarrow \sup\{\beta(\zeta(t,x)) : \zeta(t,x) = r\} \leq \beta(r)$
. (from (1))
$\Rightarrow \beta(x) \leq \beta(r)$ for all x: $\zeta(t,x) = r \Rightarrow \beta(x) \leq \beta(\zeta(t,x))$
Hence $\beta(tx) \geq \beta(x)$.

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